Fatigue crack nucleation at a stress concentration point

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ABSTRACT. *The coupled criterion, using both stress and energy conditions, satisfactorily predicts the crack nucleation starting from a stress concentration point in brittle or quasi-brittle materials under monotonic loading. But it is a priori difficult to generalize to fatigue. A first fatigue model was established based on a Dugdale cohesive zone model but in turn it proved difficult to be extended to complex loadings.*
The present work is twofold: (i) showing how to generalize the coupled criterion to take into account both shear and tensile strengths as well as mode I and II toughness to predict crack nucleation under monotonic complex loadings; (ii) extending this criterion to the crack nucleation under fatigue cycles by considering a gradual degradation along the presupposed crack path. One parameter is identified so that the rate of advance coincides with that of a Paris law in case of a pre-existing long crack. As derived from the model, the growth is intermittent which provides an explanation for the striations observed in experiments.
A relationship is established between the initial crack velocity and the exponent of the singularity characterizing the stress concentration, showing that the weaker the singularity and the smaller the crack advance rate. From these considerations one can deduce that the short crack range can be characterized by the distance required to reach a steady velocity.

INTRODUCTION

The coupled criterion, using both stress and energy conditions [1], was developed to predict crack nucleation at stress concentration points in brittle materials under monotonic loading. It was established using asymptotic expansions and theory of singularity and gave satisfactory predictions. However, in its original form, it appeared difficult to generalize to fatigue. It was a priori dedicated to brittle fracture and did not seem able to integrate concepts such as the accumulation of damage or plasticity. That is why we tried in a first step to develop a fatigue criterion [2] based on the use of Dugdale's cohesive zone model (CZM) [3]. It was originally developed as a simplified model of crack tip plasticity: the traction acting ahead of the crack tip cannot exceed a threshold value denoted here σ_c (the tensile strength), but corresponding to the plastic flow threshold in the original model. It was extended to V-notches in homogeneous

materials [4] and proved that, under symmetric monotonic loadings, such a model gave similar answers to the coupled criterion. The generalization to fatigue was obtained by exploiting the following idea: there is an accumulation of the successive elementary openings of the cohesive zone until it reaches a critical value [5]. This model is similar to a kind of cumulative plasticity. However, this approach has had to be modified because it leads systematically to a Paris exponent [6] equal to 4 which is a special case poorly adapted to quasi-brittle materials. For this purpose, an alternative interpretation of the accumulation law was made which is more consistent with the concept of cumulative damage. We proposed that only an adjustable part of the opening energy is converted into damage while the other is restored. The partition is achieved by using a parameter which can be identified in the particular case of a pre-existing long crack using a known relationship with the Paris exponent.
Unfortunately these results are difficult to generalize to complex loadings or other geometries that will inevitably require the concept of mode mixity. All these statements will be adapted to develop the model presented below.

COMPLEX MONOTONIC LOADINGS

Before focusing on fatigue, the first stage is to generalize the coupled criterion [2] to monotonic complex loadings. This was required in many situations examined in different contexts such as the study of bonded structures under complex loading using the Arcan test [7] (Fig. 1) for instance.

