On Plate Thickness Effect in Plane Problems of Elasticity

A. G. Kotousov[1], S. Harding[1] and P. Lazzarin[2]

[1] The University of Adelaide, School of Mechanical Engineering, SA, 5005, Australia; E-mail: Andrei.Kotousov@adelaide.edu.au, Steven.Harding@adelaide.edu.au
[2] University of Padova, Department of Management and Engineering, Stradella S.Nicola 3, 36100 Vicenza, Italy; E-mail: plazzarin@gest.unipd.it

ABSTRACT. *Plane theory of elasticity constitutes a foundation of many important results in science and engineering. However, the understanding of the elastic solutions derived under plane stress or plane strain assumption, is far from complete. In particular, it is not clear how adequate the classical two-dimensional solutions of the plane theory of elasticity are when applied to the analysis of actual plate components having a finite thickness. So far there is no generally accepted criterion for identifying what thickness would qualify as plane-stress or plane-strain and, in general, what effect on the stress distribution the plate thickness has. In this work we review some recent numerical studies, experimental and analytical efforts in order to throw light on to how the plate thickness, which is largely ignored by the classical plane solutions of the theory of elasticity, influences the stress and fracture of notched plate components.*

INTRODUCTION

Solutions of plane theory of elasticity, which are sometimes more than a hundred years old, still serve as a basis for many engineering design procedures, standards and failure assessment techniques. Relative simplicity is the main reason behind the popularity of these solutions as the three-dimensional equations of elasticity are not very amenable to analytical treatment. Plane theories of elasticity accommodate two basic assumptions regarding the state of stress in a plate subjected to in-plane loading: plane stress (zero transverse stresses) and plane strain (zero transverse strain components). In the literature, especially in textbooks, the dominant state of stress is often related to the plate thickness using a simple rule. If the plate is thin enough then the stress state is normally considered to be plane stress, and plane strain otherwise. However, so far there is no generally accepted criterion for identifying what thickness would correspond to plane-stress or plane-strain conditions. Consequently there is a significant level of empiricism in deciding whether a particular plate could be treated as thin or thick enough in order to apply the corresponding solutions of the plane theory of elasticity. Furthermore, despite a strong correlation between the plate thickness and the state of stress, many exact solutions do not obey this simple rule. For example, in an infinite plate with a circular hole loaded by internal pressure the state of stress is always plane stress regardless of the plate thickness. In problems with singularities e.g. cracks, angular corners and multi-material joints, it is widely accepted that the area close to the stress singularity is under plane strain conditions (in the current work we will demonstrate that this statement is incorrect). However, the actual three-dimensional state of strain in this area is much more complicated and can deviate significantly from its plane strain counterpart as demonstrated in recent theoretical, numerical and experimental studies (see the references section).

In this paper we try to summarize some previous studies and recent analytical efforts in order to identify and evaluate various three-dimensional phenomena for notched plate components and cracks. Further, the three-dimensional numerical solutions will be compared with the results obtained from plane theory of elasticity and first order plate theory. From this comparison we derive some general conclusions regarding the applicability of these theories in engineering analysis of the notched plate components. One interesting deterministic effect discovered for sharp notches loaded in shear leads to a conclusion that the increase of the plate thickness can significantly reduce the strength of such notches. This effect is much stronger than the common statistical theory of size effect, based on the concept of random strength, predicts.

NON-SINGULAR SOLUTIONS

To exemplify the plate thickness effect and difference between the three-dimensional exact solutions and its two-dimensional counterparts, consider a typical problem of a circular hole in an infinite plate having finite thickness $2h$ and loaded by uniaxial stresses on infinity as shown in Figure 1a. This problem, of course, has been studied by several researchers. At first, due to the difficulties posed by three-dimensional equilibrium equations of elasticity, most analytical studies were based on the two-dimensional linear plane theory of elasticity. In 1899, the two-dimensional solution was obtained by Kirsch [1]. Problems with multiple holes and different shapes of holes were solved by Muskhelishvilli [2] using the complex variable method and conformal mapping techniques. Comprehensive reviews on this subject have been given by Neuber [3] and Savin [4]. However, in the case where the radius of the hole, R, is of the same order of magnitude as the plate thickness 2h, these two-dimensional solutions can no longer provide an acceptable approximation due to the very thick or very thin assumptions. In this case, three-dimensional effects can significantly influence the stress distribution and, consequently, the failure of such components.

