On Coupled Fracture Modes and Three-Dimensional Fracture Mechanics

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ABSTRACT. *It is well-known that shear and anti-plane loadings of a through-the-thickness crack in a plate generate coupled three-dimensional fracture modes. These singular modes are currently largely ignored in theoretical and experimental investigations as well as in standards and failure assessment codes of structural components, where it is assumed implicitly that the intensities of these modes as well as other three-dimensional effects are negligible. In this paper we provide an overview of recent studies carried out by the authors, which demonstrate that the account for these coupled modes can totally change the classical (two-dimensional) view on many fracture phenomena. In particular, this relates to a generation of the coupled modes by non-singular shear and anti-plane stress fields and a strong effect of the plate thickness on the intensity of the coupled modes, which can influence fracture conditions.*

INTRODUCTION

Plane analytical solutions, some of which are more than a hundred years old, still serve as the foundation for many engineering disciplines, applications, design procedures, standards and failure assessment codes. In addition, two-dimensional computational solutions are currently dominating the numerical analysis of plate components because such models are normally far more computationally efficient, much easier to develop, mesh, implement and verify in comparison with the corresponding three-dimensional counterparts. Because of this dominance the fast majority of experimental studies in the past have also utilised the theoretical two-dimensional framework [1].

However, in problems with cracks (or other strong concentrators) the two-dimensional theories can often lead to peculiar results due, in part, to the fact that these are approximate theories even when the governing equations of these theories are solved exactly [2]. This is particularly pronounced when the two-dimensional (or plane) solutions are utilized to analyze very small or very large structural components. This important actuality will be illustrated and discussed later in this paper. We will start this paper with a brief introduction to the coupled fracture modes, which essentially represent a three-dimensional effect, which can not be recovered from the classical plane solutions of the theory of elasticity.

Consider a through-the-thickness crack loaded in shear or anti-plane loading. These loadings generate additional local fracture modes due to Poisson's ratio effect and the redistribution of stresses on the free surfaces as illustrated: in Fig. 1a – mode II loading and Fig.1b – mode III loading.

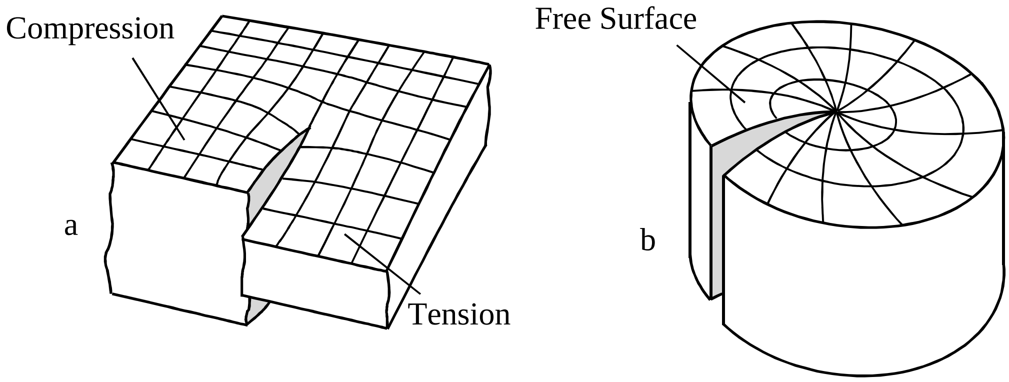

Figure1. Illustration of coupled fracture modes due to Poisson's effect and the redistribution of stresses close to the free surfaces for a crack subjected to shear (a) and anti-plane loading (b)

The coupled modes are characterized by singular stress states, which rapidly decay with the radial distance from the crack tip. Their intensities vary significantly across the plate thickness apart from the stress intensities of the primary modes (mode I, II and III), which vary in relatively narrow range. The maximum values of the intensities of the coupled modes are in the vicinity of the free surfaces. However, at a point when a corner front (crack front) intersects the free surface the singular stress states associated with the primary and coupled modes disappear. At this point a new three-dimensional corner singularity develops instead. Thus, the stress state at corner points or in the close vicinity of the free plate surfaces is very complicated.

