On a numerical scheme for curved crack propagation based on configurational forces and maximum dissipation

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ABSTRACT. *A scheme is presented to predict crack trajectories in two and three dimensional components. First a relation between the curvature in mixed-mode crack propagation and the corresponding configurational forces is derived, based on the principle of maximum dissipation. With the help of this, a numerical scheme is presented which is based on a predictor-corrector method using the configurational forces acting on the crack together with their derivatives along real and test paths. With the help of this scheme it is possible to take bigger than usual propagation steps, represented by splines. Essential for this approach is the correct numerical determination of the configurational forces acting on the crack tip. The methods used by other authors are shortly reviewed and an approach valid for arbitrary non-homogenous and non-linear materials with mixed-mode cracks is presented. Numerical examples show, that the method is a able to predict the crack paths in components with holes, stiffeners etc. with good accuracy.*

INTRODUCTION

Crack path prediction is based on three parts: a criterion for the onset of crack propagation, a criterion for the direction of propagation, a criterion for the propagation speed or step length, for fatigue or quasi-static propagation respectively. There is a wealth of criteria available, but most of the criteria available in the literature cannot consistently by derived without ad-hoc assumptions. Furthermore one has to distinguish between approaches for kinking and curving of cracks. Most criteria available will give a statement for the immediate directional change of the crack path, thus producing a kink. With small step sizes and kinking angles these criteria are then used to represent a curved crack path, as a kink will physically only be necessary, if there is an abrupt change in the loading or the material properties. Among the few approaches using curving segments is the one of Sumi et al. [1], but he is restricting himself to slightly curving cracks. Most of these criteria are unfortunately restricted to linear elastic fracture mechanics, as they are either based on the near tip stress field solution or the stress intensity factors. So for nonlinear elastic, inhomogeneous or plastic crack propagation methods based on configurational forces have drawn attention [2, 3, 4]. Unfortunately the direction of the J-Integral or configurational force vector on the crack tip as the directional criterion is not a correct choice as it does not account for the change of the configurational forces induced by the kink [5]. This can be

seen directly from the fact that this criterion predicts always straight crack propagation for pure mode II, which is in contrast with experimental observations. Also the calculation of J_2 with the help a domain integral type approach used in most of these methods is inaccurate. So these criteria are only valid for small kinking angles, which results in small steps sizes for an accurate representation of curved cracks. The aim of this paper is to present a derivation of a propagation criterion valid for strongly curved cracks with finite propagation step sizes and formulated with the help of configurational forces, so the numerical approaches presented in [3, 4] can be used to end up with a numerical scheme that can be generalized to treat inhomogeneous materials at finite deformations. Additionally a method is presented to calculate valid results for J_2 from a direct configurational nodal force approach.

CRACK CURVING IN LEFM

The derivation of the criterion for curved crack propagation is done with the help of the results obtained by Amestoy and Leblond [6] in the framework of linear elastic fracture mechanics. Linear elastic fracture mechanics is based on the near tip stress field

$$\sigma_{ij} = K_\alpha \, f_{ij}^\alpha(\theta) \, r^{-\frac{1}{2}} + T_\alpha \, g_{ij}^\alpha(\theta) + b_\alpha \, h_{ij}^\alpha(\theta) + O(r), \tag{1}$$

where the K_α are the stress intensity factors (SIFs), T_α the (non-local) T-stresses and the b_α are the coefficients of square-root stress terms also used by Sumi et al. [1]. The f^α, g^α, h^α matrices of angular functions stem from the Williams series solution [7]. A kinked and curved crack with the elongation of the crack s is described by

$$y' = a^\star x'^{\frac{3}{2}} + \frac{1}{2} C^\star x'^2 .$$

The evolution of the SIFs is given by [6] as

$$K_\alpha(s) = K_\alpha^\star + K_\alpha^{(1/2)} \sqrt{s} + K_\alpha^{(1)} \, s + O(s^{2/3}), \tag{2}$$

with

$$\begin{aligned} K_\alpha^\star &= F_{\alpha\beta}(\phi) K_\beta \\ K_\alpha^{(1/2)} &= G_{\alpha\beta}(\phi) T_\beta + a^\star H_{\alpha\beta}(\phi) K_\beta \\ K_\alpha^{(1)} &= \left[K_\alpha^{(1)} \right]_{C^\star = 0}^{\phi, a^\star} + C^\star M_{\alpha\beta}(\phi) K_\beta , \end{aligned} \tag{3}$$

