# A Fracture initiation direction influenced by an orthotropic bi-material notch

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***ABSTRACT.*** *In the present contribution the joint of orthotropic materials is modelled as an orthotropic bi-material notch. The singular stress field near the notch tip is investigated. Depending on the notch geometry and materials, the stress singularity exponents are determined. A criterion for the direction of crack nucleation is formulated based on angular dependences of the mean value of tangential stress in the notch tip vicinity. For various combinations of the orthotropic parts of the bi-material notch the crack initiation direction was investigated by means of the finite element method as well as on the basis of the analytical-numerical approach described in the paper. Results of both methods are mutually compared and discussed.*

## INTRODUCTION

Modern technical constructions very often contain composite materials (e.g. layered composite materials, constructions with protective surface layers, thermal barriers). They enable achievement of properties which could not be attained by means of homogeneous materials. On the other hand, compound structures lead to geometrical and material discontinuities and thus to singular stress concentrations with a general stress singularity exponent different from 1/2. Such stress concentrators preclude any application of the fracture mechanics approaches originally developed for a crack in isotropic homogeneous materials. Further, the orthotropic material properties seriously complicate procedures for assessment of bi-material wedge stability. Most such discontinuities can be mathematically modelled as bi-material notches composed of two orthotropic materials. The edge of a protective coating, a free edge stress singularity or other shapes of notches can be modelled for varying angles $\omega_1$ and $\omega_2$.

In the contribution the orthotropic bi-material notch is analysed from the perspective of linear elastic fracture mechanics, i.e. the validity of small-scale yielding conditions is assumed. We further assume ideal adhesion at the bi-material interface and the notch radius $R \to 0$ (a sharp bi-material notch).

The aim of the contribution is to suggest a procedure for the determination of the direction of crack initiation from a bi-material notch composed of two orthotropic parts.

The approach is based on the knowledge of the stress distribution in the place of the concentration. A criterion of the maximal tangential stress is modified and adapted to particularities of the nature of the stress concentrator. For the determination of the fracture initiation direction both analytical and numerical techniques are employed. The results are presented for specific notch geometry for the varying ratio of Young's moduli $E_{Mx}/E_{My}$ of both materials.

## STRESS DISTRIBUTION

Singular stress fields usually occur near the tip of a sharp interfacial corner, and their nature has been the subject of a number of studies. Consider the bi-material notch composed of two orthotropic parts as shown in Fig. 1. Within plane elasticity of anisotropic media the Lekhnitskii-Eshelby-Stroh (LES) formalism based on [1,2,3] can be used. Complex potentials satisfying the equilibrium and the compatibility conditions as well as the linear stress-strain dependence and given boundary conditions are the basis for the determination of stress and deformation fields. In the case of general plane anisotropic elasticity all the components of the stress and deformation tensors have to be considered. In the case of orthotropic materials symmetry in the stiffness and compliance matrices occur. Thus the stress and strain tensor is significantly reduced. According to the LES theory for an orthotropic material, the relations for deformations and stresses can be written as follows:

$$\begin{gathered} u_i = 2\,\mathrm{Re}\left\{A_{ij} f_j(z_j)\right\} \\ \sigma_{2i} = 2\,\mathrm{Re}\left\{L_{ij} f_j'(z_j)\right\}, \quad \sigma_{1i} = -2\,\mathrm{Re}\left\{L_{ij}\mu_j f_j'(z_j)\right\} \end{gathered} \tag{1}$$

where $\mu_j = \mu_j' + \mathrm{i}\mu_j''$ are the eigenvalues of the elastic constants, $z_j = x + \mu_j y$ and for matrices $A_{ij}$ and $L_{ij}$ holds:

$$\mathbf{A} = \begin{bmatrix} s_{11}\mu_1^2 + s_{12} & s_{11}\mu_2^2 + s_{12} \\ s_{12}\mu_1 + s_{22}/\mu_1 & s_{12}\mu_2 + s_{22}/\mu_2 \end{bmatrix}, \qquad \mathbf{L} = \begin{bmatrix} -\mu_1 & -\mu_2 \\ 1 & 1 \end{bmatrix} \tag{2}$$