A typical crack front of a fatigue crack is usually curved in the vicinity of a corner point. It tends to intersect a free surface at some angle. This angle is often linked to a critical angle, at which the corner singularity has the same strength (power) as the rest of the crack front i.e. square root singularity. This critical angle is a function of type of loading and Poisson's ratio, but this tendency and correlation are still not sharply defined. In the following we will consider only the case when the crack-front is perpendicular to the free surface or critical angle, if we rely on this concept, is 90^0. The latter corresponds to the theoretical values of the critical angle at small values of Poisson's ratio in the cases of shear and anti-plane loadings. Analysis of other cases will require a more substantial computational effort but it is believed that all major tendencies and effect to be presented will take place in this more general case as well.

It is obvious that within the classical plane stress or plane strain theories of elasticity these effects can not be recovered and investigated. Therefore, the systematic investigations of coupled modes accompanying the primary modes II and III become possible only with a step advance in computational approaches and computing power in the 1990's; primarily due to the requirement of a very fine and accurate meshing as the coupled modes are local and concentrated in the vicinity of the tip. In many research papers these coupling modes were named as a reason for various phenomena accompanying fracture and fatigue crack growth (see among others Nakamura and Parks [3]; Les Pook [4, 5]).

The coupled mode in shear loading, so called K_O-mode was also investigated analytically utilising Kane and Mindlin high order plate theory [6]. Recently, the coupled modes were studied for sharp and round notches of arbitrary notch opening angle by the present authors. It was demonstrated that these coupled modes have many interesting and previously unknown features, which are capable of advancing our understanding of size effects, mixed-mode fracture, crack initiation and fatigue growth phenomena [7 -13]. Among these features a generation of the coupled modes by non-singular shear or anti-plane loading (with $K_{II} = 0$ or/and $K_{III} = 0$) [10]. From the classical point of view, the quasi-brittle crack propagation is impossible in these cases as the energy release rate is zero. However, the non-singular loading still generates singular coupled fracture modes, which are capable to initiate fracture. In this situation a strong plate thickness effect was found, which predicts an increase of the intensity of the coupled mode generated by mode II loading with an increase of the plate thickness. A similar situation takes place for mode III loading. An extrapolation of these results leads to a very interesting conclusion that very thick plate components with through-the-thickness cracks have no strength if loaded in shear and anti-plane loading [2, 13]. The coupled mode generated by shear loading is strongly affected by Poisson's ratio and, in contrast the intensity of the anti-plane coupled modes does not vary much with the change of Poisson's ratio. In the next Sections, we will provide some numerical examples of the generation of the coupled modes and a detail description of the aforementioned effects.

MODELLING APPROACH

In the beginning we briefly describe the modelling methodology adapted in our numerical studies [6 – 13].

Geometry

Because the coupled singular modes are local modes and spread to the distance of approximately half of the plate thickness, the problem geometry is normally truncated to a disk with in-plane dimensions sufficient to avoid the effect of the finite boundaries on the stress state of the coupled and primary modes. The antisymmetric boundary conditions are utilised to further simplify the geometry. The final geometry is shown in Fig. 2 and appropriate displacement boundary conditions corresponding to anti-plane or shear loadings are applied on the cylindrical surface as illustrated in this figure. The origin of the Cartesian coordinate system (x, y, z) is located at the crack tip, at the mid-surface where x-direction was chosen to be the direction of the crack bisector.

In the finite element models, a denser nodal arrangement is created in the proximity of the crack tip where the mesh has to be very fine for the analysis of the coupled modes. All calculations are carried out using the ANSYS 11 code. The mesh package, utilising a mesh consisting of an initial arrangement of 15-node trapezoidal elements at the notch tip, surrounded by a radial array of 20-node brick elements, where each element spans an angular sweep of 11.25^0.