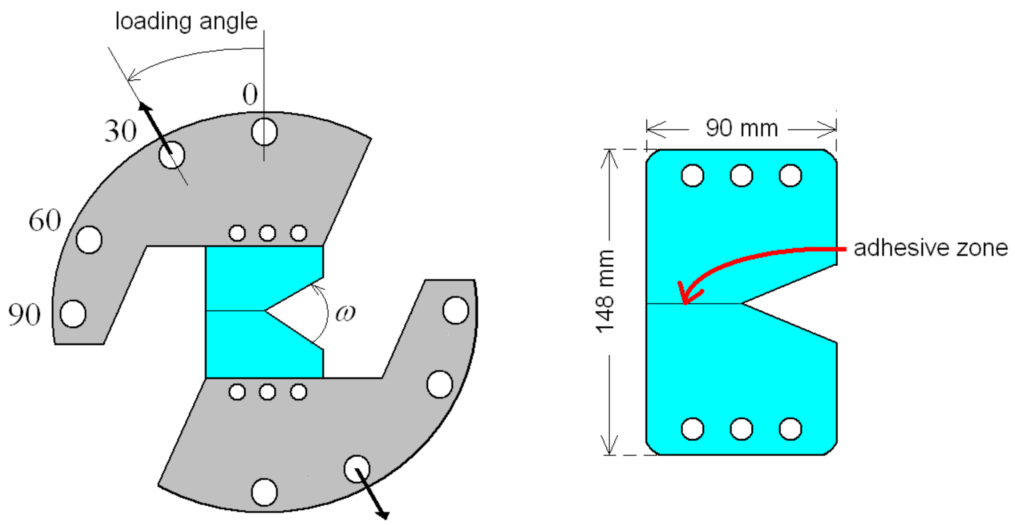

Figure 1. The Arcan test (left) and a PMMA specimen with an adhesive joint (right). Such a setup allows a complete range of mixity from a pure symmetric mode (loading angle 0°) to a pure antisymmetric one (loading angle 90°).

The Williams expansion of the elastic solution $\underline{U}(r,\theta)$, in polar coordinates emanating from the notch root, is written with two singular terms, and a mixity parameter m

$$\begin{cases} \underline{U}(r,\theta) = \underline{U}(O) + k_1 r^{\lambda_1}\left(\underline{u}_1(\theta) + m\ \underline{u}_2(\theta)\right) + ... \\ \sigma(r,\theta) = k_1 r^{\lambda_1 - 1}\left(s_1(\theta) + m\ s_2(\theta)\right) + ... \\ \tau(r,\theta) = k_1 r^{\lambda_1 - 1}\left(t_1(\theta) + m\ t_2(\theta)\right) + ... \end{cases} \quad \text{with } m = \frac{k_2}{k_1} r^{\lambda_2 - \lambda_1} \tag{1}$$

Where σ and τ hold for the tensile ($\sigma_{\theta\theta}$) and shear ($\sigma_{r\theta}$) components of the stress tensor. The coefficients k_1 and k_2 are the generalized stress intensity factors (GSIF) of the two singular terms characterized by the exponents λ_1 and λ_2 and the two modes $\underline{u}_1(\theta)$ and $\underline{u}_2(\theta)$, respectively symmetric and antisymmetric with respect to the bisector. The functions $s_1(\theta)$ and $t_1(\theta)$ (resp. $s_2(\theta)$ and $t_2(\theta)$) are associated with the components $\theta\theta$ and $r\theta$ of the stress field derived from $\underline{u}_1(\theta)$ (resp. $\underline{u}_2(\theta)$) through the elastic constitutive law. Note that the definition of m implies $k_1 \neq 0$, the particular case $k_1 = 0$ refers to the pure mode II that can be treated similarly to the pure mode I. Energy and stress conditions give

$$\delta W^P(l) \geq \int_0^l G_c(\mu)\mathrm{d}r \ ; \ \sigma \geq \sigma_f(\mu) \ ; \ |\tau| \geq \tau_f(\mu) \ \text{ for } \ 0 \leq r \leq l \text{ with } \mu = \frac{|\tau|}{\sigma} = \frac{|t_1 + mt_2|}{s_1 + ms_2} \tag{2}$$

Where G_c is the toughness, and where σ_f and τ_f are the tensile and shear strength at failure under mix mode loading (the mixity is characterized by μ). According to (1)

$$\begin{cases} k_1^2 l^{2\lambda_1}\left(A_1(\theta_0) + m(l)A_{12}(\theta_0) + m(l)^2 A_2(\theta_0)\right) \geq \int_0^l G_c(r)\mathrm{d}r \\ k_1 r^{\lambda_1 - 1}\left(s_1 + m(r)s_2\right) \geq \sigma_f(r) \ \text{ for } \ 0 \leq r \leq l \\ k_1 r^{\lambda_1 - 1}\left|t_1 + m(r)t_2\right| \geq \tau_f(r) \ \text{ for } \ 0 \leq r \leq l \end{cases} \tag{3}$$

