Crack paths in functionally graded materials

M. Steigemann[1], M. Fulland[2], M. Specovius-Neugebauer[1] and H.A. Richard[3]

[1]Institute of Applied Mathematics, University of Kassel, Germany,
Heinrich-Plett-Str. 40, 34132 Kassel, martin.steigemann@mathematik.uni-kassel.de
[2]Westfälisches Umwelt Zentrum, Universität Paderborn, Germany
[3]Institute of Applied Mechanics, Universität Paderborn, Germany

ABSTRACT. *One of the main interests of fracture mechanics in functionally graded materials is the influence of such an inhomogeneity on a crack propagation process. Using the* GRIFFITH*' energy principle, the change of energy has to be calculated, if the crack starts to propagate. In homogeneous linear-elastic structures (asymptotically precise) formulas for the energy release rate are known, but a direct transfer of the methods to functionally graded materials can lead to very inaccurate results. Moreover, the influence of the inhomogeneity on the crack path can not be seen. Here a simple model for functionally graded materials is presented. For this model, a formula for the change of potential energy is derived, giving detailed information on the effect of the gradation on the crack growth process.*

INTRODUCTION

In this contribution we present ideas how fatigue crack growth in anisotropic functionally graded materials (FGMs) can be predicted using the GRIFFITH' energy criterion for plane problems. Here FGM especially means, that the elastic properties can change continuously.

From a physical point of view the energy principle, formulated by GRIFFITH in 1921, can be applied in anisotropic and inhomogeneous materials to calculate quasi-static crack propagation:

A crack is growing in such a way that the total energy always is minimal.

The total energy Π is composed from the surface energy $\mathbf{S}$ and the potential energy $\mathbf{U}$, the latter is the difference of the elastic energy and the work performed by external forces.

For homogeneous solids, the following result is known [1]: Suppose the crack is increased by a (small) crack shoot of length h, then the change of the potential energy can be calculated asymptotically to

$$\Delta \mathbf{U} = -\frac{1}{2}\mathbf{K}^\top \cdot \mathbb{M}(h) \cdot \mathbf{K} + \mathcal{O}(h^{3/2}), \qquad h \to 0. \tag{1}$$

Thereby $\mathbf{K}$ denotes the vector of stress intensity factors and $\mathbb{M}(h)$ is a symmetric $2\times 2-$matrix, the so called energy release matrix (ERM). For a straight crack shoot, (1) is also well-known as IRWIN-RICE-formula. The ERM contains certain integral characteristics depending on the geometry of the specimen and the crack shoot as well as on the elastic properties of the material. All entries of ERM can be calculated numerically up to sufficient precision. Using the asymptotic energy release rate the kink angle of a crack can be determined in arbitrary plane anisotropies and the crack path can be approximated piecewise by polygons [2].

The derivation of formula (1) requires the asymptotic (WESTERGAARD) representation of the displacement field at the crack tip. If the specimen under consideration consists of a FGM, the first asymptotic term of the near field is the same as in the homogeneous case with material properties frozen at the crack tip. It suggest itself to use formula (1) with this material data to calculate energy release rates and this method is known as "local homogenization" [3]. However, if the material properties vary to much near the crack tip, the resulting approximation (1) of the energy release can be very inaccurate.

In the following we discuss ideas to detect the influence of a local gradation on the developing crack path. Introducing a function δ in the material properties, which can be interpreted as a measure for the level of inhomogeneity, we single out asymptotic formulae for the change of the potential energy, if the crack propagates along a small shoot. The influence of the function δ will be shown.