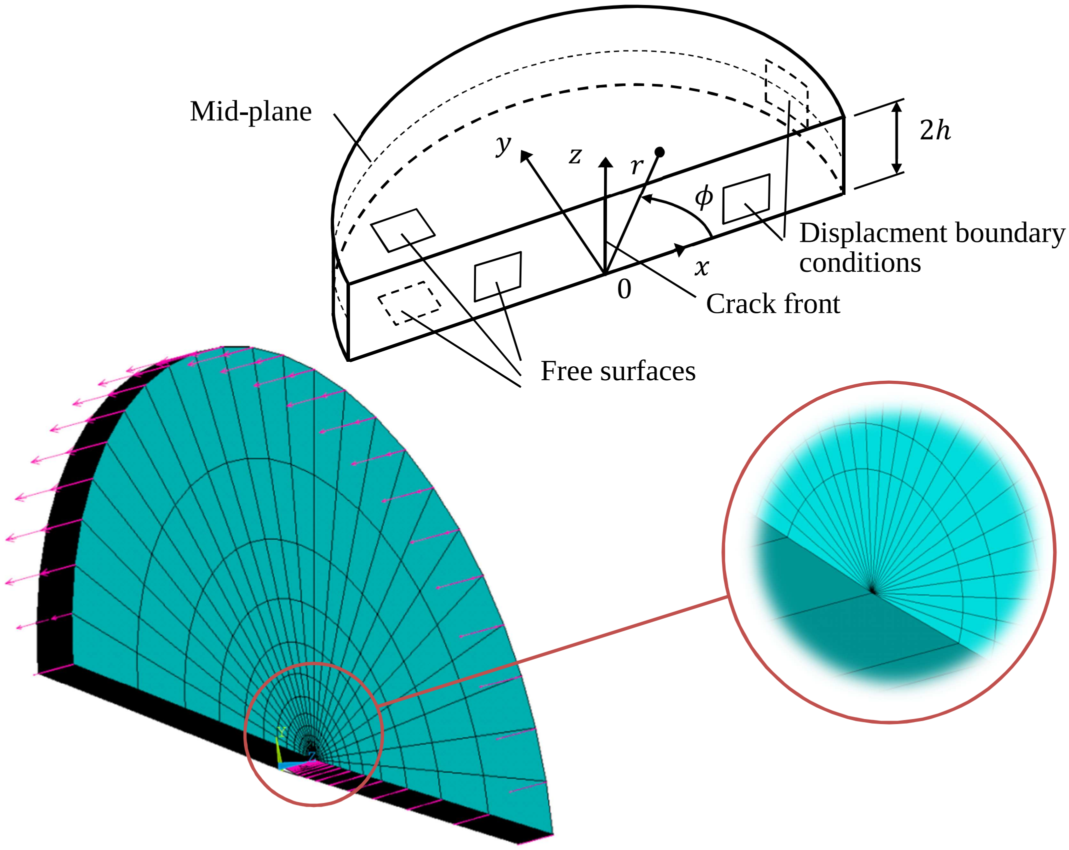

Figure 2. Finite element model geometry and boundary conditions

Boundary Conditions

The displacement plane stress boundary conditions are applied beyond the area of 3D effects (this area is confined within a cylinder with the axis of symmetry along the crack front and radius equal to half of the plate thickness). In shear loading the displacement field far from the crack tip can be represented in a series as [14]

$$u_x = -\sum_{n=0}^{\infty} \frac{r^{n/2}}{2\mu} b_n \left[\left(k + \frac{n}{2} - (-1)^n \right) \sin\frac{n}{2}\phi - \frac{n}{2} \sin\left(\frac{n}{2} - 2\right)\phi \right] \tag{1a}$$

$$u_y = \sum_{n=0}^{\infty} \frac{r^{n/2}}{2\mu} b_n \left[\left(k - \frac{n}{2} + (-1)^n\right) \cos\frac{n}{2}\phi + \frac{n}{2}\cos\left(\frac{n}{2} - 2\right)\phi\right] \tag{1b}$$

where μ is the transverse elastic modulus and k the Kolosov's constant for plane stress conditions, which prevail far from the crack tip:

$$k = \frac{3 - \nu}{1 + \nu}$$

ν being Poisson's ratio.