In order to express the stress compoments in polar coordinates the stress function $\phi = [\varphi_i]$ is used, where $\varphi_i = 2\,\mathrm{Re}\left\{L_{ij} f_j(z_j)\right\}$, and the radial and tangential stresses are then expressed as:

$$\sigma_{rr} = \mathbf{n}\cdot\mathbf{t}_r, \quad \sigma_{r\theta} = \mathbf{m}\cdot\mathbf{t}_r = \mathbf{n}\cdot\mathbf{t}_\theta, \quad \sigma_{\theta\theta} = \mathbf{m}\cdot\mathbf{t}_\theta, \tag{3}$$

where $\mathbf{n}^T = [\cos\theta, \sin\theta]$, $\mathbf{m}^T = [-\sin\theta, \cos\theta]$ and $\mathbf{t}_r = -\frac{1}{r}\phi_{,\theta}, \mathbf{t}_\theta = \phi_{,r}$. In the case of the studied notch, the potential $f_j$ has the following form:

$$\mathbf{f} = H\left\langle z_*^{\delta}\right\rangle \mathbf{v}, \tag{4}$$

where $H$ is the generalized stress intensity factor (it can generally be complex), $v_i$ is a complex eigenvector corresponding to the eigenvalue δ, where $1-\delta = 1-(\delta' + \mathrm{i}\delta'')$ represents the exponent of the stress singularity at the notch tip. Eigenvector $v_i$ and eigenvalue δ are the solution of the eigenvalue problem leading from the prescribed notch boundary and compatibility conditions. The expression $\left\langle z_*^{\delta}\right\rangle$ is a diagonal matrix for which $\left\langle z_*^{\delta}\right\rangle = \mathrm{diag}\left[z_1^{\delta}, z_2^{\delta}\right]$.

Then the relation between the polar coordinates $(r,\theta)$ and the coordinates $(R_i,\Psi_i)$ in the plane of $\mathrm{Re}(z_i)$, $\mathrm{Im}(z_i)$:

$$R_i^2 = (\cos\theta + \mu_i'\sin\theta)^2 + (\mu_i''\sin\theta)^2$$

$$\Psi_i \begin{cases} 0 & \text{for } \theta = 0 \\ \mathrm{arccot}((\cos\theta + \mu_i'\sin\theta)/\mu_i''\sin\theta) & \text{for } \theta \in (0,\pi) \\ \mathrm{arccot}((\cos\theta + \mu_i'\sin\theta)/\mu_i''\sin\theta) - \pi & \text{for } \theta \in (-\pi,0) \\ -\pi & \text{for } \theta = -\pi \end{cases} \tag{5}$$

From the relations (2) and (4) we get the potentials:

$$\begin{aligned} \varphi_1(r,\theta) = 2H\,\mathrm{Re}\{ &-(\mu_1' + \mathrm{i}\mu_1'')(v_1' + \mathrm{i}v_1'')r^{\delta'}R_1^{\delta'}\mathrm{e}^{-\delta''\Psi_1}\mathrm{e}^{\mathrm{i}(\delta''\ln r + \delta''\ln R_1 + \delta'\Psi_1)} + \\ &-(\mu_2' + \mathrm{i}\mu_2'')(v_2' + \mathrm{i}v_2'')r^{\delta'}R_2^{\delta'}\mathrm{e}^{-\delta''\Psi_2}\mathrm{e}^{\mathrm{i}(\delta''\ln r + \delta''\ln R_2 + \delta'\Psi_2)}\} \end{aligned} \tag{6}$$

$$\begin{aligned} \varphi_2(r,\theta) = 2H\,\mathrm{Re}\{ &(v_1' + \mathrm{i}v_1'')r^{\delta'}R_1^{\delta'}\mathrm{e}^{-\delta''\Psi_1}\mathrm{e}^{\mathrm{i}(\delta''\ln r + \delta''\ln R_1 + \delta'\Psi_1)} + \\ &+(v_2' + \mathrm{i}v_2'')r^{\delta'}R_2^{\delta'}\mathrm{e}^{-\delta''\Psi_2}\mathrm{e}^{\mathrm{i}(\delta''\ln r + \delta''\ln R_2 + \delta'\Psi_2)}\} \end{aligned} \tag{7}$$

