Stress intensity factors in three-dimensional planar cracks subjected to uniform tensile stress

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ABSTRACT. *In this paper we present a closed-form solution for mode I Stress Intensity Factors (SIF) in three-dimensional planar flaws based on homotopy transformations of a disk. The SIF is given for each point of the crack border under the hypothesis of an isolated crack under tensile loading. The solution is proposed in terms of the Fourier series and the first order approximation of the coefficients is given using the explicit form. The results indicate that the proposed equation is very accurate when the flaw is a small deviation from a circle. Our solution is used to predict the SIF of many types of planar flaws and the results are compared with numerical predictions taken from the literature.*

INTRODUCTION

Compliance in Stress Intensity Factor (SIF) evaluations in planar three-dimensional crack is usually overcome using numerical applications. In fact, apart from some particular geometrical cases under simplified stress conditions [1], an analytical solution for generic crack shape contours has not been discussed in the literature. In order to avoid this problem, Oore-Burns [2] introduced a two-dimensional weight function which gives an exact solution in the case of circular or tunnel crack. However, when an elliptical crack is assumed, the authors recently showed [3] that, under remote uniform tensile loading, the Oore-Burns integral gives a first order approximation of SIF along the whole crack front. Furthermore, this first order equation is very close to the first order approximation of the Irwin [4] exact solution. In particular, when the eccentricity e of the ellipse tends to be zero, the principal contribution $\frac{e^2}{20\sqrt{\pi}}$ to the discrepancy is very small.

Murakami and Endo [5] proposed the $\sqrt{area}$ as an empirical parameter for the evaluation of the fatigue limit linked to the maximum stress intensity factors under mode I loadings ($K_{I,max}$) of small convex cracks. On the basis of several examples of flaw shapes, Murakami ad Nemat-Nasser [6] proposed the simple formula $K_{I,max} = Y\,\sigma\sqrt{\pi\sqrt{\text{area}}}$, where Y is a coefficient which is evaluated as the best fitting of the numerical and analytical results (Y=0.629 for surface crack [6]). So that, an explicit analytical equation for SIF calculations could be useful for estimating the fatigue limit of internal irregular small defects or irregular cracks.

The aim of this paper is to propose the evaluation of the SIF along the whole crack front, based on an analytical approach. More precisely, we are able to compute a first order approximation of the Oore-Burns integral using the closed form. The solution is precise to the first order of deviation from a circular shape and under the hypothesis of uniform tensile stress. In addition, the coefficient Y related to the maximum stress intensity factors can be evaluated and we make a comparison with those proposed in the literature.

THEORETICAL BACKGROUND

In reference [2] Oore and Burns proposed the following general expression for the evaluations of the mode I stress intensity factor for embedded cracks Ω in an infinite solid, under arbitrary normal tension $\sigma(Q)$:

$$K_O(Q') = \frac{2}{\sqrt{\pi}} \int_\Omega \frac{\sigma(Q)}{\sqrt{f(Q)}\, |Q-Q'|^2} d\Omega \ , \qquad Q' \in \partial\Omega \tag{1}$$

where Q' is the point on the crack border $\partial\Omega$, Q is a generic point inside the flaw and f(Q) is defined as

$$f(Q) = \int_{\partial\Omega} \frac{ds}{|Q-P(s)|^2} \qquad Q \in \Omega \tag{2}$$

s being the arch-length on $\partial\Omega$. The integral (2) is convergent and the proof is reported in reference [8] and is based on the following approximation of f(Q) near $\partial\Omega$:

$$f(Q) \approx \frac{\pi}{\Delta} \tag{3}$$

where Δ is the distance between Q and $\partial\Omega$.

Let Ω be an open bounded simply-connected subset of the plane as reported in Fig. 1. Therefore, we consider a C^2-function $R=(\varepsilon, \psi)$, where $0 \leq \varepsilon \leq 1$ is a parameter and $0 \leq \psi \leq 2\pi$ is the angle and require

$$R(0,\psi) \equiv 1 \tag{4}$$

Hence, we emphasise the dependence of R on parameter ε, by writing the integral (1) on the form

$$K_O(\varepsilon,\alpha) = \frac{\sqrt{2}}{\pi} \int_{x^2+y^2 \leq 1} \frac{h(\varepsilon, R(\varepsilon,\psi)(x,y))\, R^2(\varepsilon,\psi)}{|R(\varepsilon,\psi)\ (x,y) - R(\varepsilon,\alpha)\ (\cos\alpha, \sin\alpha)|^2} dx\, dy \tag{5}$$

If Ω is a disk with radius R and $\sigma \equiv 1$, f(Q) and $K_O(Q')$ can be easily evaluated:

$$K_O(Q') \equiv \frac{2}{\sqrt{\pi}}\sqrt{R} \tag{6}$$

From (5), by means of a Taylor expansion, we have

$$K_O(\varepsilon,\alpha) = \frac{2}{\sqrt{\pi}} + \varepsilon \frac{\partial K_O}{\partial \varepsilon}(0,\alpha) + O(\varepsilon^2) \tag{7}$$

The detailed calculation of $\frac{\partial K_O}{\partial \varepsilon}(0,\alpha)$ is reported in reference [9].

