Second-Order Deformation of the Front of a Mode I Crack Propagating in a Heterogeneous Material

J.B. Leblond[1], L. Ponson[2] and M. Vasoya[3]

[1] UPMC Univ Paris 6 and CNRS, UMR 7190, Institut Jean Le Rond d'Alembert, F-75005 Paris, France; e-mail: jbl@lmm.jussieu.fr
[2] Same address; e-mail: laurent.ponson@upmc.fr
[3] Same address; e-mail: manishvasoya36@gmail.com

ABSTRACT. *We calculate the distribution of the stress intensity factor for a semi-infinite tensile crack with a slightly curved front embedded in some infinite medium, up to second order in the deviation from straightness. From there, we determine the equilibrium shape of the front of the crack when it propagates along a heterogeneous fracture plane, up to second order in the toughness fluctuations. As a first application, we show that the "apparent fracture toughness" experienced by a crack propagating in a randomly heterogenous material is slightly less than the rigorous average value of the local toughness. As a second application, we determine the equilibrium shape of a crack front penetrating into an infinitely elongated harder obstacle.*

INTRODUCTION

In a celebrated paper, Rice [1] derived a formula for the first-order variation of the mode I stress intensity factor (SIF) resulting from some small but otherwise arbitrary coplanar perturbation of the front of a semi-infinite tensile crack in an infinite body. This formula has been applied many times since, to study the propagation of cracks in materials having an inhomogeneous distribution of fracture toughness; see e.g. the works of Gao and Rice [2] and Chopin [3] on the trapping of crack fronts by obstacles.

In the present work, we shall extend Rice's formula to second order in the deviation of the crack front from straightness. This will be done through basically straightforward application of general formulae for the first-order variations of the stress intensity factor and fundamental kernel (FK, to be defined below) due to Rice [4].

Two applications of the extended formulae found will be envisaged:

- Calculation of the "apparent toughness" of a heterogeneous material, that is the toughness of some suitably defined "equivalent homogeneous material". It will be shown that this apparent toughness is slightly less than the rigorous average value of the local toughness, as a result of the fact that strict stability of crack propagation demands that the unperturbed stress intensity factor decrease when the front moves in the direction of propagation under constant loading.

- Determination of the equilibrium shape of a crack front penetrating into a harder obstacle of infinite length in the direction of propagation, up to second order in the gap of toughness between the matrix and the obstacle.

RICE'S FORMULAE FOR AN ARBITRARY PLANAR CRACK

Consider an isotropic elastic body Ω containing a planar crack with arbitrarily shaped contour. Assume the body and the loading to be symmetric about the crack plane. The crack is then in a situation of pure mode I all along its front; let s and $K^0(s)$ denote a curvilinear abscissa along this front and the local SIF, respectively.

Now displace the crack front, within the crack plane, by some infinitesimal distance $\delta a(s)$ perpendicularly to itself, while keeping the loading unchanged. The resulting infinitesimal variation $\delta K(s_1)$ of the local SIF is given by Rice's *first formula* [4]:

$$\delta K(s_1) = \left[\delta K(s_1)\right]_{\delta a(s)=\delta a(s_1),\forall s} + PV\int_{CF} Z(s_1,s)K^0(s)\left[\delta a(s) - \delta a(s_1)\right]ds \qquad (1)$$

where the integral over the crack front (CF) is to be understood as a Cauchy principal value (PV). In this expression, $\left[\delta K(s_1)\right]_{\delta a(s)=\delta a(s_1),\forall s}$ denotes the value of $\delta K(s_1)$ for a *uniform* advance of the front equal to $\delta a(s_1)$, and $Z(s_1,s)$ the FK of the cracked geometry considered. This quantity, which is tied to Bueckner's mode I crack-face weight function, has no dependence upon the loading other than on which portions of Ω and $\partial\Omega$ have forces versus displacements imposed, and verifies the following properties:

$$Z(s_1,s_2) = Z(s_2,s_1) \quad ; \quad Z(s_1,s_2) \sim \frac{1}{2\pi(s_1-s_2)^2} \quad \text{for} \quad s_1 - s_2 \to 0. \qquad (2)$$