In the case of anti-plane loading the out-of-plane displacement, w, beyond the area of 3D effects can be expressed as [15]:

$$w = \sum_{n=0}^{\infty} \frac{r^{n+\frac{1}{2}}}{\mu} c_n \sin\left(\frac{1}{2} - n\right)\phi \tag{2}$$

Coefficient b_1 and c_0 in (1) and (2) are tied to the applied mode II and mode III stress intensity factors, respectively, by the following relationships [14, 15]

$$b_1 = -\frac{K_{II}^{\infty}}{\sqrt{2\pi}} \tag{3a}$$

and

$$c_0 = K_{III}^{\infty}\sqrt{\frac{2}{\pi}} \tag{3b}$$

The displacement filed corresponding to b_0 is related to rigid body translation at the crack tip and do not contribute into the stresses and strains; similar, the term in the asymptotic expansion (1) with b_2 represents a rigid rotation of the body about the crack tip also producing no stress field. Therefore, these terms are omitted in the numerical studies.

To systematically investigate the generation of coupled modes we will apply far from crack tip the displacement field, which corresponds to a single term in the asymptotic expansions. In the case of complex loading with several terms, the solution can always be found by simple superposition of the solutions corresponding to the each asymptotic term.

SHEAR AND ANTI-PLANE LOADING BY LEADING TERMS OF FAR FILD EXPANSION

In the beginning we will provide a formal definition of the stress intensities for the coupled modes. The stress intensity factor of the anti-plane coupled mode can be defined similar to mode II, or as

$$K_{II}^{c}(\mathrm{z}) = \sqrt{2\pi}\ \lim_{x\to 0} \tau_{xy}(x,0,z)\,\sqrt{x} \tag{4a}$$

and the definition of the stress intensity of the shear coupled mode is similar to mode III, or

$$K_{III}^{c}(\mathrm{z}) = \sqrt{2\pi}\ \lim_{x\to 0} \tau_{yz}\ (x,0,z)\sqrt{x} \tag{4b}$$

where x is the distance from the crack tip along the bi-sector direction.

To determine the stress intensities, the corresponding stress components are first calculated and extracted from FE analysis and then substituted into the above equations to identify the value of the stress intensity for the coupled modes at certain z - coordinate along the crack front. Some conditions have to be met in order to get correct values of the stress intensities, which are exhaustively described in the literature.

First, we consider the case when a through-the-thickness crack is loaded by shear or anti-plane loading when only the first (singular) term in the asymptotic expansion is non-zero ($b_1 \neq 0$ and $c_0 \neq 0$) and all other terms are zero i.e. $b_n = c_n = 0$ for all other n. Figures 3a and 3b show the results of careful 3D FE studies for shear and anti-plane loading of a through-the-thickness crack, respectively.