There are many experimental evidences revealing the effect of the plate thickness for plate problems. One of such evidences is that for relatively thin plates subjected to an in-plane tensile loading the crack either originates at the corner, where the hole meets the free surface of the plate, or at the centre of the plate as shown in Fig. 1b. On the other hand for relatively thick plates the crack almost always originates in the vicinity of the corner [5], see Fig. 1c as an illustration. Of course, cracks in general originate from small imperfections or discontinuities that may be present in the structure. Thus, one of the explanations for this phenomenon is that such discontinuities are most likely to be present in the vicinity of the free surface. Another possible contributing factor into this phenomenon is that in the corner or mid-plate region the stress level may actually be higher or lower than those predicted by the plane theory of elasticity, i.e. the usual stress concentration factor of 3.

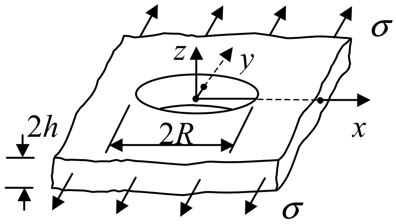

Fig.1a. Plate with a circular hole

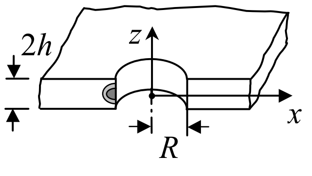

Fig.1b. Thin plate: failure location

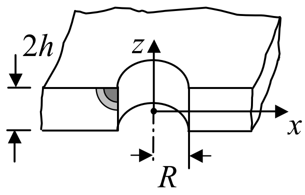

Fig.1b. Thick plate: failure location

In order to obtain a more accurate solution, many three-dimensional studies have been undertaken in the past including Green's solution in an infinite series form [6], approximate solution given by Sternberg and Sadowsky [7] and numerical solutions by Alblas [8]. In 1962, Reiss [9] using a perturbation analysis was able to obtain a solution, which yielded three-dimensional corrections to those of the plane stress. The results, which are valid for small values of thickness to radius ratios, substantiate the findings by Alblas. Kotousov and Wang [10] utilized the first order plate theory and obtained an exact analytical solution of the problem within this theory. Finally, an analytical solution utilizing Kantorovish and Krylov [11] and Fourier transform methods were derived by Folias and Wang [12]. Results obtained by Folias and Wang for the maximum stress concentration factor distribution across the thickness at the critical location $x = \pm R$ and $y = 0$, see Fig.1a, are shown in Fig. 2, where $SCF_{max} = \sigma_{max}/\sigma$.

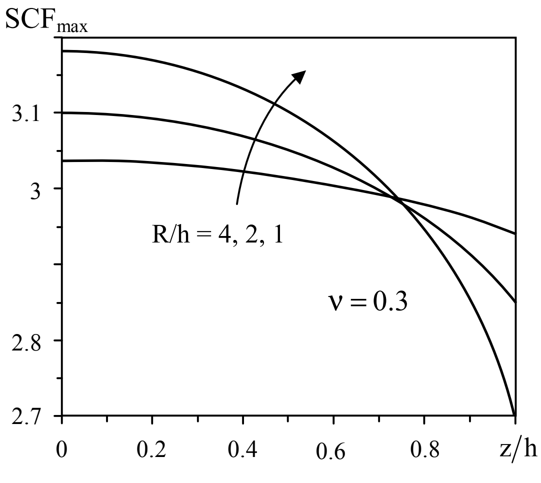

Fig.2a. Thin plate

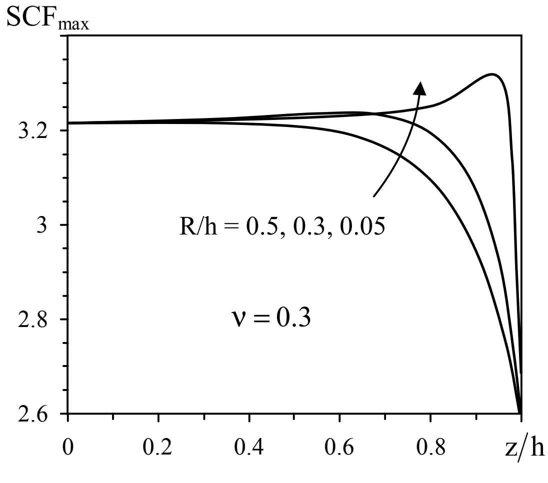

Fig. 2b. Thick plate

Dependence of the stress concentration factor (SCF) across the plate thickness at critical location ($x = \pm R$ and $y = 0$, see Fig.1a)