In addition, if $\delta a(s)$ vanishes at s_1 and s_2, the infinitesimal variation $\delta Z(s_1,s_2)$ of the FK at these points is given by Rice's *second formula* [4], which involves two principal values, at the points s_1 and s_2:

$$\delta Z(s_1,s_2) = PV\int_{CF} Z(s_1,s)Z(s,s_2)\delta a(s)ds. \qquad (3)$$

APPLICATION OF RICE'S FORMULAE TO A SEMI-INFINITE CRACK

Generalities

We now consider (Figure 1) a semi-infinite tensile crack located in some infinite body subjected to prescribed forces only. The crack front is assumed to be slightly curved, its equation in the plane Oxz being of the form

$$x(z) = a + \varepsilon\phi(z), \tag{4}$$

where a is the distance from the axis Oz to some "reference straight front", ε a small parameter and $\phi(z)$ a given function. The position of the front is thus specified by the parameters a and ε, and the position of a current point along it by the coordinate z.

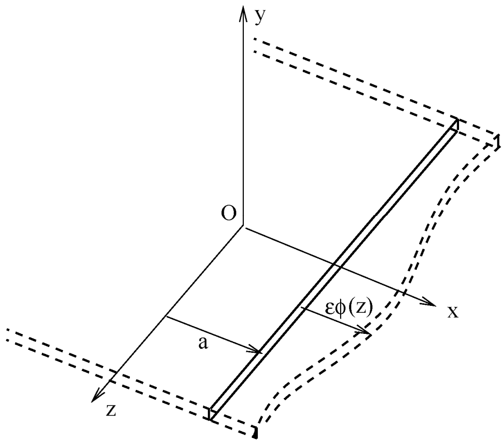

Figure 1: A semi-infinite crack with a slightly perturbed front in an infinite body.

The SIF for a given, fixed loading and the FK for this cracked geometry are denoted $K(a,\varepsilon;z_1)$ and $Z(\varepsilon;z_1,z_2)$ respectively. (The FK is independent of a because the geometry is insensitive to translatory motions of the crack front in the direction x). The topic of interest here is the expansions of $K(a,\varepsilon;z_1)$ and $Z(\varepsilon;z_1,z_2)$ in powers of ε, and more precisely the second-order expression of the former quantity and the first-order expression of the latter:

$$\begin{cases} K(a,\varepsilon;z_1) = K^0(a) + \varepsilon K^1(a;z_1) + \varepsilon^2 K^2(a;z_1) + O(\varepsilon^3) \\ Z(\varepsilon;z_1,z_2) = Z^0(z_1,z_2) + \varepsilon Z^1(z_1,z_2) + O(\varepsilon^2). \end{cases} \tag{5}$$

(The loading is assumed to have a translatory invariance in the direction z so that the unperturbed SIF $K^0(a)$ depends on the position a of the (straight) front but not on the position of the point of observation along it).

Expression of the fundamental kernel at order 0

No general expression can be provided for the unperturbed SIF $K^0(a)$ since it depends on the loading, but the expression of the unperturbed FK $Z^0(z_1,z_2)$ is (Rice [1]):

$$Z^0(z_1,z_2) = \frac{1}{2\pi(z_1 - z_2)^2}. \tag{6}$$

Expressions of the stress intensity factor and fundamental kernel at order 1

The expression of $K^1(a;z_1)$ is obtained by applying Rice's first formula (1) to the straight configuration of the front using Eq. (6), and integrating by parts:

$$K^1(a;z_1) = \frac{dK^0}{da}(a)\phi(z_1) + \frac{K^0(a)}{2\pi} PV\int_{-\infty}^{+\infty} \frac{\phi'(z)}{z-z_1} dz. \tag{7}$$

The expression of $Z^1(z_1,z_2)$ may be obtained in a similar way from Rice's second formula (3) and Eq. (6). An arbitrary perturbation $\varepsilon\phi(z)$ generally violates the conditions $\varepsilon\phi(z_1) = \varepsilon\phi(z_2) = 0$, necessary for Eq. (3) to be applicable, but this difficulty may be overcome by using Rice's suggestion [4] to decompose this perturbation in the form $\varepsilon\phi(z) = [\varepsilon\phi(z) - \varepsilon\phi_*(z)] + \varepsilon\phi_*(z)$ where $\varepsilon\phi_*(z)$ is a suitable combination of a translatory motion and a rotation (having no effect on the FK) such that $\varepsilon\phi_*(z_1) = \varepsilon\phi(z_1)$ and $\varepsilon\phi_*(z_2) = \varepsilon\phi(z_2)$. One thus gets:

$$\begin{aligned} Z^1(z_1,z_2) = \frac{1}{4\pi^2(z_1-z_2)^2} PV\int_{-\infty}^{+\infty} \Bigg[&\left(\frac{1}{z-z_1} + \frac{1}{z-z_2}\right)\phi'(z) \\ &+ \frac{2}{z_1-z_2}\left(\frac{1}{z-z_2} - \frac{1}{z-z_1}\right)\phi(z)\Bigg] dz. \end{aligned} \tag{8}$$