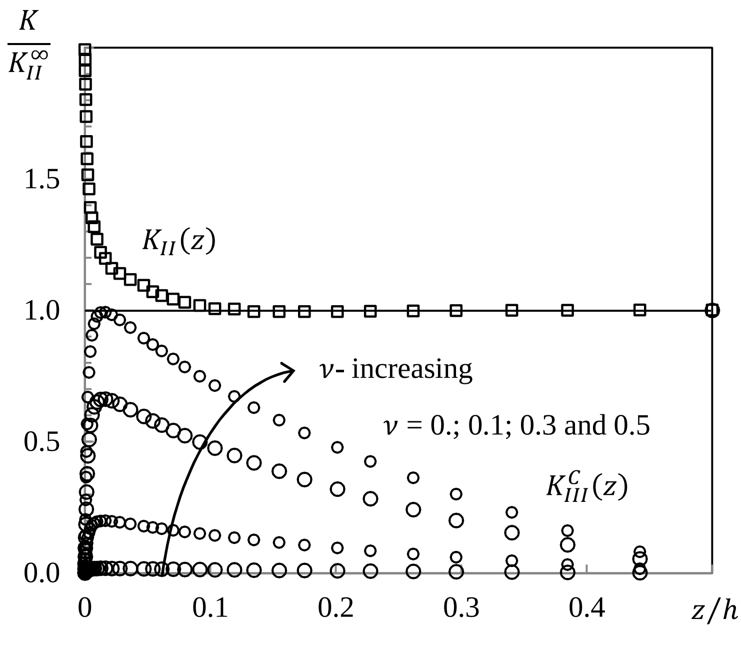

Fig.3a The dependence of the intensity of the primary (mode II) $K_{II}(z)$ and the coupled mode $K_O(z) = K_{III}^{C}(z)$ across the plate thickness in the area near the crack tip. The influence of Poisson's ratio on the intensity of the primary mode is very weak and results are not shown for the sake of the clarity of the figure.

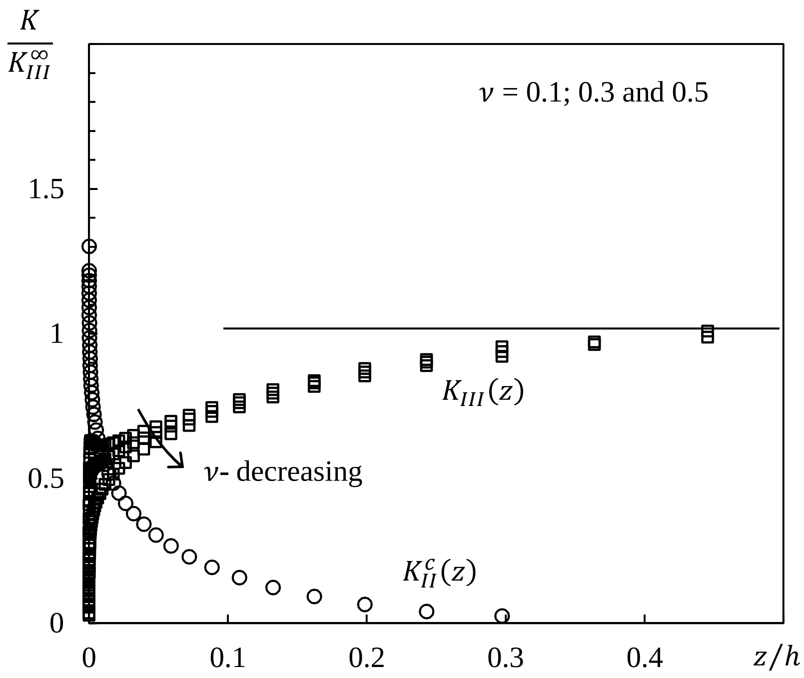

Fig.3b The dependence of the intensity of the primary (mode II) $K_{III}(z)$ and the coupled mode $K_{II}^{C}(z)$ across the plate thickness in the area near the crack tip. The influence of Poisson's ratio on the intensity of the coupled mode is very weak and results are not shown for the sake of the clarity of the figure.

From these figures it is clear seen that the intensities of the coupled modes are comparable with the intensities of the primary modes. For example, in the case of shear loading (Fig. 3b) the intensity of the coupled mode is even higher than the intensity of the applied mode. The maximum values of the intensities of the coupled modes are located in the vicinity of the plate free surfaces. This feature of the coupled modes can explain as to why for shear and anti-plane loadings experimental observations indicate that fatigue crack growth tends to take place in this region [4].

The intensities in the case of shear loading are significantly affected by Poisson's ratio, and the intensity of the coupled mode vanishes when $\nu = 0$, see Fig. 3a. This is because the Poisson's effect is the main mechanism in the generation of this coupled mode. In contrast, the intensities of the primary and coupled modes in the case of anti-plane loading (mode III loading) are not significantly affected by Poisson's ratio as this coupled mode is generated by a mechanism assotiated with a redistribution of the transverse shear stresses close to free plate surfaces, see Fig. 3b. These transverse stresses have to be negated due to the stress-free boundary conditions on the plate surfaces.