Finally, the stress components in polar coordinates are obtained via the relations (3). Further we assume the existence of more than one singular term. In the following the subscript $k$ denotes the pertinence to the specific stress singularity exponent 1-$\delta_k$:

$$\sigma_{rr} = \sum_k (n_1 t_{r1}(\delta_k) + n_2 t_{r2}(\delta_k)),\ \sigma_{r\theta} = \sum_k (m_1 t_{r1}(\delta_k) + m_2 t_{r2}(\delta_k)),\ \sigma_{\theta\theta} = \sum_k (m_1 t_{\theta 1}(\delta_k) + m_2 t_{\theta 2}(\delta_k)). \tag{8}$$

In most practical cases, as well as in the cases studied in the paper, there are two singular terms corresponding to two stress singularity exponents 1-$\delta_1$ and 1-$\delta_2$. Note that for the final determination of the stress field in the bi-material notch vicinity the generalized stress intensity factors have to be estimated by means of numerical approaches. Their values result from a numerical solution for a certain construction with given material properties, geometry and boundary conditions.

## GENERALIZED STRESS INTENSITY FACTORS DETERMINATION

In order to determine the final stress distribution around a bi-material notch, it is important to find out the value of the generalized stress intensity factors (GSIFs) $H$ from the numerical solution to a concrete situation with a given geometry, materials and boundary conditions. In contrast with the determination of the $K$ factor for a crack in an isotropic homogeneous medium, for the ascertainment of a GSIF $H$ there is no procedure incorporated in the calculation systems. The calculation of $H$ is not trivial and requires certain experience. In the case of an orthotropic bi-material notch the GSIFs can be determined using the so-called Ψ-integral [4]. This method is an implication of Betti's reciprocity theorem which in the absence of body forces states that the following integral is path-independent.
The definition of the Ψ-integral:

$$\Psi(\mathbf{u},\hat{\mathbf{u}}) = \int_{\Gamma} (\sigma_{ij}(\mathbf{u}) n_i \hat{u}_j - \sigma_{ij}(\hat{\mathbf{u}}) n_i u_j) \mathrm{d}s \quad \text{for } i, j = 1, 2 \tag{9}$$

The contour Γ surrounds the notch tip and $\mathbf{u}, \hat{\mathbf{u}}$ are two admissible displacement fields. The displacements $u_j$ are considered as the regular solution and $\hat{u}_j$ as the auxiliary solution of the eigenvalue problem of the notch, for the eigenvalues it holds $\hat{\delta} = -\delta$. A major advantage of the integral is its path independence for the case of multimaterial wedges. For the contour Γ closely surrounding the notch tip, the Ψ-integral:

$$\Psi(\mathbf{u},\hat{\mathbf{u}}) = -\int_{-\omega_2}^{\omega_1} (\varphi_{j\theta}(\mathbf{u})\hat{u}_j - \varphi_{j\theta}(\hat{\mathbf{u}})u_j)\mathrm{d}\theta = H(c_1^I + c_2^I - c_3^I - c_4^I + c_1^{II} + c_2^{II} - c_3^{II} - c_4^{II}) \tag{10}$$

where the constants $c_1^M$, ..., $c_4^M$ are given by definite integrals independent of $r$. The superscript $M = I, II$ corresponds to the material regions $I$ and $II$ bounded by the angles $(0, \omega_1)$ for the material region $I$, and $(-\omega_2, 0)$ for the material $II$.