FORMULATION OF THE PROBLEM

Let Ω be a domain in the plane $\mathbb{R}^2$ with polygonal boundary Γ. We consider the problem of 2-dimensional linear elasticity theory in the domain $\Omega_0 := \Omega \setminus \Xi_0$, where $\Xi_0 := \{x \in \overline{\Omega} : x_1 \leq 0, x_2 = 0\}$ is a rectilinear edge cut:

$$\begin{aligned} -\nabla \cdot \sigma(u;x) &= 0, && x \in \Omega_0, \\ \sigma^{(n)}(u;x) = \sigma(u;x) \cdot n(x) &= 0, && x \in \Xi_0^+ \cup \Xi_0^-, \\ \sigma^{(n)}(u;x) = \sigma(u;x) \cdot n(x) &= p(x), && x \in \Gamma. \end{aligned} \tag{2}$$

$n = (n_1, n_2)^\top$ is the outward normal, $u = (u_1, u_2)^\top$ the displacement field and $p = (p_1, p_2)^\top$ denotes the vector of surface load, assumed to be self-balanced. With

Ξ_0^+ and Ξ_0^- we denote the upper and lower sides of the crack, considered to be tension-free. The coordinate system is chosen in such a way, that the crack tip x_0 is at the origin. The strain tensor (in CARTESIAN coordinates) $\varepsilon_{ij}(u;x) = \frac{1}{2}\left(\partial_i u_j(x) + \partial_j u_i(x)\right)$, $i,j = 1,2$, is related to the stress tensor by HOOKE's law:

$$\sigma_{ij}(u;x) = \sum_{k,l=1}^{2} a_{ij}^{kl}(x)\varepsilon_{kl}(u;x), \qquad i,j = 1,2.$$

$a(x) = a_{ij}^{kl}(x)$ is a symmetric rank-4 tensor, i.e. in an anisotropic material there are 6 different elastic moduli. For the strain tensor, we also use the vector notation

$$\varepsilon(u;x) := \Big(\varepsilon_{11}(u;x), \varepsilon_{22}(u;x), \sqrt{2}\varepsilon_{12}(u;x)\Big)^\top.$$

Then, for the stress tensor (in vector notation) the relation holds

$$\sigma(u;x) = A(x)\cdot\varepsilon(u;x) = \Big(\sigma_{11}(u;x), \sigma_{22}(u;x), \sqrt{2}\sigma_{12}(u;x)\Big)^\top$$

with the matrix function

$$A(x) = \begin{pmatrix} a_{11}(x) & a_{21}(x) & \sqrt{2}a_{31}(x) \\ a_{21}(x) & a_{22}(x) & \sqrt{2}a_{32}(x) \\ \sqrt{2}a_{31}(x) & \sqrt{2}a_{32}(x) & 2a_{33}(x) \end{pmatrix} \tag{3}$$

symmetric and positive definite in every point $x \in \Omega$, containing the elastic moduli. $A(x)$ is called the mathematical HOOKE tensor or HOOKE matrix (VOIGT notation). The factor $\sqrt{2}$ ensures, that strains (and stresses) have the same EUCLIDEAN norm, in vector and in tensor notation.

A model for functionally graded materials.
Let us assume, that the specimen under consideration is composed of a FGM. With this notion we relate the following: The HOOKE matrix depends (continuously) on the space coordinates. Our main interest is to detect the influence of such an inhomogeneity on the fracture process. But if the material properties depend on six (different!) functions, there is no real chance to see which of them cause an effect on the crack path. Therefore, we simplify the problem and introduce just one (scalar) function in the material properties:

$$A(x) := A + \delta(x)B$$

where δ is a smooth and bounded function. $A, B \in \mathbb{R}^{3\times 3}$ are (constant) symmetric matrices of type (3). We always suppose that the matrix function $A(x)$ is symmetric and positive definite in every point $x \in \mathbb{R}^2$. The function δ can be understood as a measure of the level of the gradation.