SHEAR OR ANTI-PLANE LOADING WITH K $_{II}$ =0 AND K $_{III}$ = 0

In the following examples of FE simulations we demonstrate that the terms in the asymptotic expansions (1) and (2), which corresponds to non-singular stress fields can also generate singular coupled modes. Figure 4a display the dependence of the intensity of the coupled modes across the plate thickness generated by single term b_3, when the corresponding displacement field is applied to the FE model as the boundary condition far from the area of 3D effects. In this case the intensity of the primary mode across the plate thickness is zero or $K_{II}(z) = 0$. It is seen from Fig. 4a that the intensity of the coupled mode generated by non-singular loading is significantly affected by Poisson's ratio [10].

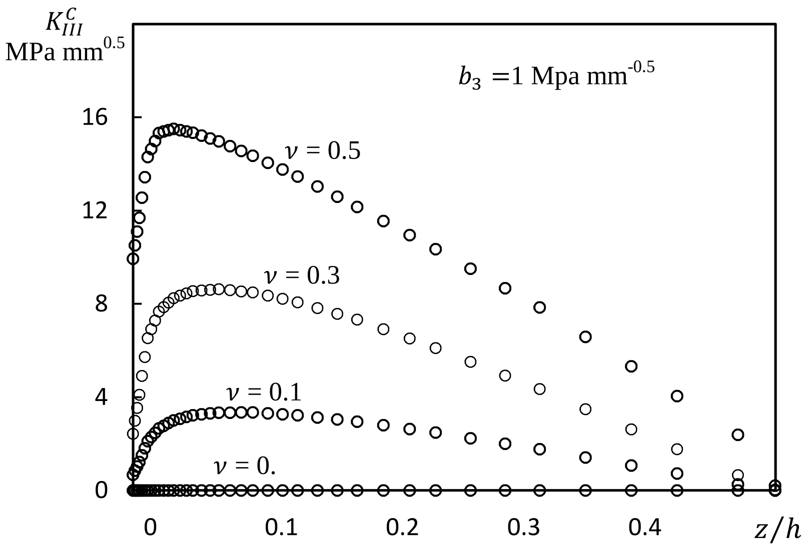

Fig.4a The dependence of the singular coupled mode, K_{III}^{C}, across the plate thickness for $h = 10$ mm and b_3= 1 MPa mm$^{-0.5}$

Similar computational results for the anti-plane loading are shown in Figure 4b. As in the case of singular loading described in the previous Section, the intensity of the anti-plane coupled mode is not significantly affected by Poisson's ratio.

These figures, demonstrate that the coupled modes can be generated by non-singular shear and anti-plane loadings, i.e. when the applied $K_{II}^{\infty} = K_{III}^{\infty} = (K_{I}^{\infty}) = 0$. These features of the coupled modes have a direct implication to failure assessment of plate components. Shear and anti-plane loadings are capable to initiate brittle fracture by crack propagation due to the generation of the coupled modes, even when the intensities of the primary modes are negligible. Thus, in the case of sufficiently brittle material, the coupled modes can totally dominate the stress state at the crack tip, contribute to the energy release rate and, therefore, initiate brittle fracture. The same comment relates to the case of the fatigue crack growth.