These theoretical results provide a convincing explanation of the phenomenon associated with the preferable crack formation region as for relatively thin plates R/h >1 the maximum of stress is located on the mid-plane (z = 0) and the failure is expected to initiate from this location. At very low ratios R/h (relatively thick plates) the stress distribution across the thickness has a maximum in the vicinity of the free surface and hence the preferred crack initiation site of thick plates is in this region.

In the last two decades the major progress in the analysis of three-dimensional solutions is associated with the application of the finite element (FE) method. The finite element results presented in various papers confirm the main features of the in-plane stress distribution and provide a relatively easy way to analyze the effect of various parameters. For example, Fig. 3 shows the maximum and minimum stress concentration factors at the same critical location ($x = \pm R$ and $y = 0$) as a function of Poisson's ratio, ν [13]. Despite the relatively small variation of the maximum stress concentration factor from its plane stress or plane strain counterpart (SKF = 3), the variation of the stresses across the plate thickness (or ratio of the maximum to minimum SCF) is significantly affected by Poisson's ratio, ν, and can reach 50 percent for high values of Poisson's ratios. Three-dimensional numerical investigations of elliptical holes subjected to uniaxial tension show that the difference between the solution of the plane theory of elasticity and the corresponding three-dimensional solutions in terms of the maximum SCF can reach 30 percent and the variation of SCF across the thickness up to 100 percent [13].

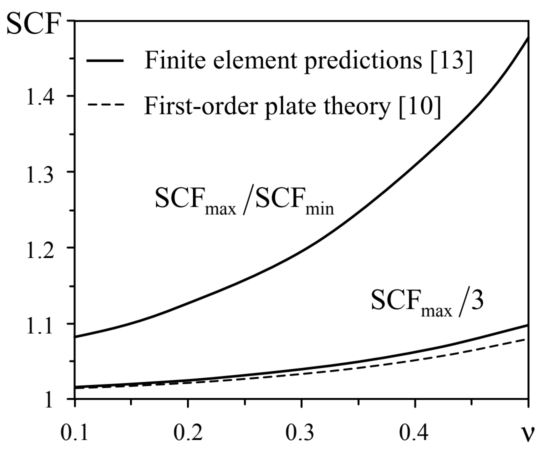

Fig.3. Maximum and minimum stress concentration factor at $x = \pm R$, $y = 0$ as a function of Poisson's ratio, ν.

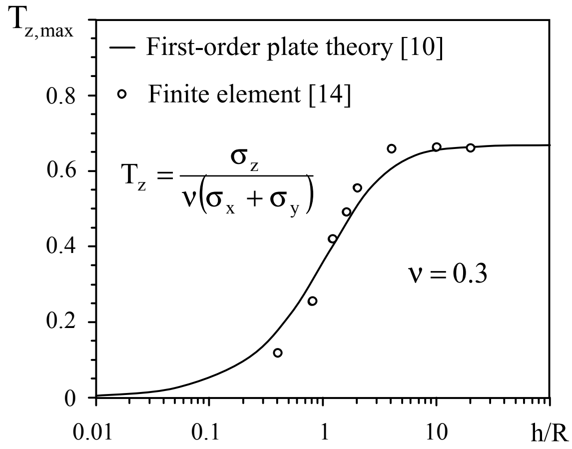

Fig.4. Maximum constraint factor as a function of half-thickness to hole radius ratio.

To characterize the out-of-plane stress components we introduce the constraint factor as a ratio of the out-of-plane stress, σ_z to the sum of the in-plane stress components,

$$T_z = \sigma_z / \nu(\sigma_x + \sigma_y). \quad (1)$$

The constraint factor is widely used to characterize the dominant stress state in plate components [10-13] especially in fracture mechanics and composites where, the main failure mechanisms, delamination, is often associated with the transverse stress components. The constraint factor can vary between zero, this corresponds to the plane stress assumption and one, which corresponds to the plane strain state. Numerical and analytical results from the first order plate theory for Poisson's ratio of 0.3 are shown in Fig. 4 demonstrating that for the practically important situations ($0.1 < h/R < 10$) neither plane stress or plane strain assumption provide reasonable approximation for the transverse stresses.