The scaling functions A_i were defined in [8] and θ_0 is the crack direction. With the additional relation ($q = 2$ holds for an elliptical criterion)

$$\begin{gathered} \left(\frac{\sigma_f(\mu)}{\sigma_c}\right)^q + \left(\frac{\tau_f(\mu)}{\tau_c}\right)^q = 1 \ ; \ \tau_f(\mu) = \mu\sigma_f(\mu) \\ \Rightarrow \sigma_f(\mu) = \frac{\sigma_c \tau_c}{\left(\tau_c^q + \mu^q \sigma_c^q\right)^{1/q}} \ ; \ \tau_f(\mu) = \mu\sigma_f(\mu) \end{gathered} \tag{7}$$

With the Hutchinson and Suo [9] condition, setting $\psi = \tan^{-1}(\mu)$ and where G_{Ic} and G_{IIc} respectively denote mode I and II toughness

$$
\begin{aligned}
&G_c = G_{Ic}\left(1+\tan^2(c\psi)\right) \\
&\text{if } \psi=0 \; G_c = G_{Ic} \\
&\text{if } \psi=\pm\pi/2 \; G_c = G_{IIc} \;\Rightarrow\; c = \frac{2}{\pi}\tan^{-1}\left(\sqrt{\frac{G_{IIc}}{G_{Ic}}-1}\right)
\end{aligned}
\tag{8}
$$

The general form of the criterion can be written

$$
k_1 \ge k_{1f} = \left(\frac{\overline{G}_c}{A_1(\theta_0)+m_0A_{12}(\theta_0)+m_0^2A_2(\theta_0)}\right)^{1-\lambda_1}\left(\frac{\sigma_c\tau_c}{\left((s_1+m_0s_2)^q\tau_c^q+\left|t_1+m_0t_2\right|^q\sigma_c^q\right)^{1/q}}\right)^{2\lambda_1-1}
\tag{9}
$$

Where $\overline{G}_c = \frac{1}{l}\int_0^l G_c(r)\mathrm{d}r$ corresponds to the mean toughness along the presupposed crack path and $m_0 = m(l_0)$, l_0 being the solution to the implicit equation

$$
l = \frac{\overline{G}_c}{\sigma_c^2\tau_c^2}\frac{\left((s_1+m(l)s_2)^q\tau_c^q+\left|t_1+m(l)t_2\right|^q\sigma_c^q\right)^{2/q}}{A_1(\theta_0)+m(l)A_{12}(\theta_0)+m^2(l)A_2(\theta_0)}
\tag{10}
$$

A more general form of the criterion (yet slightly heavier to handle) was established by Garcia and Leguillon [10], it avoids the asymmetry between mode I and mode II that appears through the mixity parameter m defined in (4).

EXTENSION TO FATIGUE LOADINGS

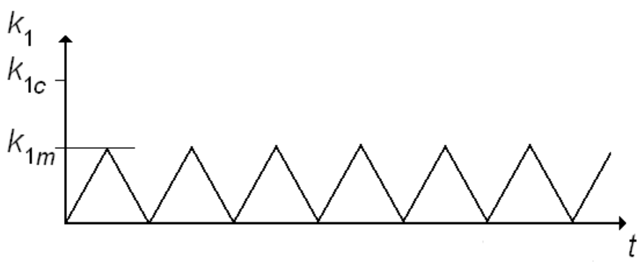

Figure 2. Schematic view of a fatigue loading vs. time *t*. The load intensity is characterized by the GSIF k_1 of the singularity at the root of the V-notch, k_{1c} is its critical value under monotonic loading and k_{1m} the maximum reached during a cycle.