where the greek indices run over I, II, III the three crack modes and the matrices *F, G, H, M* are universal functions, depending only on the kinking angle ϕ and not the special crack problem under consideration. The first term in eq. (3c) involves the *b*-coefficients of eq. (1), but also some non-universal parts, that means it can only be determined for a special crack problem in a finite body. A special note on the non-universal characteristic of the second-order term for curving cracks seems to be missing in Sumi et al.'s [1] approach, but is pointed out in [6]. Amestoy & Leblond have derived in [6] also the consequences for the crack path of the criterion of local symmetry [8] (K_{II}=0). Here, using the same series approach, the consequences of a maximum dissipation postulate should be derived, motivated by the work of Le et al. [5], where they have shown, that from the variational principle of a body containing a crack the maximum dissipation (or maximum driving force) criterion follows without any ad-hoc assumptions. Furthermore the energetic approach has the advantage that the crack propagation rate and the driving force acting on the crack can accurately be determined for crack kinking and curving and also remain the correct thermodynamic dual quantities for these cases.

Maximum Dissipation for regular curved cracks

As the criterion for kinking cracks has already been derived in [5], we will here restrict ourselves to the case of regular crack propagation, i.e. curving without kinking. This implies for all criteria, that mode II has to vanish for the initial crack configuration as it would immediately lead to crack kinking. The starting point is thus the dissipation of a growing crack, based on the driving force acting on the propagating crack tip. Following [5] we introduce this driving force with the help of the actual SIFs

$$\begin{aligned} G(s) &= K_\alpha(s)\ K_\beta(s)\Lambda_{\alpha\beta} \\ &= G^\star + G^{(1/2)}\sqrt{s} + G^{(1)}s + O(s^{(3/2)}) \end{aligned}, \qquad \Lambda_{\alpha\beta} = \frac{1-\nu^2}{E}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{1-\nu} \end{pmatrix} \tag{4}$$

The terms in the series can be given with the help of eq. (2) as

$$\begin{aligned} G^\star(\phi) &= K^\star_\alpha K^\star_\beta \Lambda_{\alpha\beta} = F_{\alpha\gamma}\Lambda_{\alpha\beta}F_{\beta\delta}K_\gamma K_\delta \\ G^{(1/2)}(\phi, a^\star) &= \left(2K^\star_\alpha K^{(1/2)}_\beta + K^{(1/2)}_\alpha K^{(1/2)}_\beta\sqrt{s}\right)\Lambda_{\alpha\beta} \\ G^{(1)}(\phi, a^\star, C^\star) &= \left(2K^\star_\alpha K^{(1)}_\beta + 2K^{(1/2)}_\alpha K^{(1)}_\beta\sqrt{s} + K^{(1)}_\alpha K^{(1)}_\beta s\right)\Lambda_{\alpha\beta} \end{aligned} \tag{5}$$

The consequences of the postulate of maximum dissipation are here for the sake of simplicity derived from the maximum driving force principle. The consequences of the two are the same, as long as the fracture resistance force does not explicitly depend on the direction crack propagation, e.g. through the kinking angle. In the following we restrict ourselves to the two-dimensional case.

Zeroth order approximation The zeroth order term depending only on the kinking angle is the driving force acting on a kinked crack already determined in [5]. Introducing $K_{II}=0$ into this solution leads to a vanishing kinking angle $\phi=0$.

First order approximation To derive the first curvature parameter a^* the maximum driving force is determined for the series eq. (4) is cut after the square-root term.

$$G(s) = G^\star(\phi = \phi^\star) + G^{(1/2)}\sqrt{s} + O(s) \tag{6}$$

The postulate of maximum driving force leads to a vanishing first order curvature parameter and thus a vanishing first order driving force

$$a^\star = 0 \Rightarrow G^{(1/2)} = 0. \tag{7}$$

Second order approximation The driving force series eq. (4) is reduced with the help of eq. (10) and $\phi=0$ to

$$\begin{aligned} G(s) &= G^\star + G^{(1/2)}\sqrt{s} + G^{(1)}s + O(s^{3/2}) \\ &= G + G^{(1)}s + O(s^{3/2}) \end{aligned} \tag{8}$$

The second order driving force term can be further simplified with the help of a* = 0 to

$$G^{(1)} = \left(2K_\alpha K_\beta^{(1)} + K_\alpha^{(1)} K_\beta^{(1)} s\right)\Lambda_{\alpha\beta} \tag{9}$$