Finally, after some calculations, we obtained the following approximation for Oore-Burns integral (1) as a function of angle α:

$$K_O(\varepsilon,\alpha) = \frac{2}{\sqrt{\pi}}\left\{1+\varepsilon\sum_{-\infty}^{+\infty} c_n E_n e^{in\alpha}\right\} + O(\varepsilon^2) \tag{8}$$

Where c_n are the Fourier coefficients of the first order crack front position $\alpha \to S(\alpha) = \frac{\partial R}{\partial \varepsilon}(0,\alpha)$ in the sense that $S(\alpha) = \sum_{-\infty}^{+\infty} c_n e^{in\alpha}$.

The E_n coefficients, are independent from the homotopy $R(\varepsilon,\alpha)$ and are reported in table 1.

In general, by taking into account that an *a*-dilatation of Ω under uniform normal tension σ produces factor $\sqrt{a}$ in the expression of K_o, from (8) we are able to state the following final equation:

$$K_O(\varepsilon,\alpha) = \frac{2\sigma\sqrt{a}}{\sqrt{\pi}}\left\{1+\varepsilon\sum_{-\infty}^{+\infty} b_n E_n e^{in\alpha}\right\} + O(\varepsilon^2) \tag{9}$$

where b_n are the Fourier coefficients of $\alpha \to \frac{1}{a}\frac{\partial S}{\partial \varepsilon}(0,\alpha)$ and $S(\varepsilon,\alpha)$ describes the boundary of $\partial\Omega(\varepsilon)$

Table 1. E_n coefficients

n	E_n	n	E_n
0	½	6	-1.58042
1	0	7	-1.81911
2	-0.4	8	-2.04377
3	-0.74286	9	-2.2566
4	-1.04762	10	-2.45929
5	-1.32468	11	-2.65318

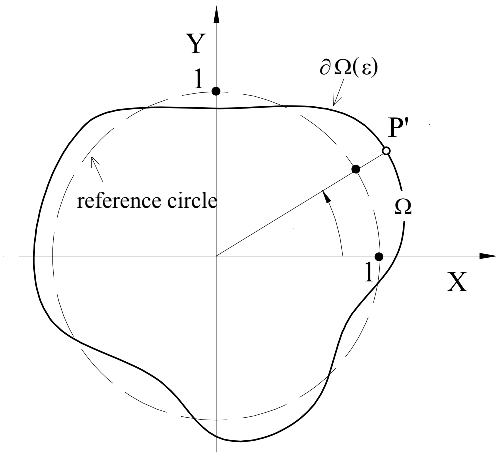

Figure 1. Perturbation of the circular flaw

COMPARISON WITH THE NUMERICAL EXAMPLES IN THE LITERATURE

In order to compare the results given by Eq. (9), in the following we have considered four cases of plane crack in an infinite solid under tensile loading which has contour shaped elliptical cracks, square cracks, half circle-half ellipse and circle-like sinusoidal cracks. These cracks have convex contours like the cracks considered in references [5, 6]. The first case was tested by means of the classic solution proposed by Irwin [4] and the others convex cracks were checked by referring to Mastrojannis et al. [7] who proposed a new integral equation for SIF under generic stress distribution over the cracks. However, the integral equation was solved numerically.

Elliptical crack

As is well known, the SIF for elliptical cracks in the semi-axis (*a,b*) was given by Irwin using the closed form in Ref. [4] :

$$K_{Irw}(\beta) = \frac{\sigma\sqrt{\pi b}}{E(k)} \left(\sin^2\beta + \frac{b^2}{a^2} \cos^2\beta \right)^{1/4} \tag{10}$$

with $tg\beta = \frac{a}{b} tg\alpha$, $k = \sqrt{1 - \frac{b^2}{a^2}}$ and E(k) is an elliptical integral of the second kind.

Figure 2 shows the comparison between the Irwin formula (10) and Eq. (9) for a ratio a/b equal to 0.6. The average error is around 1.4 % and becomes less than 1% when the b/a ratio is between 1 and 0.6.