Expression of the stress intensity factor at order 2

The expression of $K^2(a;z_1)$ may be obtained by applying Rice's first formula (1) to a configuration of the front deduced from the straight one through the perturbation $\varepsilon\phi(z)$, and further perturbed by the amount $\delta\varepsilon\,\phi(z)$ where $\delta\varepsilon$ is an infinitesimal quantity. The formula provides an integral expression of the derivative $\partial K(a,\varepsilon;z_1)/\partial\varepsilon$ accurate to first-order in ε if the first-order expressions of $Z(\varepsilon;z_1,z)$ and $K(a,\varepsilon;z)$ are employed; and the second-order expression of $K(a,\varepsilon;z_1)$ follows through integration. One thus gets:

$$\begin{aligned} K^2(z_1) = &\frac{1}{2}\frac{d^2K^0}{da^2}(a)[\phi(z_1)]^2 + \frac{1}{2\pi}\frac{dK^0}{da}(a)\, PV\int_{-\infty}^{+\infty} \frac{\phi(z)\phi'(z)}{z-z_1} dz \\ &+ \frac{K^0(a)}{8\pi^2} PV\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Bigg[\left(\frac{1}{z'-z_1} + \frac{2}{z'-z}\right)\phi'(z') \\ &+ \frac{2}{z-z_1}\left(\frac{1}{z'-z_1} - \frac{1}{z'-z}\right)\phi(z')\Bigg] \frac{\phi(z)-\phi(z_1)}{(z-z_1)^2} dz dz'. \end{aligned} \tag{9}$$

Expression of the energy-release-rate at order 2 in Fourier's space

From there, one may calculate the second-order expression of the energy-release rate (ERR) $G(a,\varepsilon;z_1)$. The result is best expressed in Fourier's space. The definition of the Fourier transform $\hat{\psi}(k)$ of an arbitrary function $\psi(z)$ adopted in this work is

$$\psi(z)=\int_{-\infty}^{+\infty}\hat{\psi}(k)e^{ikz}dk \quad \Leftrightarrow \quad \hat{\psi}(k)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\psi(z)e^{-ikz}dz. \tag{10}$$

With these notations, one gets, $G^0(a)$ denoting the unperturbed ERR:

$$\hat{G}(a,\varepsilon;k_1)=G^0(a)\delta(k_1)+\varepsilon\widehat{G^1}(a;k_1)+\varepsilon^2\widehat{G^2}(a;k_1)+O(\varepsilon^3), \tag{11}$$

$$\begin{cases}\widehat{G^1}(a;k_1)=G^0(a)\left[\dfrac{dG^0}{G^0da}(a)-|k_1|\right]\hat{\phi}(k_1)\\ \widehat{G^2}(a;k_1)=G^0(a)\displaystyle\int_{-\infty}^{+\infty}Q(a;k,k_1-k)\hat{\phi}(k)\hat{\phi}(k_1-k)dk,\end{cases} \tag{12}$$

$$\begin{aligned}Q(a;k,k')=\frac{1}{2}\frac{d^2G^0}{G^0da^2}-\frac{1}{4}\frac{dG^0}{G^0da}\left(|k+k'|-|k|-|k'|\right)+\frac{1}{8}\{\mathrm{sgn}(kk')(k+k')^2\\ +\left[\mathrm{sgn}(k)-\mathrm{sgn}(k')\right]|k+k'|(k-k')-(|k|-|k'|)^2\}.\end{aligned} \tag{13}$$

EQUILIBRIUM CRACK FRONT SHAPE IN A HETEROGENEOUS MEDIUM

We now consider crack propagation governed by Griffith's criterion in a material having a heterogeneous fracture toughness $G_c(x,z)$ given by

$$G_c(x,z)=\overline{G_c}\left[1+\varepsilon g_c(x,z)\right] \tag{14}$$

where $\overline{G_c}$ is a "mean fracture toughness", ε a small parameter and $g_c(x,z)$ a given function.