ASYMPTOTIC DECOMPOSITION AT THE CRACK TIP

In a solid consisting of a functionally graded material with HOOKE matrix $A(x) = A + \delta(x)B$ the displacement field has an asymptotic (WESTERGAARD) expansion of the following type at the crack tip $x_0 = 0$ [4]:

$$u(x) = K_I U_0^I(x) + K_{II} U_0^{II}(x) + \dots, \qquad |x| \to 0. \tag{4}$$

K_I and K_{II} are the stress intensity factors (SIFs) and the (generalized) eigenfunctions

$$U_0^j(x) = r^{1/2}\Phi^j(\varphi), \qquad j = I, II,$$

where (r, φ) are plane polar coordinates, are solutions of the homogeneous elasticity problem in the whole plane with a semi-infinite cut:

$$\begin{aligned} -\nabla \cdot \sigma^0(U_0^j; x) \;&=\; 0, \qquad x \in \mathbb{R}^2 \setminus \Xi_\infty, \qquad \Xi_\infty := \{x : x_1 \leq 0, x_2 = 0\}, \\ \sigma_{12}^0(U_0^j; x_1, 0) \;&=\; 0, \qquad \sigma_{22}^0(U_0^j; x_1, 0) = 0, \qquad x_1 < 0. \end{aligned} \tag{5}$$

Here, $\sigma^0 = A_0 : \varepsilon$ is the stress tensor with material properties $A_0 = A + \delta(x_0)B$ frozen at the crack tip. Due to [5], the functions U_0^j can be normalized to the following condition for every plane anisotropy:

$$[U_0^I](-r) = \frac{8}{\sqrt{2\pi}}(A_0^{-1})_{11}\sqrt{r}\begin{pmatrix}0\\1\end{pmatrix}, \qquad [U_0^{II}](-r) = \frac{8}{\sqrt{2\pi}}(A_0^{-1})_{11}\sqrt{r}\begin{pmatrix}1\\0\end{pmatrix}. \tag{6}$$

Here, $[u](x_1) = u(x_1, +0) - u(x_1, -0)$ is the jump over the crack surfaces. For this normalization of the near field, SIFs are given by the limit

$$K_j = \frac{\sqrt{2\pi}}{8(A_0^{-1})_{11}} \lim_{x \to -0} \frac{1}{\sqrt{r}}[u_{3-j}](x_1), \qquad j = 1, 2.$$

Generalized eigenfunctions are known explicitly for isotropic materials and some classes of anisotropic ones [6]. For general anisotropies, they can be computed numerically up to arbitrary precision.

GRIFFITH' ENERGY CRITERION

For calculating quasi-static crack growth in FGMs, the energy principle can be used, formulated in the introduction. The total energy Π is the sum of the surface energy $\mathbf{S}$ and the potential energy $\mathbf{U}$:

$$\Pi = \mathbf{S} + \mathbf{U} = \mathbf{S} + \frac{1}{2}\int\limits_{\Omega_0} \sigma_{ij}(u)\varepsilon_{ij}(u)\,dx - \int\limits_{\Gamma} g \cdot u\,ds = \mathbf{S} - \frac{1}{2}\int\limits_{\Gamma} g \cdot u\,ds.$$

Here, we use the sum convention and $g \cdot u = g_i u_i$. The last equation follows from CLAPEYRON's theorem. Let u^h be the solution to problem (2) in the solid Ω_h, where the crack has propagated along a small shoot Υ_h of (small) length h. For simplicity, we suppose that the shoot is a linear polygon, starting from the tip of the crack Ξ_0 at an angle $\theta \in (-\pi, \pi)$:

$$\Upsilon_h(\theta) := \{x : 0 \le x_1 \le h\cos(\theta), x_2 = x_1 \tan(\theta)\}$$

The change in the potential energy produced by crack propagation can be calculated using CLAPEYRON's theorem:

$$\Delta \mathbf{U} = \mathbf{U}^h - \mathbf{U} = -\frac{1}{2}\int\limits_{\Gamma} (u^h - u)\cdot g\, ds = \frac{1}{2}\sum_{\pm}\int\limits_{\Upsilon_h^{\pm}(\theta)} u^h \cdot \sigma^{(n)}(u)\, ds. \tag{7}$$