We assume that G_c remains approximately constant along the crack path (it is not a strong assumption since the crack extension lengths are very small especially in fatigue, which is easy to check a posteriori). For simplicity we note $G_f = G_c = \overline{G}_c$ and G_m is the toughness of a material having the same elastic moduli but a failure corresponding to k_{1m} (the GSIF at the peak of a fatigue cycle) and we set $\alpha = k_{1m} / k_{1f}$.
To be consistent with the fatigue model derived from the Dugdale CZM, it is assumed that the toughness is degraded at each cycle of a quantity ΔG proportional to G_m (β is an adjustable parameter), then for n cycles during the nucleation phase

$$G_m = G_f - n\Delta G, \text{ if } \Delta G = \alpha^\beta G_m \text{ then } G_m\left(1 + n\alpha^\beta\right) = G_f \tag{11}$$

According to (9), it comes

$$\begin{cases} k_{1f} = \left(\dfrac{G_f}{A_f^*}\right)^{1-\lambda_1} \left(\dfrac{\sigma_c \tau_c}{S_f^*}\right)^{2\lambda_1 - 1} \\ k_{1m} = \left(\dfrac{G_m}{A_m^*}\right)^{1-\lambda_1} \left(\dfrac{\sigma_c \tau_c}{S_m^*}\right)^{2\lambda_1 - 1} \end{cases} \text{ with } \begin{cases} A_a^* = A_1 + m_a A_{12} + m_a^2 A_2 \;;\; S_a^* = \left((s_1 + m_a s_2)^q \tau_c^q + \left|t_1 + m_a t_2\right|^q \sigma_c^q\right)^{1/q} \\ m_a = \dfrac{k_2}{k_1} l_a^{\lambda_2 - \lambda_1} = \dfrac{k_{2a}}{k_{1a}} l_a^{\lambda_2 - \lambda_1} \quad a = f, m \end{cases} \tag{12}$$

Where l_f and l_m are the extension lengths derived from the coupled criterion, they are solution to (10) where G_f and G_m successively replace $\overline{G}_c$

$$l_a = \frac{G_a}{\sigma_c^2 \tau_c^2} \frac{S_a^{*2}}{A_a^*} \quad a = f, m \tag{13}$$

It is important to note that the intensity of the cyclic loading occurs only through G_m. From (12), we deduce

$$\alpha = \left(\frac{G_m}{G_f}\right)^{1-\lambda_1} \left(\frac{A_f^*}{A_m^*}\right)^{1-\lambda_1} \left(\frac{S_f^*}{S_m^*}\right)^{2\lambda_1 - 1} \tag{14}$$

Inserting in (11) leads to

$$n = \frac{1}{\alpha^{\beta}}\left(\frac{1}{\alpha^{\frac{1}{1-\lambda_1}} F^*} - 1\right) \approx \frac{1}{\alpha^{\beta+\frac{1}{1-\lambda_1}} F^*} \quad \text{where } F^* = \frac{A_m^*}{A_f^*}\left(\frac{S_m^*}{S_f^*}\right)^{\frac{2\lambda_1-1}{1-\lambda_1}} \tag{15}$$

The case of a symmetric loading (pure mode I) corresponds to $m_m = m_f = 0$ then $\sigma_f = \sigma_c$, $\tau_f = 0$, $G_f = G_{Ic}$, $A_m^* = A_f^* = A_1$, $S_m^* = S_f^* = \tau_c$ and $F^* = 1$, through proper normalization of the singular modes.

The stress conditions (6) and (7) lead to

$$\begin{cases} k_{1m} l_m^{\lambda_1 - 1}\left(s_1 + m_m s_2\right) \geq \sigma_f(l_m) = \dfrac{\sigma_c \tau_c}{\left(\tau_c^q + \mu_m^q \sigma_c^q\right)^{1/q}} \\ k_{1f} l_f^{\lambda_1 - 1}\left(s_1 + m_f s_2\right) \geq \sigma_f(l_f) = \dfrac{\sigma_c \tau_c}{\left(\tau_c^q + \mu_f^q \sigma_c^q\right)^{1/q}} \end{cases} \quad \text{with } \mu_a = \frac{\left|t_1 + m_a t_2\right|}{s_1 + m_a s_2} \quad a = f, m \tag{16}$$

These inequalities refer only to tensions but by virtue of (7) inequalities in tension and shear are equivalent. It comes finally

$$\alpha = \left(\frac{l_m}{l_f}\right)^{1-\lambda_1} \frac{S_f^*}{S_m^*} \quad \Rightarrow \quad l_m = l_f \alpha^{\frac{1}{1-\lambda_1}} \left(\frac{S_m^*}{S_f^*}\right)^{\frac{1}{1-\lambda_1}} \tag{17}$$

$$\dot{a} = \frac{l_m}{n} = l_f \alpha^{\beta + \frac{2}{1-\lambda_1}} \frac{A_m^*}{A_f^*}\left(\frac{S_m^*}{S_f^*}\right)^{\frac{2\lambda_1}{1-\lambda_1}} = l_f \alpha^{\beta + \frac{2}{1-\lambda_1}} H^* \tag{18}$$