Assuming G to be equal to G_c at every point of the crack front, the distribution of toughness determines the shape of this front in the form

$$x=a+\varepsilon\phi^1(a;z)+\varepsilon^2\phi^2(a;z)+O(\varepsilon^3) \tag{15}$$

where a, $\phi^1(a;z)$ and $\phi^2(a;z)$ are a parameter and functions to be determined. To do so, it suffices to equate the Fourier transform of the expansion of the ERR, deduced from Eqs. (11), (12) and (13) with $\phi=\phi^1+\varepsilon\phi^2$, to the Fourier transform of the expansion of the local toughness,

$$G_c(x,z)=\overline{G_c}\left[1+\varepsilon g_c(a,z)+\varepsilon^2\frac{\partial g_c}{\partial x}(a,z)\phi^1(a;z)\right]+O(\varepsilon^3).$$

One thus gets the following conditions:

$$\begin{cases} G^0(a)=\overline{G_c} \quad , \quad \widehat{\phi^1}(a;k_1)=-\dfrac{\widehat{g_c}(a,k_1)}{|k_1|-\dfrac{dG^0}{G^0da}(a)}, \\ \widehat{\phi^2}(a;k_1)=\dfrac{1}{|k_1|-\dfrac{dG^0}{G^0da}(a)}\left\{\displaystyle\int_{-\infty}^{+\infty}Q(a;k,k_1-k)\frac{\widehat{g_c}(a,k)}{|k|-\dfrac{dG^0}{G^0da}(a)}\times\right. \\ \qquad \left.\times\dfrac{\widehat{g_c}(a,k_1-k)}{|k_1-k|-\dfrac{dG^0}{G^0da}(a)}dk+\displaystyle\int_{-\infty}^{+\infty}\frac{\partial\widehat{g_c}}{\partial x}(a,k)\frac{\widehat{g_c}(a,k_1-k)}{|k_1-k|-\dfrac{dG^0}{G^0da}(a)}dk\right\}. \end{cases} \tag{16}$$

The first condition determines the position a of the reference straight front as a function of the loading applied, and the second and third conditions then determine the first- and second-order perturbations of this front.

APPARENT TOUGHNESS OF A HETEROGENEOUS MATERIAL

In this section and the next one, we introduce the following hypotheses on the variation of the unperturbed ERR:

$$\frac{dG^0}{da}(a)<0 \quad ; \quad \frac{d^2G^0}{da^2}(a)>0. \tag{17}$$

The first hypothesis is necessary for crack propagation to be strictly stable, and the second is reasonable since $G^0(a)$ is then a positive decreasing function of a.

We consider here a material with a randomly heterogeneous, but statistically homogeneous distribution of fracture toughness. The “apparent toughness” $G_c^{\mathrm{eff}}(a)$ of this material is defined as the value of the ERR for a fictitious straight crack having the same average position as the real, curved one:

$$G_c^{\mathrm{eff}}(a)=G^0\left(a+\varepsilon\left\langle\phi^1(a;z)\right\rangle+\varepsilon^2\left\langle\phi^2(a;z)\right\rangle+O(\varepsilon^3)\right) \tag{18}$$

where $\left\langle\phi^1(a;z)\right\rangle$ and $\left\langle\phi^2(a;z)\right\rangle$ are the average values of $\phi^1(a;z)$ and $\phi^2(a;z)$.

The calculation of these average values is made easier by assuming the distribution of toughness to be periodic in the direction z, so that $g_c(x,z)$ is of the form

$$g_c(x,z) = \sum_{m=-\infty}^{+\infty} c_m(x) e^{imk_0 z} \tag{19}$$

where k_0 is a positive wavenumber and the $c_m(x)$ coefficients; $c_0(x)$ may be assumed to be zero for every x since the degree of arbitrariness left on the definition of $\overline{G_c}$ permits to consider it as identical to the exact average value of the local toughness.

Calculating then $\left\langle \phi^1(a;z) \right\rangle$ and $\left\langle \phi^2(a;z) \right\rangle$ and using the definition (18), one gets

$$G_c^{\text{eff}}(a) = \overline{G_c} + \varepsilon^2 \sum_{m=1}^{+\infty} \frac{\dfrac{dG^0}{da}(a) m k_0 - \dfrac{d^2 G^0}{da^2}(a)}{\left[m k_0 - \dfrac{dG^0}{G^0 da}(a) \right]^2} \left\langle \left| c_m(a) \right|^2 \right\rangle + O(\varepsilon^3) \tag{20}$$

where the $\left| c_m(x) \right|^2$ have been averaged over a distance much larger than the typical distance of fluctuation of the local toughness, though still much smaller than the distance over which $G^0(a)$ varies significantly. By inequalities (17), each term in the series is negative, so that $G_c^{\text{eff}}(a)$ is slightly less than the exact average toughness $\overline{G_c}$; the effect is tied to the dependence of the unperturbed ERR $G^0(a)$ upon the position a of the straight crack front and disappears in the limit $\dfrac{dG^0}{da}(a) \to 0$.