At the new crack tip $x_{\text{tip}} = h(\cos(\theta), \sin(\theta))^\top$, the displacement field u^h has an expansion similar to (4):

$$u^h(y) = u^h(x_{\text{tip}}) + K_1^h \widehat{U}_{\text{tip}}^{1,1}(y) + K_2^h \widehat{U}_{\text{tip}}^{2,1}(y) + \ldots, \qquad |y| \to 0, \tag{8}$$

where y denote local CARTESIAN (crack) coordinates at x_{tip} directing along $\Upsilon_h(\theta)$. $\widehat{U}_{\text{tip}}^{j,k}$ are related to the homogeneous elasticity problem (5) with HOOKE matrix $\widehat{A}(x_{\text{tip}}) = \widehat{A}(\theta) + \delta(x_{\text{tip}})\widehat{B}(\theta)$, where $\widehat{A}, \widehat{B}$ are the HOOKE tensors A, B, rotated to crack coordinates y (see e.g. [6] for more details). K_i^h are the SIFs at the new tip. To evaluate formula (7), we replace u^h and u by their asymptotic expansions at the new crack tip x_{tip} and at x_0 respectively. Using the jump relations (6) and expansions (4) and (8), short calculations lead to

$$\begin{aligned}
\Delta \mathbf{U} &\approx C(\theta)\sum_{i=1}^{2} K_{3-i}^h \int\limits_{\Upsilon_h(\theta)} (h - |x|)^{1/2}\sigma_i^{(n)}(u; x)\, ds \\
&= C(\theta)\sum_{i,j=1}^{2} K_{3-i}^h K_j^0 \left(\widetilde{\Phi}_i^{j,1}(A;\theta)h + \widetilde{\Phi}_i^{j,1}(B;\theta)\int\limits_0^h \sqrt{\frac{h-r}{r}}\delta(r,\theta)\, dr \right) + \mathcal{O}(h^{3/2})
\end{aligned}$$

where $r^{-1/2}\widetilde{\Phi}_i^{j,k}$ are components related to the normal stresses of the eigenfunctions:

$$\begin{aligned}
r^{-1/2}\widetilde{\Phi}_i^{j,k}(A;\theta) &= -\sin(\theta)\sigma_{i1}^A(U_0^{j,k}; x) + \cos(\theta)\sigma_{i2}^A(U_0^{j,k}; x), \qquad x \in \Upsilon_h(\theta), \\
r^{-1/2}\widetilde{\Phi}_i^{j,k}(B;\theta) &= -\sin(\theta)\sigma_{i1}^B(U_0^{j,k}; x) + \cos(\theta)\sigma_{i2}^B(U_0^{j,k}; x), \qquad i, j = 1, 2.
\end{aligned}$$

We write $\delta(r,\theta) := \delta(r\cos(\theta), r\sin(\theta))$. The integral on the right hand side can be calculated by partial integration:

$$\int\limits_0^h \sqrt{\frac{h-r}{r}}\delta(r,\theta)\, dr = \frac{\pi}{2}\delta(x_{\text{tip}})h - \int\limits_0^h \psi(r)\Big(\cos(\theta)\partial_{x_1}\delta(r,\theta) + \sin(\theta)\partial_{x_2}\delta(r,\theta)\Big)\, dr$$

where $\partial_{x_j}\delta(r,\theta) = \partial_{x_j}\delta(x)\big|_{x\in\Upsilon_h(\theta)}$ and ψ is the monotone increasing and positive function

$$\psi(r) = \sqrt{h-r}\sqrt{r} + h\arctan\left(\frac{\sqrt{r}}{\sqrt{h-r}}\right) \qquad \text{thus} \qquad 0 \le \psi(r) \le \frac{\pi}{2}h.$$

The function $C(\theta) = -\frac{4}{\sqrt{2\pi}}\left(\widehat{A}(x_{\text{tip}})^{-1}\right)_{11}$ (see condition (6)) can be calculated exactly and depends on the function δ as well as on the crack shoot $\Upsilon_h(\theta)$. Finally, we get

$$\begin{aligned}\Delta\mathbf{U} \approx\ & C(\theta)\sum_{i,j=1}^{2} K_{3-i}^{h} K_j^0 \Bigg(\left(\widetilde{\Phi}_i^{j,1}(A;\theta) + \widetilde{\Phi}_i^{j,1}(B;\theta)\frac{\pi}{2}\delta(x_{\text{tip}})\right) h \\ & -\widetilde{\Phi}_i^{j,1}(B;\theta)\int_0^h \psi(r)\big(\cos(\theta)\partial_{x_1}\delta(r,\theta) + \sin(\theta)\partial_{x_2}\delta(r,\theta)\big)\,dr\Bigg) + \mathcal{O}(h^{3/2}).\end{aligned} \tag{9}$$