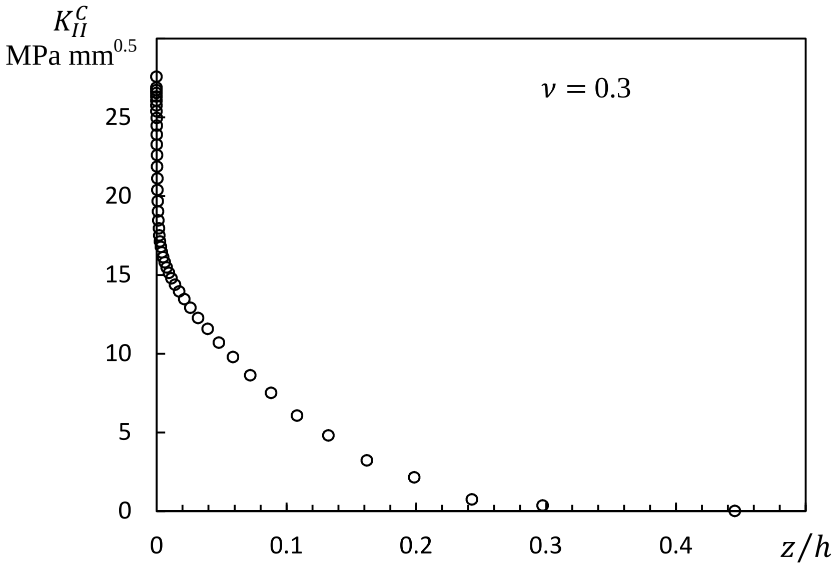

Fig.4b The dependence of the singular coupled mode, K_{II}^C, across the plate thickness for $h = 10$ mm and c_1= 1 MPa mm$^{-0.5}$

SCALE EFFECTS ASSOCIATED WITH INCREASE OF PLATE THICKNESS

As mentioned above the coupled modes are local modes and propagate in the plane direction to approximately a half of the plate thickness. It means if the stress fields corresponding to the coupled modes are encapsulated by the area, where the primary load is dominating then the intensity of the applied mode must be a linear function of the intensity of the primary loading, or, K_{II}^C and K_{III}^C have to be proportional to c_n and b_n, respectively. Further, from dimensionless considerations, and taking into account that the plate thickness is the only other dimensional parameter of the problem with a length dimension, we arrive to the following interesting theoretical dependences [2, 13]:

$$K_{III}^C(z/h) = f_{IIIn}(z/h\,,\nu) b_n h^{\frac{n-1}{2}} \tag{5a}$$

and

$$K_{II}^C(z/h) = f_{IIn}(z/h\,,\nu) c_n h^n \tag{5b}$$

where f_{IIn} and f_{IIIn} are dimensionless functions of the position along the crack front and Poisson's ratio, such as those shown in Figs.3 and 4.

These dependences (5) mean that the intensities of the coupled modes increase with an increase of the plate thickness except for the singular loading or loading by the first singular term. For higher order terms an increase of the plate thickness (or overall sizes of plate structure) will lead to much stronger effect and even a small variation of thickness can cause large variation in the intensity of the coupled mode. This thickness effect is also has been confirm by direct numerical simulations. Typical results for the anti-plane loading for two different plate thicknesses are shown in Fig.5.