EQUILIBRIUM SHAPE OF A CRACK FRONT MEETING AN OBSTACLE

We finally consider a matrix of toughness G_c^{M} containing an obstacle of width $2d$ in the direction z, infinite length in the direction x and toughness $G_c^{\text{O}} > G_c^{\text{M}}$. The toughness distribution may be represented by Eq. (14) with

$$\overline{G_c} = G_c^{\text{M}} \quad ; \quad \varepsilon = \frac{G_c^{\text{O}} - G_c^{\text{M}}}{G_c^{\text{M}}} \quad ; \quad g_c(x,z) = \begin{cases} 1 & \text{if} \quad |x| < d \\ 0 & \text{if} \quad |x| > d. \end{cases} \tag{21}$$

We are especially interested here in the first- and second-order perturbations of the crack front in the limit $\dfrac{dG^0}{da}(a) \to 0$ (meaning that the characteristic distance of variation of the unperturbed ERR is much larger than that of toughness fluctuations).

The integral expressions of $\phi^1(a;z)$ and $\phi^2(a;z)$ are found to diverge in this limit, but not those of $\phi^1(a;z)-\phi^1(a;0)$ and $\phi^2(a;z)-\phi^2(a;0)$ which suffice to characterize the deviations of the front from straightness and are given by

$$\begin{cases} \phi^1(a;z)-\phi^1(a;0)=\dfrac{d}{\pi}\left[(1+u)\ln\left(\left|1+u\right|\right)+(1-u)\ln\left(\left|1-u\right|\right)\right] \\ \phi^2(a;z)-\phi^2(a;0)=\begin{cases} -\dfrac{d}{2\pi}\left[(1+u)\ln(1+u)+(1-u)\ln(1-u)\right] & \text{if } |u|\leq 1, \\ -\dfrac{d}{2\pi}\left[(u-\operatorname{sgn}(u))\ln\left(\dfrac{u+1}{u-1}\right)+2\ln 2\right] & \text{if } |u|\geq 1 \end{cases} \end{cases} \quad u=\frac{z}{d}. \tag{22}$$

Figure 2 shows the results obtained for various values of ε. The perturbation $\varepsilon\phi^1(a;z)+\varepsilon^2\phi^2(a;z)$ has been divided here by ε to evidence the non-proportionality of the two quantities, and the curves have been made to coincide on the boundaries of the obstacle ($z=\pm d$) rather than at its center ($z=0$) to facilitate their comparison.

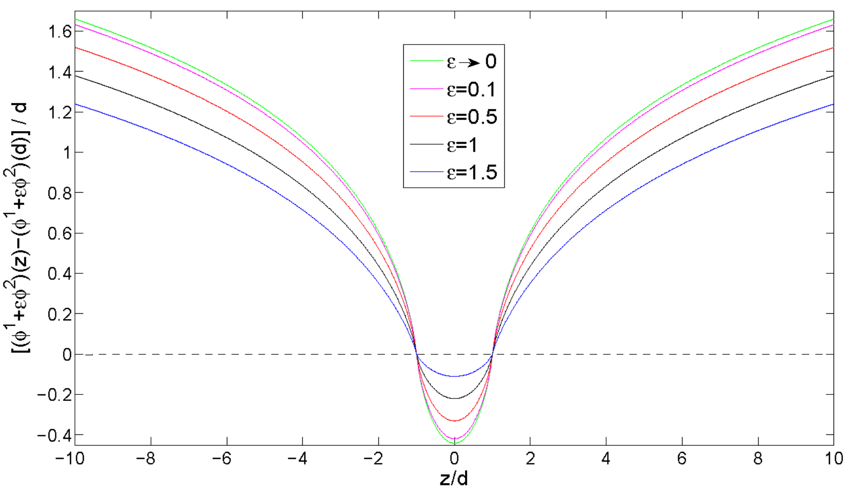

Figure 2: The shape of a crack front deformed by the presence of an obstacle.

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