## CRACK INITIATION DIRECTION

The stress field around a bi-material notch inherently covers combined normal and shear modes of loading. For mixed mode fields a crack may grow along the interface or at a certain angle $\theta_0$ with the interface into material I or II. In the present paper where the two orthotropic materials are assumed as perfectly bonded, only crack propagation into materials I or II will be supposed. Erdogan and Sih [5] proposed and Smith et al. [6] modified the MTS theory in a study on the slant crack under mixed mode I/II loading, see also [7]. This criterion states that the crack is initiated in the direction $\theta_0$

where the circumferential stress $\sigma_{\theta\theta}$ at some distance from the crack tip has its maximum and reaches a critical tensile value. The local maximum of the tangential stress $\sigma_{\theta\theta}$ in the case of a bi-material orthotropic notch depends on the radial distance $r$ from the notch tip. In order to suppress the influence of the distance $r$, the mean value of the tangential stress is evaluated over a certain distance $d$:

$$\overline{\sigma_{\theta\theta}}(\theta) = \frac{1}{d}\int_0^d \sigma_{\theta\theta}(r,\theta)\mathrm{d}r = \frac{1}{d}\sum_k \left( m_1\varphi_1(d,\theta,\delta_k) + m_2\varphi_2(d,\theta,\delta_k) \right) \tag{11}$$

The distance $d$ has to be chosen in dependence on the mechanism of a rupture, e.g. as a dimension of a plastic zone or as a size of material grain. The distance $d$ can also be chosen by means of the theory of critical distances, see [8]. The mean value of the tangential stress is determined in dependence on the polar angle $\theta$.

The potential direction of crack initiation is determined from the maximum of the mean value of tangential stress in both materials. The following two conditions have to be satisfied:

$$\left( \frac{\partial \overline{\sigma_{M\theta\theta}}}{\partial \theta} \right)_{\theta_0} = 0, \quad \left( \frac{\partial^2 \overline{\sigma_{M\theta\theta}}}{\partial \theta^2} \right)_{\theta_0} < 0 \tag{12}$$

Using (8), (11) and (12) the first derivation it follows:

$$\frac{\partial \overline{\sigma_{\theta\theta}}}{\partial \theta} = \sum_k \left( -n_1\varphi_1(d,\theta,\delta_k) - n_2\varphi_2(d,\theta,\delta_k) + m_1\frac{\partial \varphi_1}{\partial \theta} + m_2\frac{\partial \varphi_2}{\partial \theta} \right)_{\theta_0} = 0 \tag{13}$$

It can be shown that in the case of existence of twofold singularity ($k$ = 1, 2) the crack initiation angle $\theta_0$ is independent of the absolute values of GSIFs, but it depends only on their ratio $H_2/H_1$ (obtained from FEM). The maximum of $\overline{\sigma_{\theta\theta M}}$ can exist in both material I in the interval (0; $\omega_1$) and material II in the interval (-$\omega_2$; 0). If there are more than one direction of possible crack initiation, it is necessary to consider all of them.

## NUMERICAL EXAMPLE

A rectangular bi-material notch composed of two orthotropic parts was considered in the numerical example. The geometry of the notch is characterized by the angles $\omega_1$ = 90°, $\omega_2$ = 180° and is shown in the fig. 1. The combination of materials was chosen in order to gain the influence of orthotropy of the substrate on the crack initiation angle. The upper layer has constant material characteristics described by Young's moduli in the longitudinal and transversal directions $E_{x\mathrm{I}}$ = 100 GPa, $E_{y\mathrm{I}}$ = 50 GPa. The following four pairs of elastic moduli of the substrate were considered: $E_{x\mathrm{II}}/E_{y\mathrm{II}}$ = {50/50; 100/50;

200/50; 400/50} GPa. The varying input values of the elastic moduli in the directions $x$ and $y$ can be achieved by a varying volume percentage and orientation of fibres in a matrix. Poisson's constants of both the layer and the substrate were taken as $\nu_{xyI} = \nu_{xyI} = 0.3$. The stress singularity exponents follow from the geometry and material characteristics and they are stated in the table 1 below.