At this point we briefly outline the main features of the first order plate theory, which is also known as the Kane and Mindlin theory or generalized plane strain theory [10]. It was first introduced by Kane and Mindlin in 1956 in their work [15] on high frequency extensional vibrations of short cylinders where plane stress and plane strain assumptions respectively under and over estimate the natural frequencies observed in experimental studies. The governing equations of this theory include the transverse stress components and retain the simplicity of a two-dimensional model and have provided a very good approximation of the experimental results. The importance of this theory for the analysis of three-dimensional plate problems streams from the very fact that the first-order plane theory is, probably, the only elementary extension of the classical plane theory of elasticity, which allows for analytical three-dimensional solutions to be obtained for non-trivial geometries. Many analytical and semi-analytical solutions derived within this theory are now available in the literature. As it can be seen from Fig.4, for example, the theory provides a very good estimate of the effect of the plate thickness on the stress distribution in a plate component [10].

3D SINGULAR SOLUTIONS

In this section we consider the classical singular problem of the theory of elasticity, i.e. an elastic wedge (prismatic corner) subjected to in-plane loading, which is also known as the classical Williams' problem. In 1952 Williams was the first to show that the in-plane stress components at the apex of an isotropic corner can be singular [16], see Fig. 5a. Later, based on the plane theory of elasticity, many researchers used an eigenfunction expansion approach, Mellin, or Fourier transform method to investigate this singular mode for multi-material junctions [17], as well as inelastic, anisotropic and inhomogeneous materials subjected to various boundary conditions. The solution of this problem plays the fundamental role in asymptotic methods of failure assessments of various structures (bi-materials, welded and adhesive joints), machines (contact and sliding pairs) as well as in failure mechanisms (fracture, fatigue and fretting fatigue) [18-20].

In the late 70s and early 80s Benthem and a number of other researchers who employed a finite difference scheme and the eigenfunction expansion method, demonstrated that at the intersection/vertex of the crack front and a free surface the square root singularity disappears, and at such a point one has to deal with a 3D corner singularity [21], see Fig. 5b. For example, it was established that the in-plane singularity disappears at the vertex of the wedge corner front and free surface, where a previously unknown corner singularity develops instead. This work was repeatedly generalised for various geometries, materials models and multi-material systems. For crack-like geometries it was found experimentally that the corner singularity has a strong influence on the fatigue crack growth behaviour [22].

In 2005 Kotousov, with reference to the first order plate theory, first identified and investigated the out-of-plane singular mode in a corner of arbitrary vertex angle subjected to in-plane loading [23], see Fig. 5c. This singular mode relates to the out-of-plane shear stress components and is associated with Poisson's effect. Fig. 6 illustrates the mechanism of formation of this singular mode for a particular configuration: a sharp corner with zero vertex angle (crack) subjected to anti-symmetric (shear) loading. Intuitively, such loading will create compressive and tensile zones along two opposite free edges; and Poisson's effect will lead to a scissoring motion of the faces generating conditions similar to the tearing mode in fracture mechanics but symmetric with respect to the mid-plane of the plate (see Fig.6). This intuitive analysis can be further verified if one considers the classical asymptotic equations of the plane stress solution for a crack stressed in the shear mode leading to the same general conclusion with regard to the out-of-plane displacements. However, the plane stress theory of elasticity predicts infinite displacements at the crack tip and cannot be utilised in the quantitative analysis of this singular mode, which is why, presumably, this type of singularity has been overlooked in the past. It is clear that the same mechanism or symmetric out-of-plane loading, which is less common in practice, can generate the out-of-plane singularities at other vertex angles.