In the symmetric case $H^* = 1$. Parameter β can be identified so that (18) coincides with the Paris law of the material in case of a pre-existing long crack: $\lambda_1 = 1/2$ then $\beta = p - 4$ where p is the Paris exponent ($p \geq 4$).

Remark 1: If $k_{1m} = k_{1f}$ then $G_m = G_f$, $\alpha = 1$, $F^* = 1$ and $n = 1$, failures occurs at the first cycle. In other terms, the fatigue criterion remains valid for a monotonic loading.

Remark 2: In this model, the crack growth under fatigue loading is intermittent, every n cycles, the crack length increases by l_m. We put forward the idea that this length could be identified to the striations spacing observed in fatigue in quasi-brittle materials [12].

AN EXAMPLE

Numerical simulations have been carried out for a V-notch in a PMMA specimen under symmetric loading (loading angle 0° in Fig. 1). Parameters for PMMA are: $E = 3250$ MPa (Young's modulus), $\nu = 0.3$ (Poisson's ratio), $\sigma_c = 70$ MPa (tensile strength), $G_{Ic} = 0.35$ MPa.mm (toughness), $p = 6$ (Paris' exponent [2,12]).

Table 1 shows the invariant parameters that do not depend on the intensity of loading and table 2 those specifically related to fatigue. The remarkable features of these results are, first that the parameter l_m seems to remain approximately constant and depends only on the intensity of the loading, except for wide openings; and second that the crack velocity varies more with respect to the opening angle for low intensity loadings than for higher ones.

Table 1. The singular exponent λ, the dimensionless scaling coefficient $A_1' = E\,A_1$ and the characteristic length at failure l_c under monotonic loading for different openings ω of the V-notch.

ω (°)	λ	A_1'	l_c (mm)
0	0.500	6.28	0.033
30	0.502	6.16	0.033
60	0.512	5.85	0.035
90	0.545	5.26	0.038
120	0.616	4.26	0.047
160	0.819	2.65	0.076

Table 2. The characteristic length at failure under fatigue loading l_m (the striations spacing) and the crack advance rate $\dot{a}$, function of the loading intensity α and the opening ω of the V-notch.

α	0.3		0.6	
ω (°)	l_m (mm)	$\dot{a}$ (mm/cycle)	l_m (mm)	$\dot{a}$ (mm/cycle)
0	0.0029	$2.4\ 10^{-5}$	0.0117	0.0015
30	0.0029	$2.3\ 10^{-5}$	0.0118	0.0015
60	0.0029	$2.2\ 10^{-5}$	0.0121	0.0015
90	0.0027	$1.7\ 10^{-5}$	0.0125	0.0015
120	0.0021	$0.8\ 10^{-5}$	0.0126	0.0012
160	0.0001	$1.0\ 10^{-8}$	0.0045	0.0001

Obviously for wide openings, the initial crack velocity is low and the crack tip accelerates to reach a steady state growth rate corresponding to long cracks. Starting from asymptotic considerations, we can show that this regime is reached for $\omega = 160°$ (resp. $\omega = 120°$) when the crack length exceeds $5\, l_m^0$ (resp. $2\, l_m^0$), where l_m^0 holds for l_m at $\omega = 0°$. For smaller openings, the steady state seems to be reached very quickly.

CONCLUSION

The theoretical results established here still await experimental confirmation for both complex monotonic loadings and fatigue. Investigations are underway to perform tests such as those illustrated in Figure 1 with specimens made of PMMA.
Only mechanical loadings were included in the above presentation, but the generalization to thermal loads should not introduce any insuperable difficulties except for some minor technical problems almost solved [11]. The special case of complex exponents (including interface cracks) has not yet been considered.

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