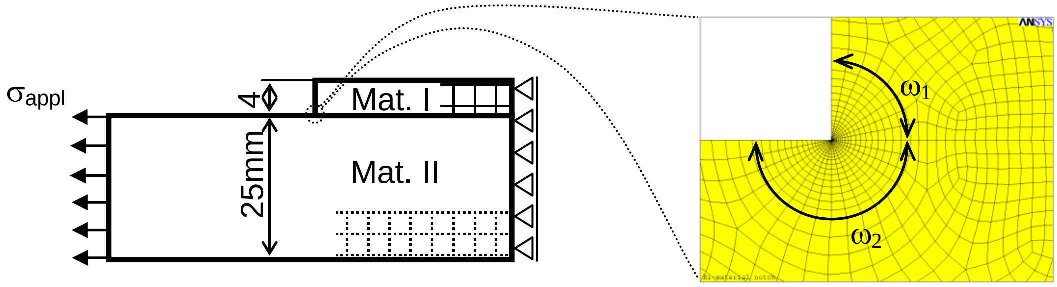



Figure 1. Configuration of a bi-material orthotropic notch with a detail of the finite element mesh.

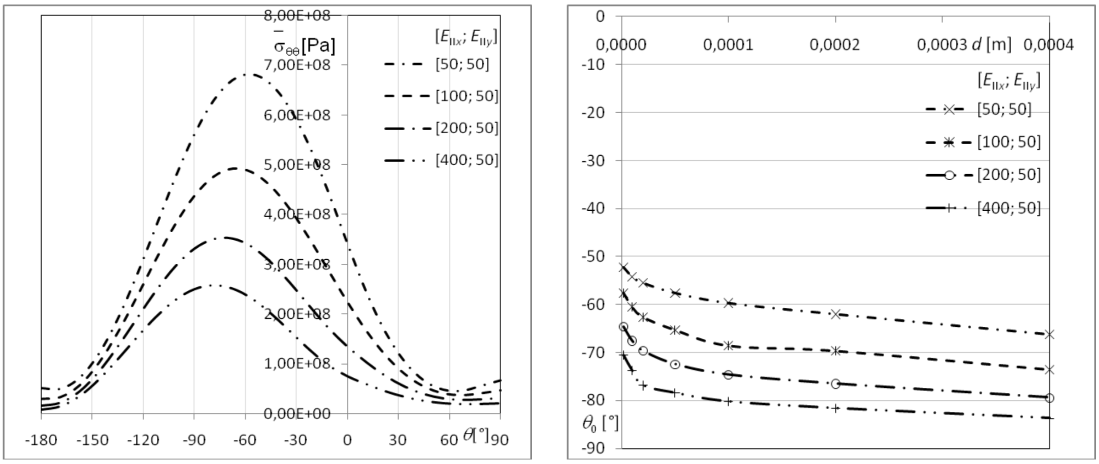
The numerical study was performed in the finite element method (FEM) system ANSYS. All the material combinations were subjected to applied load $\sigma_{appl} = 100$ MPa in the direction as shown in fig. 1. The stress field around the notch tip was analysed and the mean values of the tangential stress $\overline{\sigma_{\theta\theta M}}$ were evaluated in dependence on $\theta$.

The averaging distance $d$ was chosen according to the size of the region with a significant stress gradient in front of the notch tip. In order to outline the effect of sensitivity of the criteria on the distance $d$, all the calculations were also performed for a varying distance $d$.