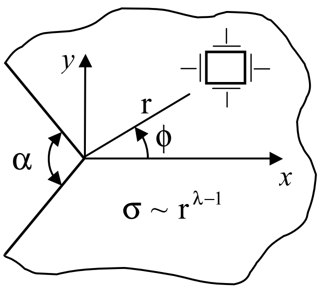

Fig.5a In-plane singularity

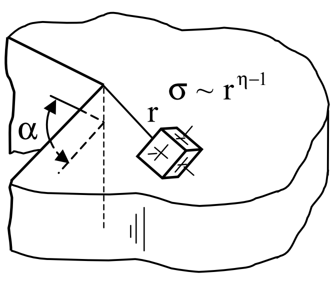

Fig.5b Corner singularity

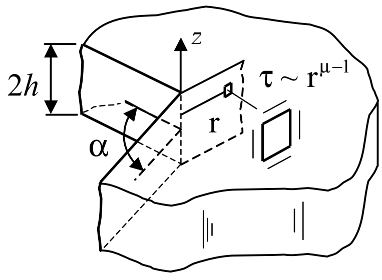

Fig.5c. Out-of-plane singularity

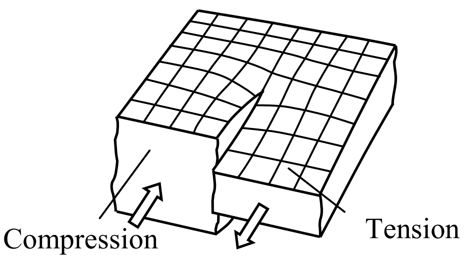

Fig.6 Crack and Poisson's effect

The strongest power of the singularity (minimum value of λ, $\lambda > 0$) for in-plane normal and shear modes (mode I and II) as well as for out-of-plane mode (O - mode) is shown in Fig. 7a. It can be realized that the dependence of the strength of the singularity of the out-of-plane mode is exactly the same as for fracture mode III. However, there are essential differences between these two singular modes, which will be discussed later in this paper.

The out-of-plane singular mode is coupled with the shear mode (mode II) and has been recently investigated by Harding and Kotousov [24] and by Harding *et al.* [25].

The dependence of the notch stress intensity factors defined as $K = \lim_{r \to 0} \tau(2\pi r)^{1-\lambda}$ are shown in Fig. 7b for a 90° degree notch and various Poisson's ratios.

The values for the notch stress intensity factor are significantly affected by Poisson's ratio and can reach almost the applied magnitude of the shear mode for crack problems (notch angle, α= 0°). The most interesting finding of the numerical results is the establishing of the fact of the existence of the out-of-plane singularity for the vertex angles above 102.4° despite the disappearance of the coupled in-plane shear singular mode [24]. This is contrary to many studies stated that above this critical angle there is no line singularity which would contribute to the failure of the notch loaded in mode II.

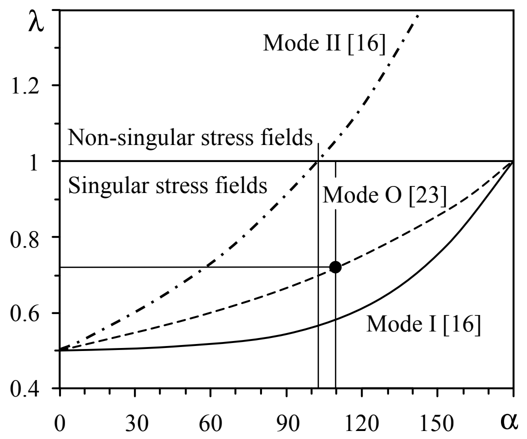

Fig.7a. Strength of singularity for in-plane mode I and II and O-mode

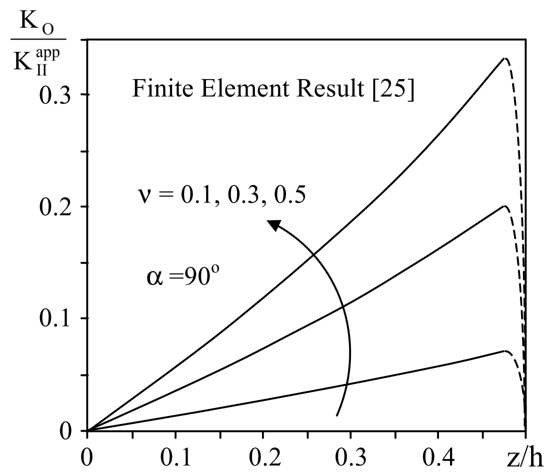

Fig. 7b. Normalized O-mode NSIF, in $(m)^{0.242}$, across the thickness for α=90° and various Poisson's ratios

SOME 3D EFFECTS IN CRACK PROBLEMS

The crack geometry, probably, is one of the most important for practical applications and will be considered next in some detail. Below we discuss two cases: a crack stressed in mode I and mode II, in other words we consider a through-the-thickness crack in an infinite plate of thickness 2h subjected to K_I^{app} or K_{II}^{app} stress intensity factors. Because of the space limitations we will discuss only the most interesting features of the three-dimensional solutions for both these problems.