And the second order SIF term appearing here reduces to

$$K_\alpha^{(1)} = K_\alpha^{(1)}\big|_{\text{straight}} + C^\star M_{\alpha\beta}(\phi = 0)K_\beta \tag{10}$$

The first non-universal term in eq. (10) is to be understood as the first order term that would appear for a straight (not kinked, not curved) crack propagation. The maximum driving force gives then an equation for the second curvature parameter C^*

$$\frac{\partial G(s)}{\partial C^\star} = \frac{\partial G^{(1)}}{\partial C^\star} = 0 = 2\Lambda_{\alpha\beta}\left(K_\alpha M_{\beta\delta} K_\delta + K_\alpha^{(1)} M_{\beta\delta} K_\delta s\right) \tag{11}$$

with the solution

$$C^{\star} = \frac{-2K_{II}^{(1)}\big|_{\text{straight}}}{K_I} \Rightarrow C^{\star} = \frac{-2\frac{\mathrm{d}K_{II}}{\mathrm{d}s}\big|_{\text{straight}}}{K_I}, \tag{12}$$

where the second interpretation in the above equation is possible because of the vanishing K_{II} for the initial crack. This is the same result as derived in [6] from the principle of local symmetry. Also Sumi presented in [5] a similar result for slightly curved cracks. To be able to use a numerical approach based on configurational nodal forces in the framework of an FEM simulation, this result has to be reformulated in terms of configurational forces.

TRANSITION TO CONFIGURATIONAL FORCES

Configurational forces are to be understood as the forces in material space (in opposite to physical space) resulting from the variation in energy due to the change in position of the singularity arising at the crack tip (cf. [9]). For the configurational or Eshelby-stress tensor

$$\mu_{ij} = \psi\delta_{ij} - u_{k,j}\frac{\partial\psi}{\partial u_{k,i}}, \tag{13}$$

with ψ the free energy density, u_i the displacement vector and δ_{ij} the Kronecker delta the following balance of material momentum equation is valid

$$\mu_{ij,j} = -\frac{\partial\psi}{\partial X_i}\Big|_{\text{exp}}, \tag{14}$$

where the right hand side term is only non-vanishing, if there exists an explicit dependency of the free energy density with respect to the position X in the material. This is only the case for non-homogeneos materials, e.g. functionally graded materials. For homogenous materials the divergence in eq. (14) is vanishing, giving rise to a path-independent conservation integral, the first component of which is the widely know J-Integral

$$\mathfrak{F}_i = J_i = \lim_{\Gamma\to 0}\oint_{\Gamma}\mu_{ij}n_j ds. \tag{15}$$

Please note, that eq. (15) shows only an asymptotical path-independency, since the integrand for J_2 is not necessarily vanishing on the crack surfaces. This will be discussed in detail in connection with the numerical approach for the accurate determination of the configurational forces in a finite element framework. In a linear elastic fracture mechanics framework we have for a plane case the following connection between configurational forces, J-integral vector components and the stress intensity factors

$$\mathfrak{F}_t = J_1 = \frac{1-\nu^2}{E}\left(K_I^2 + K_{II}^2\right)$$
$$\mathfrak{F}_n = J_2 = \frac{1-\nu^2}{E}\left(-2K_I K_{II}\right) \quad . \tag{16}$$

With the help of these relations the main result of the preceeding section, the curvature resulting from maximum dissipation can be rewritten as

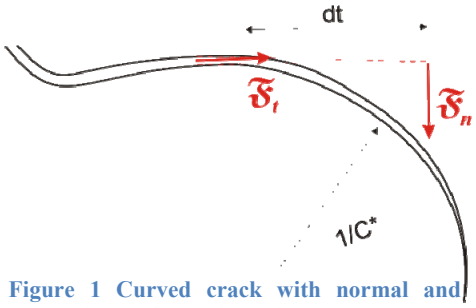

Figure 1 Curved crack with normal and tangential configurational forces

$$C^\star = \frac{\partial_t \mathfrak{F}_n}{\mathfrak{F}_t} = \frac{\partial_t \mathfrak{F}_n}{\|\mathfrak{F}\|} ,$$

where ∂_t is the tangential derivative (compare Fig. 1). This tangential derivative is to be understood as the derivative of the normal component of the configurational force along a straight crack elongation. The second interpretation of (16) is valid because for the real crack the normal component will vanish in all points.