The results of the FEM analysis are shown in the figure 2. The left graph a) shows the dependences of the mean value of the tangential stress $\overline{\sigma_{\theta\theta}}$ on the polar coordinate $\theta$ for all the material combinations. Data for these curves were calculated for the distance $d = 5\times10^{-5}$ m. The positions of the extremes, namely the maxima, indicate potential crack initiation directions. The directions for all the material combinations are oriented into the substrate and depend on the ratio of Young's moduli in the axes $x$ and $y$.

The right graph, figure 2b) shows the dependence of the presumed crack initiation angles $\theta_0$ for all the material combinations on the averaging distance $d$. The distance $d$ was varied in the interval $\langle 2\times10^{-6}; 4\times10^{-4}\rangle$ m.

Parallel to the FEM analysis of the stress field, the analytical-numerical approach of the crack initiation direction was performed. This approach followed the considerations mentioned above. The stress singularity exponents $1-\delta$ were ascertained from the notch geometry and elastic constants of both materials. The GSIFs were estimated on the basis of $\Psi$-integral, see eq. (10). Finally, the initiation direction was estimated from the relation (13) from the ratio of GSIFs $H_2/H_1$. Table 1 states the eigenvalues $\delta_k$. Although they can generally be complex, in the studied cases $\delta_k$ was real only, $\mathrm{Im}(\delta_k) = 0$. From the ratio of the GSIFs $H_2/H_1$ the theoretical crack initiation direction was solved for $d$ chosen as $d = 2\times10^{-6}$. The size of $d$ (within the theoretical study) was taken according to the size of the region where the first (singular) term of the stress series plays a significant role. In this region the dependence of the theoretical solution of $\theta_0$ on $d$ was weak, i.e. $\theta_0$ was practically constant in the notch tip vicinity.




a) b)

Figure 2. a) Dependence of the mean value of the tangential stress $\overline{\sigma_{\theta\theta}}$ on the polar coordinate $\theta$, b) Presumed crack initiation angle $\theta_0$ vs. the averaging distance $d$.

Table 1. Inputs and results of the theoretical study

| Material I: | $E_{Ix}$ = 100 GPa, $E_{Iy}$ = 50 GPa | | |
|---|---|---|---|
| Material II: | | | |
| [$E_{IIx}$;$E_{IIy}$] (GPa) | [100; 50] | [200; 50] | [400; 50] |
| $\delta_1$ | 0.544 | 0.557 | 0.573 |
| $\delta_2$ | 0.910 | 0.926 | 0.941 |
| $H_2/H_1$ [$m^{\delta_1-\delta_2}$] | 11.76 | 8.34 | 6.22 |
| $\theta_0$ [°] | -52.1 | -60.2 | -66.7 |

As to the directions resulting from FEM, it is evident from the fig. 2 that the crack initiation directions are influenced by $d$ especially for smaller values of the distance $d$. But at the same time, for the small $d$, the results of FEM studies covering all terms of the stress series tend to the values obtained from the theoretical solution considering the first singular term only. For larger regions, used for the mean value evaluation, the influence of the notch tip on the stress field weakens and the values of $\theta_0$ tend to the direction perpendicular to the applied stress. As far as $d$ is concerned, it is necessary to consider the failure mechanism as well. It can be said that it relates to the first increment of the newly initiated crack.

## CONCLUSIONS

The procedure for the determination of the direction of crack initiation from an orthotropic bi-material notch based on the knowledge of angular distribution of the mean value of the tangential stress has been presented. The expression for the distribution of the mean value of the tangential stress is derived as a function of the generalized stress intensity factors $H_1$ and $H_2$. It is concluded that for the estimation of the crack initiation direction $\theta_0$ both existing stress singular terms have to be taken into account, and $\theta_0 = \theta_0\ (H_2/H_1)$. The determination of the initial crack propagation angle is one of the necessary steps for service-life evaluation of constructions containing compound materials. The procedure makes it possible to assess to which material component and in which direction the initiated crack will propagate. Consequently, the behaviour of a crack growing in composite materials is predicted and it can be used to increase the reliability of service-life estimation.

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