Crack stressed in mode I

Fig. 8a shows the transition from plane strain (T_z = 1) to plane stress conditions (T_z = 0) for a semi-infinite crack. Despite that the stress field in the vicinity of a crack tip follows the plane strain solution the basic assumption of this theory (ε_z = 0) is not valid. Fig. 8b summarises results of various numerical, analytical and experimental studies demonstrating that the free surface has finite displacements at the crack tip. Thus, the three-dimensional (actual) displacement and strain fields in the vicinity of the crack cannot be approximated by either plane stress or plane strain assumptions.

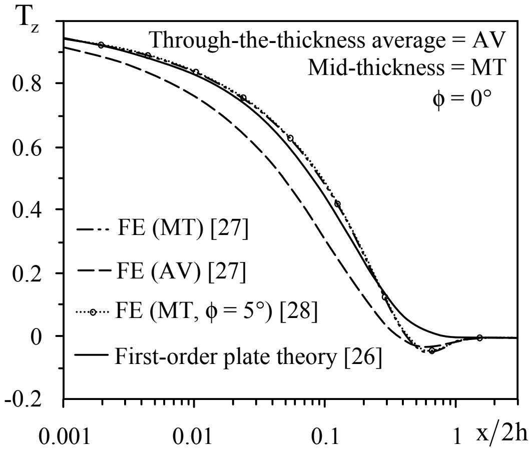

Fig. 8a. Constraint factor in the vicinity of the crack tip (x is the distance from the crack tip)

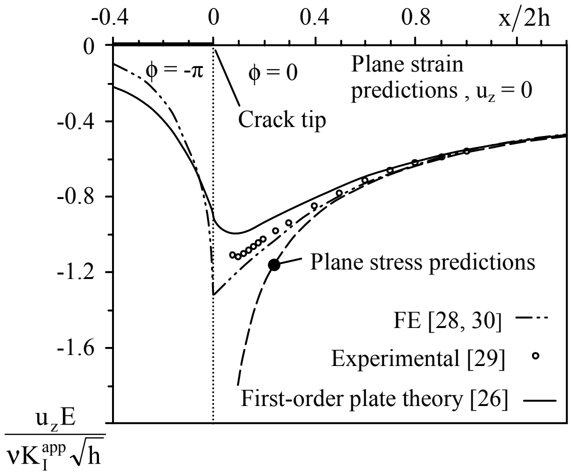

Fig.8b. Out-of-plane displacement, u_z, on the free surface, $z = \pm h$ near the crack tip

Crack stressed in mode II

Among various results available for cracks stressed in mode II we show the dependence of the O-mode as a function of the crack length (Fig. 9a) for through the thickness crack of length 2a. Fig. 9b shows the variation of the stress intensity factor of mode II in the vicinity of the plate surface [24] for semi-infinite crack.

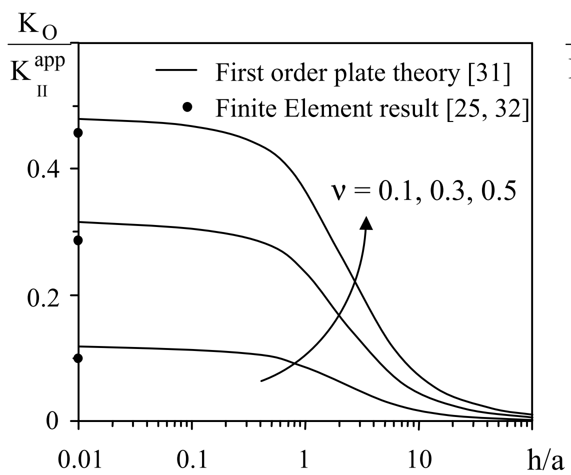

Fig. 9a. Intensity of O-mode as a function of the ratio of the crack length to the plate thickness