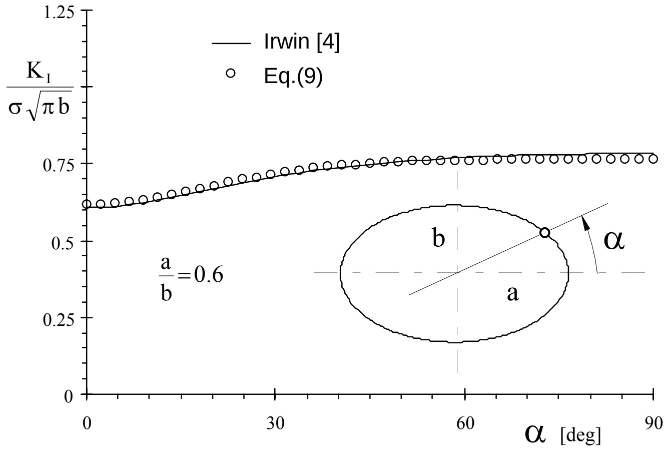

Figure 2. Comparison between Irwin solution [4] and Eq. (9) (σ is the remotely uniform tensile stress).

Square crack

The solution for square crack has been discussed in the literature only by using numerical forms [7, 10]. Figure 3 reports the predicted values for SIF by means of Eq. (9) and the numerical results obtained by Mastrojannis et al. [7]. The average error was around 4%. Note that in Ref. [5] the corner was slightly rounded, as happens in the case of Eq. (9), because we considered a limited number of waves (in Figure 3, 15 terms were used).

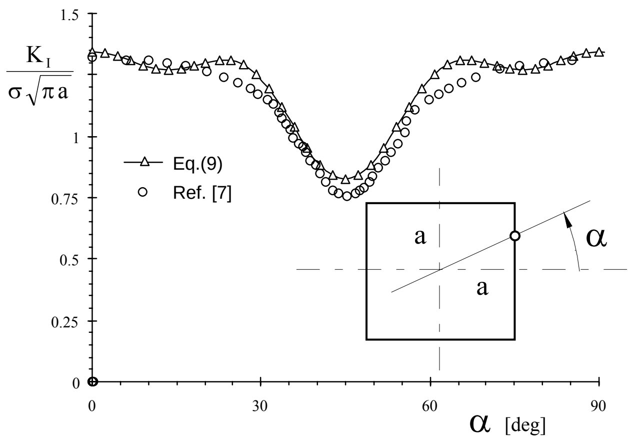

Figure 3. Comparison between the Mastrojannis et al. numerical solution [7] and Eq. (9) (σ is the remotely uniform tensile stress).

Half circle-half ellipse

A two-dimensional crack which took the shape of a half circle-half elliptical crack was also analysed in Ref. [7]. Figure 4 shows the comparison between the numerical results

[7] and those predicted by means of Eq. (9). In this case the comparison was very satisfactory and the average error was around 1%.

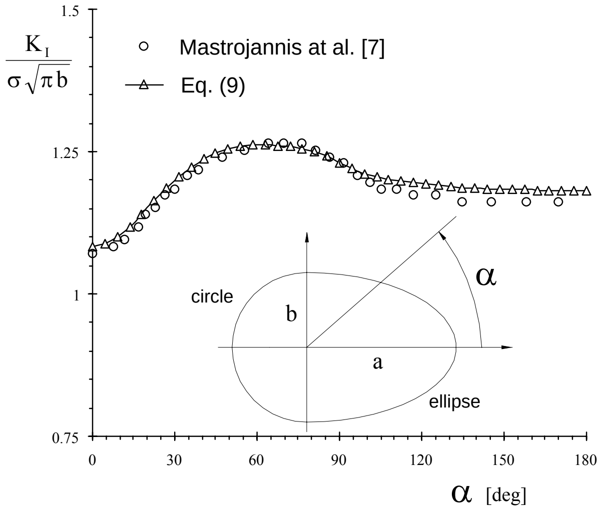

Figure 4. Comparison between the Mastrojannis et al. numerical solution [7] and Eq. (9) (a/b=1.5; σ is the remotely uniform tensile stress).

Half circle-like sinusoidal contour

This type of crack has a boundary in the form of a half circle and half shape whose polar equation is given as [7]

$$r = \frac{A}{\sqrt{1+\left(\frac{A^2}{R_m{}^2}-1\right)\left|\sin\alpha\right|}} \tag{11}$$

The SIF of the mode I loading which had been numerically solved in Ref. [7] is compared with Eq. (9) in Fig. 5. The error in the prediction of the maximum SIF was 1.4% while the average error along the whole contour was around 3%.