Equation (16) means the local curvature of the crack trajectory is the same as the local curvature of the material force field. This result seems to be such a natural and straightforward result, that the author believes, it will hold in general without the underlying assumptions made earlier in the linear elastic fracture mechanics framework. This criterion will be used as a local criterion for each point lying on the three dimensional crack front. The only influence of mode III on the curvature is assumed to be via the K_{III} component entering the tangential component of the configurational force (J_1).

FINITE ELEMENT FRAMEWORK

A finite element framework making use of nodal configurational forces is used, similar to the ones described in [3,4]. The essential part is, that this approach gives in a simple post-processing step the configuration forces as the thermodyamical dual quantitiy to a variational change of the position of the corresponding node with respect to the material

$$\mathfrak{F}^h = \mathop{\mathbf{A}}_{e=1}^{E} \sum_{n=1}^{n_{en}} \int_{\mathcal{B}_0^e} \boldsymbol{\mu} \cdot \nabla_{\mathbf{X}} N^e \, \mathrm{d}A \quad . \tag{17}$$

Without discretization errors the finite element results for a body with a crack would produce only configurational forces acting on the nodes representing the crack front (or one force acting on the crack tip). Due to the failure of the shape-functions normally used in an FEM based approach to accurately represent the two singularities involved at the crack tip (namely the stress singularity and the singularity of the Eshelbian-stress, which are of

different order) also spurious configurational nodal forces are produced in the vincinity of the crack front (or tip). The accuracy of the forces acting on the tip usually is very low. Thus many authors [3,4,5] have adopted some method similar to the domain integral method, which in this framework consists simply in adding up the contributions of the nodes contained in a certain area surrounding the crack. As only J_1 is path-independent, the value for J2 is not converging, when the size of the domain is increased. Because of this an extrapolation back to a zero area domain is necessary, as suggested by the limit value appearing in eq. (15) hinting to the *asymptotic* path independence of the J-integral vector.

STEP BY STEP PROPAGATION SCHEME

A step by step numerical scheme has been implemented in the commercial FEM-code ANSYS. After each step the geometry has been created newly and a new mesh has been created. After that the following scheme has been adopted for each propagation step

- *small test step to determine the curvature*

$$\partial_t \mathfrak{F}_n \approx \mathfrak{F}_n^{\mathsf{test,end}} / \mathrm{d}s^{\mathsf{test}}$$

- *"forward sensing" the ratio σ_{nt}/σ_{nn} along the predicted crack path to determine the maximum length*
- *constant curvature propagation*

$$C^\star = \frac{\partial_t \mathfrak{F}_n}{\mathfrak{F}_t} = \frac{\partial_t \mathfrak{F}_n}{\|\mathfrak{F}\|}$$

- *small change of end slope of spline to get vanishing J_2*

$$\phi^{\mathsf{corr,end}} = \frac{\mathfrak{F}_n^{\mathsf{end}}}{\mathfrak{F}_t^{\mathsf{end}}}; \qquad \frac{\mathfrak{F}_n^{\mathsf{end}}}{\mathfrak{F}_t^{\mathsf{end}}} << 1$$

- *or cut back, if J_2/J_1 is too big*

With the help of this scheme the experiments from Bittencourt et al. [10] have been simulated. Figure 2 illustrates, that highly accurate results can be attained with a small number of propagation steps.

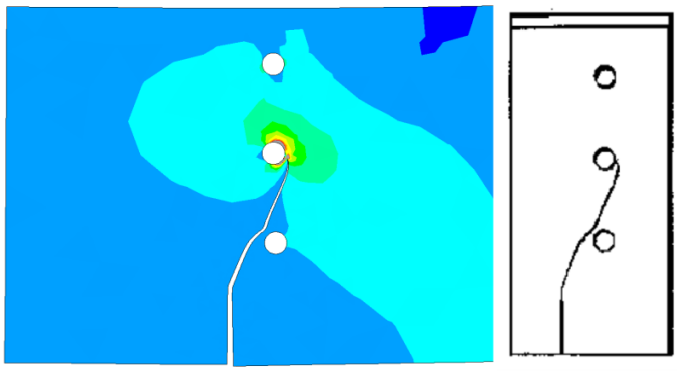

Figure 2 Computational result from 20 propagation steps

CONCLUSIONS

A criterion to describe the crack trajectory of a curved crack has been derived. The formulation in terms of configurational forces opens the door to a applicability to a wider range of inhomogenous materials or finite deformations. The necessity of an extrapolation technique to determine accurate J_2 values has been shown. The numerical scheme based on splines and a predictor-corrector method enables to take large steps in an FEM simulation as thus save computational effort, while keeping or increasing the accuracy of the predicted path.

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