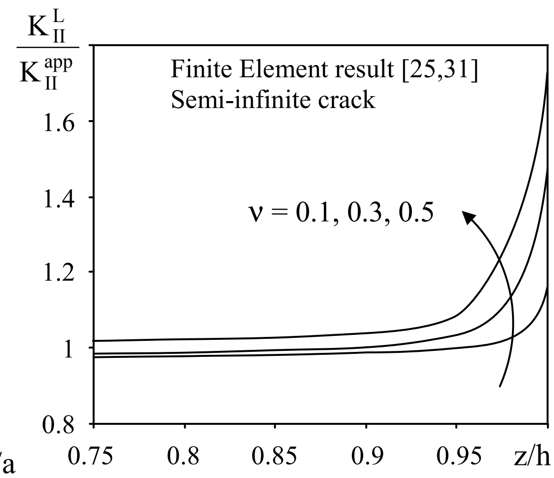

Fig.9b. Rise of the intensity of shear mode II in the vicinity of the plate surface

Normally, experimental results correspond to h/a < 1. Fig 9a and 9b indicate a strong presence of three-dimensional effects and influence of the plate thickness: an increase of the local stress intensity factor, K_{II}^{L} and a significant magnitude of K_O in the vicinity of the plate surface. These effects are arguably responsible for the preferable location of failure initiation of a crack loaded in mode II, which is known to be close to the plate free surface, and the decrease of the fracture toughness in mode II in comparison with mode I. This difference according to various experimental studies can reach up to 50% depending on the geometry of the specimen and material.

FRACTURE AT SHARP NOTCHES

The inability of conventional approaches based on nominal stresses to describe failure in high-gradient stress and strain fields such as those which occur in the vicinity of notches, corners and cracks has been known, probably, since A.A. Griffith published his pioneering work on the fracture of solids in 1920. The systematic application of asymptotic methods for failure assessment of cracked components was then initiated by work of George Irwin in 1950s. The use of the crack-tip stress intensity factor, which is a characteristic of the elastic asymptotic stress field, is ubiquitous, principally because of the powerful way it correlates the behaviour of the process zone associated with failure observed in experiments and engineering components.

Applications of the classical linear-elastic solutions to V-shaped notches are based on the following loose argument, which is similar to the one used in the classical linear-elastic fracture mechanics. Consider a notched plate made from reasonably brittle materials such that the elastic solution admits the presence of singular stress state. Three zones can be identified: a process zone, where the materials are subjected to nonlinear and non-elastic deformations; K-dominance zone, where a linear elastic asymptotic stress field of the form $K \cdot r^{\lambda-1}$ (r is the distance from the tip of the notch and λ depends on the notch opening angle α) might be expected to be accurate; and the zone of general stress state, where the stress field depends on the geometry of the solid and boundary conditions. Material failure often starts in the process zone. When the process zone is completely contained within the K-dominance zone, the state in the process zone is likely to be controlled mainly by the fields in the region of K-dominance. If this is true, the conditions for the failure initiation will be only a function of the characteristics of the singular stress state (K and λ) as obtained from the corresponding elastic solution and nothing else [18-20].

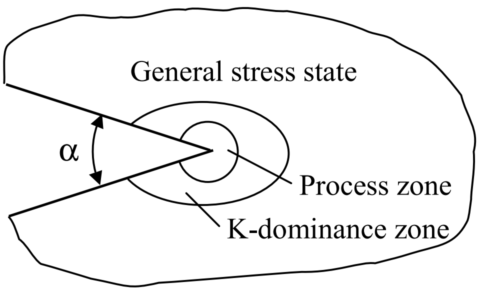

Fig. 10 Notched plate

In the past two decades there were a number of major developments in the notch analysis field. It was shown analytically that the asymptotic solutions obtained for sharp V-notches were closely linked to the non-singular field of blunt notches with a notch tip radius small but different from zero [33]. Also, several asymptotic methods for failure and durability assessment were developed based on various characteristics of the asymptotic stress field such as average stress, strain or energy density, critical distance and energy release rate. These approaches allow the comparison and evaluation of the strength of notched plates made of the same material but having different notch opening angles, and, also, stressed in arbitrary mixed fracture modes.

In a large number of publications it was demonstrated that when the notch tip radius is small enough the notch stress intensity factors adequately control the critical loads of brittle materials under static loads as well as the fatigue crack initiation at sharp V-notches (see, among others [34-39]). Such factors can be applied also for total fatigue life assessments when a large amount of crack propagation life is consumed with a short crack in a zone governed by the V-notch singular solution [18,37].