Remark

In Refs [5, 6, 11] the $\sqrt[4]{\text{area}}$ was proposed as a parameter which is approximately proportional to the maximum SIF $K_{I,max}$ along the crack, which can be evaluated as

$K_{I,max} = Y \sigma \sqrt{\pi \sqrt{area}}$, where Y is a shape factor. Murakami and Endo [5] obtained a value of 0.629 for Y (0.5 for internal crack [11]), provided that the crack contour was not concave and the crack was not too slender, as in the case of elliptical cracks, with the ratio for the two semi-axes being greater than 5 [6] on the basis of the best fitting result taken from many numerical results. Table 2 reports the Y coefficients evaluated by means of Eq. (9) and those reported in the literature (internal cracks). Although Eq. (9) was obtained with a first order theory, the error in the Y prediction is around 2-3%. However, in terms of first order theory, Eq. (9) is also suitable for non-convex shaped cracks, as shown in Fig. 6. The two-dimensional crack in figure 6 has a Y coefficient of 0.572. In this case, the maximum SIF is located in the zone where we have a re-entrant corner.

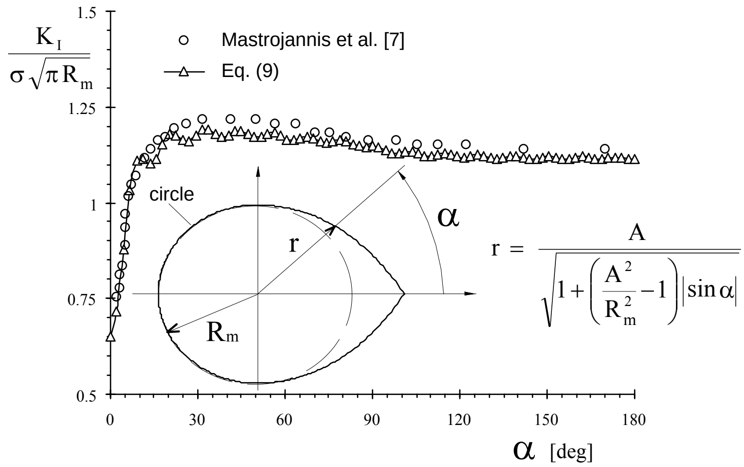

Figure 5. Comparison between the numerical solution of Mastrojannis et al. [7] and Eq. (9) (A/R_m=1.5; σ is the remotely uniform tensile stress).

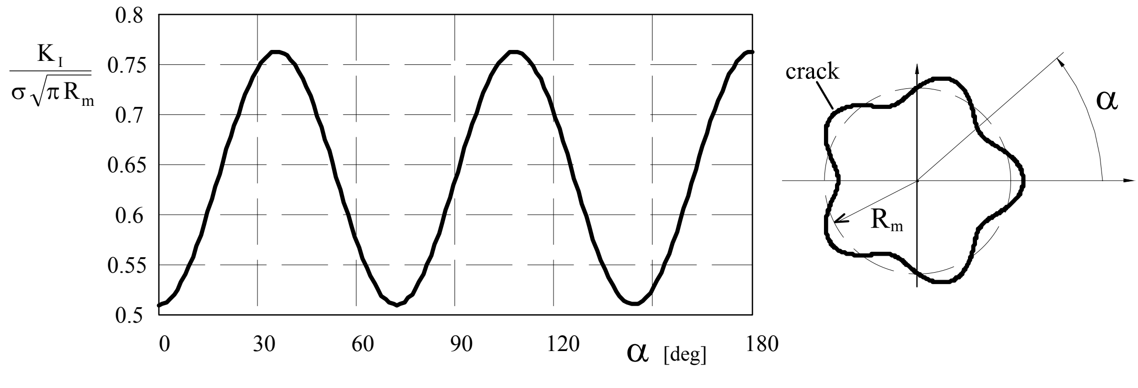

Figure 6. Prediction of irregular crack (σ is the remotely uniform tensile stress)

Table 2. Y coefficients

crack shape	Y literature	Y Eq. (9)	Error [%]
Ellipse Fig. 2	0.518	0.505	2.5
Square Fig. 3	0.527	0.536	1.7
Half circle-half ellipse Fig. 4	0.620	0.620	0.0
Half circle-like sinusoidal contour Fig. 5	0.609	0.618	1.5
Fig. 6	-	0.572	-

CONCLUSION

The stress intensity factors (SIF) of a crack subjected to remote tensile loading were analyzed by means of the Fourier series. An evaluation of the SIF along the whole crack contour was carried out using many cracks. A comparison with the literature results showed errors around of only a few per cent in both the maximum and average SIF predictions.
The satisfactory results obtained in the case of convex contours indicates that the proposed method could be used for estimating the maximum SIF when the crack has a complex contour. For example, the proposed equation could be used for estimating the fatigue limit of a material with small defects or cracks with irregular shapes provided that the crack has a somewhat circular shape.

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