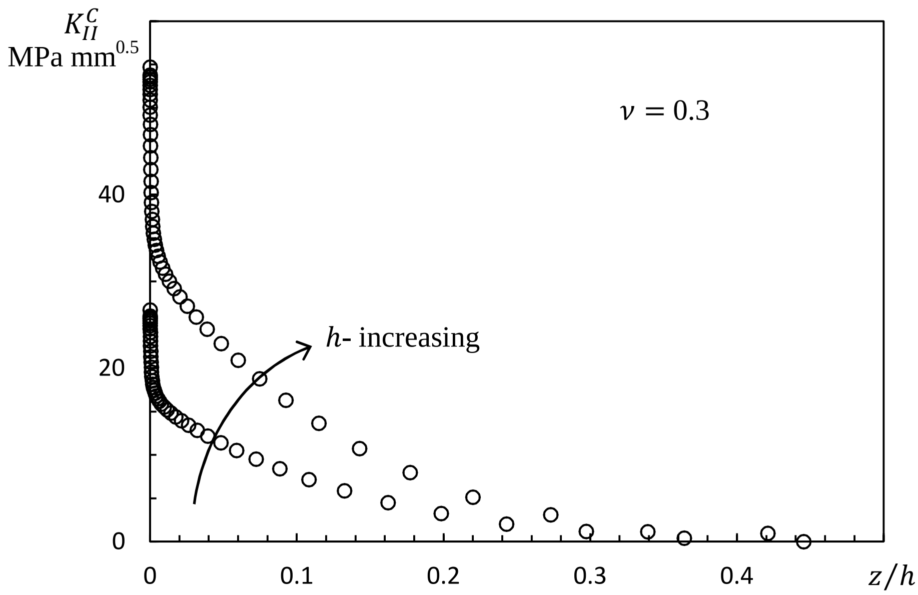

Fig.5. Intensities of the coupled modes for two different plate thicknesses h = 10 and 20 mm (c_1= 1 MPa mm$^{-0.5}$)

In the case of general loading when few asymptotic modes are applied the effect of the plate thickness can be obtained by simple superposition due to the linearity of the elastic problem as

$$K_{III}^{C}(z/h) = \sum_{n=1}^{\infty} f_{IIIn}(z/h\,,\nu) b_n h^{\frac{n-1}{2}} \tag{6a}$$

and

$$K_{II}^{C}(z/h) = \sum_{n=0}^{\infty} f_{IIn}(z/h\,,\nu) c_n h^{n} \tag{6b}$$

The dependence of the intensity of the coupled modes from the plate thickness, see equations (6), becomes quite complicated.

Currently, industrial and international standards on fracture toughness testing ignore the coupled modes and other 3D phenomena and set the thickness requirements for valid fracture testing based on the smallness of the plastic zone in comparison the plate thickness. In this paper we tried to demonstrate that the 3D effects could lead to a significant rise of the singular stress states corresponding to coupled modes, which in turn, are significantly affected by the plate thickness. This is specifically important for testing fracture toughness in mode II and III or mixed mode of loadings. The literature results indicate a large scattering in such tests and large variation of experimentally obtained values with the specimen size. The reason behind these inconsistencies can be the influence of the three-dimensional singular stress states on the failure initiation conditions. Therefore, the future work can look into these issues, specifically for typical sample geometries. The same comment relates to the failure assessment codes, which largely ignore the three-dimensional effects.

CONCLUSION

In the conclusion we will provide a summary table, which describe the most important features of the coupled modes.

Table 1: Classical versus three-dimensional theories

	Classical (2D) theories	Three-Dimensional theory
Fracture Modes*	3 primary fracture modes, modes I, II and III	5 modes (3 primary and 2 coupled local modes)
Extension of singular stress states	Extension of primary modes is not affected by plate thickness	Extension of coupled modes is strongly affected by plate thickness
Plate thickness effect	It is negligible for quasi-brittle fracture and fatigue	Marginal effect on primary and strong influence on coupled modes
Loading by non-singular shear and antiplane loading ($K_{II\,,\,III} = 0$)	No energy release rate – no fracture by crack propagation	Non-zero coupled modes can initiate brittle fracture even when $K_{II,\,III} = 0$
Effect of Poisson's ratio	No effect in problems with stress type boundary condition	Marginal effect on primary and a strong influence on coupled modes
Scale effect	Stochastic, fractal, etc nature	Also predicts a strong scale effect of deterministic nature

*Meaning the fracture modes which contribute to the energy release rate apart from the 3D corner singularities [16], which are concentrated in a very small area (point).

As it can be seen from Table 1, the theoretical research recently conducted by the authors has identified significant and fundamental differences between the actual three-dimensional world and the simplified classical (two-dimensional) theories of quasi-brittle fracture leading to essentially different tendencies and predictions; specifically for very thick and very thin plate or shell-like structures, as the intensity of the coupled modes grow or decay as a power function of the plate thickness in the simplest case, see Eq. (5) [17]. It is recognised that the future work has to be directed to the experimental confirmation of the above described theoretical tendencies and effects.

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