Because the function δ is smooth and bounded, the integral in (9) can be estimated and is of order h. But this would be a loss of information. In principle, this integral is the change of δ along the shoot $\Upsilon_h(\theta)$. All terms can be calculated for any smooth function δ and any angle θ. This formula is a first step to detect the influence of local inhomogeneities on the crack path.

EXAMPLES AND CONCLUSIONS

Finally, we show first numerical results and consider a symmetric compact tension (CTS-)specimen subjected to a Mode-II-loading (see Figure 1). The length units are selected to $w = 90mm$, the thickness of the specimen is $10mm$ and we apply a force $F = 10000N$. We choose a local gradation only in one space direction (see Fig. 2):

$$A(x) = (1+\delta(x))A, \qquad \delta(x) := \begin{cases} 0.5\sin\left(\frac{(x_1-20)}{5}\pi\right) & 20. < x_1 < 25. \\ 0 & \text{otherwise}\end{cases}$$

Because this function is not smooth, we flatten out $A(x)$ at the points $x_1 = 20$ and $x_1 = 25$. This is only technical and we go not into details. The elastic moduli are $a_{11} = a_{22} = \lambda + 2\mu$, $a_{21} = \lambda$, $a_{31} = a_{32} = 0$, $a_{33} = \mu$ with $\lambda = 56023N/mm^2$, $\mu = 26364N/mm^2$, corresponding to aluminium alloy $7075 - T651$. Our motivation of this simple example is just: "What can happen, if the material is locally functionally graded?"

Numerical computations are done with deal.II (dealii.org) and the meshgenerator Cubit 11.0 (cubit.sandia.gov). To calculate SIFs, we use weight functions and solve a pure NEUMANN problem without clamping the specimen, see [2, 7] for more details.

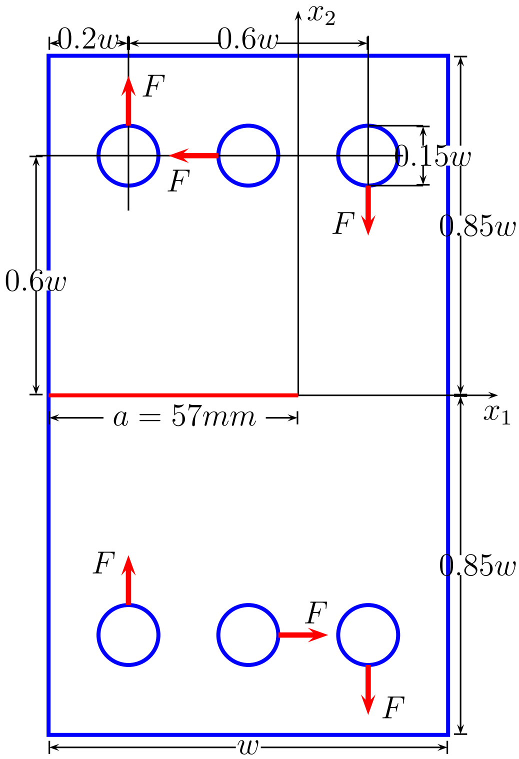

Figure 1: CTS-specimen

For the isotropic material, the computed crack path is shown in Fig. 3. We set $h = 0.5mm$ and the simulation is stopped after 47 steps, when a critical SIF $K_V = 0.5K_I + 0.5\sqrt{K_I^2 + 5.336K_{II}^2} = 972N/mm^{3/2}$ is reached [8]. The crack path in the functionally graded specimen is shown in Fig. 2. We stop the simulation after 60 steps and it seems that the gradation "pushes away" the crack.
We want to emphasize, that for the simulation of a crack growth process one has to take into account surface energy. It is the nature of this problem, that crack propagation depends on fracture toughness or in this context called surface energy. In an isotropic material, surface energy can be assumed to be constant and this is taken into account in the isotropic example. But in an inhomogeneous anisotropic structure, surface energy depends on the direction of the crack and the position of the crack tip to. The crack path shown in Fig. 2 is calculated without taking into account surface energy! Our simulation gives no information about the speed of the crack, because we do not have any data for surface energy here. One can assume that surface energy is constant in the isotropic homogeneous part of the specimen and this would not have an influence on the shape of the crack path itself. But we are not sure, if or if not a gradation influences fracture toughness in the whole structure. This example only shows the influence of an inhomogeneous HOOKE tensor.

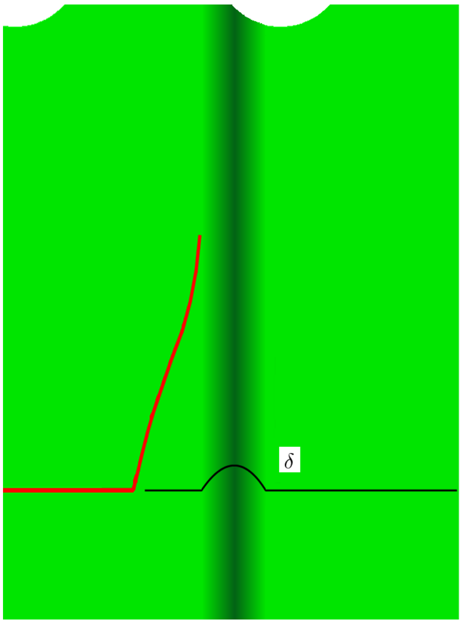

Figure 2: Functionally graded

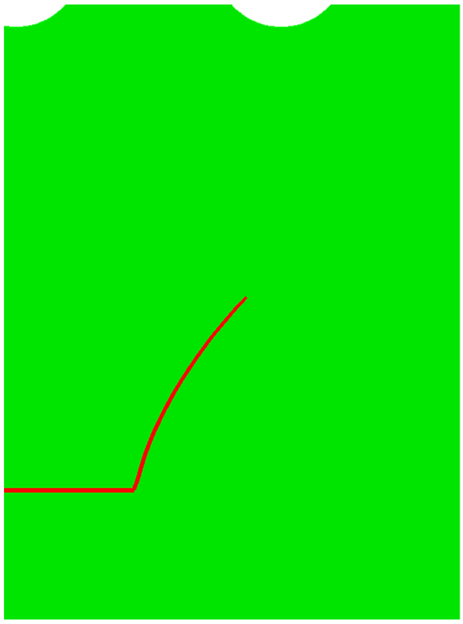

Figure 3: Aluminium 7075-T651

REFERENCES

1. Argatov, I. and Nazarov, S. (2002) *J. Appl. Maths. Mechs.* **66**, 491–503.
2. Steigemann, M. Verallgemeinerte Eigenfunktionen und lokale Integralcharakteristiken bei quasi-statischer Rissausbreitung in anisotropen Materialien. PhD thesis University of Kassel (2008).
3. Kim, J.-H. and Paulino, G. (2007) *Mech. Adv. Mat. Struct.* **14(4)**, 227–244.
4. Costabel, M. and Dauge, M. (2002) *Math. Nachr.* **235**, 29–49.
5. Nazarov, S. (2005) *J. Appl. Mech. Techn. Physics* **36(3)**, 386–394.
6. Steigemann, M. and Specovius-Neugebauer, M. (2008) *ZAMM* **88(2)**, 100–115.
7. Steigemann, M. and Fulland, M. (2007) *Int. J. Frac.* **146(4)**, 265–277.
8. Richard, H. and Sander, M. (2008) Ermüdungsrisse, Vieweg+Teubner-Verlag, Wiesbaden.

ACKNOWLEDGEMENT

This contribution is based on investigations of the collaborative research center SFB/TR TRR30, which is kindly supported by the DFG.