PLATE THICKNESS AND SIZE EFFECT FOR V-SHAPED NOTCHES

Consider an infinite plate with a sharp notch (V-notch) of thickness 2h and opening angle, α, loaded in shear mode characterised by the applied notch stress intensity factor on infinity, K_{II}^{app}. The applied stress intensity factor K_{II}^{app} will generate a local notch stress intensity factor describing the local stress state along the plate thickness and out-of-plane stress intensity factor as described in the previous sections of this paper, see, for example, Figs 7b and 9b. Each singular mode has its own strength of singularity as shown in Fig. 7a. Based on the dimensionless analysis, the local notch stress intensity factor K_{II}^{L} has to be a function of the z/h ratio only, where z is the distance from the mid-plane in the lateral (out-of-plane) direction i.e.

$$K_{II}^{L} = K_{II}^{app} \cdot f(z/h) \qquad (2)$$

where f is a function.The notch stress intensity factor characterizing the out-of-plane mode, K_O, has to be a function of the applied load (which is proportional to K_{II}^{app}), the geometry (2h – the only one geometry parameter for this problem) and position (z). Then, it can be written in the following form

$$K_O = K_{II}^{app} \cdot h^{\lambda_{II}-\lambda_O} g(z/h) \quad , \qquad (3)$$

where g is a function and exponents λ_{II} and λ_O depend on the notch angle α as shown in Fig. 7a. The maximum value of K_O might be also given in the form

$$K_O = k_O \cdot h^{1-\lambda_O} \cdot \tau_{nom} \quad , \qquad (4)$$

where k_O is a dimensionless shape function and τ_{nom} the nominal value of the shear stress. Under mode II loading, the size effect does depend both on K_{II} and K_O resulting in a mixed-mode fracture where not only the degree of singularity of the stress distributions is important but also the intensity of the relevant NSIFs. The incidence of the two modes varies as a function of the opening angle and the Poisson’s ratio.
When the V-notch angle is greater than 102.4 degrees, only the O-mode is singular and then it is expected to dominate the stress state in the vicinity of the apex. Kept constant the notch-opening angle, K_O increases as the plate thickness increases, with a

progressive reduction of the strength of the notched component expressed in terms of nominal shear stresses, see Eq. (3). This is in agreement with the experimental results obtained from sharply notched components under Mode I, Mode II and Mode III loads. For example, dealing with fatigue strength of welded joints of thickness t, the reference curve for fatigue failures from the welded toe (with an angle of 135°) is [18,37]:

$$\Delta K_{1A} = k_1 \cdot t^{1-\lambda_1} \cdot \Delta\sigma_n = 210\ \text{MPa} \cdot \text{mm}^{0.326}. \qquad (5)$$

Thus, these simple considerations reveal a new thickness effect for V-notched components. The out-of-plane mode alone results in a size effect that is proportional to $(1-\lambda_O)$. For a notch angle of 110^o the scale effect has a power of 0.28 (see Fig. 7a), which is lower than the classic value of 0.5 provided from LEFM but significantly higher than the statistical theory of size effect, based on the concept of random strength, predicts [40]. For example, for concrete, the typical value of the exponent in a power law describing the size effect is approximately 1/6 [41].

CONCLUSION

In this short paper we made an attempt to summarize some important features of the elastic three-dimensional solutions of plate problems. We demonstrated that there significant differences exist between exact three-dimensional solutions and the corresponding solutions of the plane theory of elasticity. The distribution of the in-plane stresses can vary significantly through the plate thickness especially for problems with singularities. The plane theory of elasticity is unable to provide estimates of the transverse stress components for many practically important problems. It is not surprising that the application of this theory often leads to peculiar results due, in part, to the fact that it is an approximate three-dimensional theory even when the plane stress equations are solved exactly.

The first order plate theory normally provides a very good estimate of the both in-plane and out-of-plane stress components averaged through the plate thickness. The first order plate theory also led to the discovery of a new singular mode for notched components – the out-of-plane singular mode – which is coupled with the shear mode and related to the transverse shear stress components. This mode is symmetric with respect to the mid-plane of the plate, significantly affected by Poisson's ratio and has zero magnitude at the plate surface and the mid-plane. Recent numerical studies confirm the theoretical findings and indicate that this mode can contribute significantly to fracture of structural components with cracks, V-shaped notches as well as bi-material and adhesive joints, contact and sliding mechanisms.

REFERENCES

1. Kirsch, G. (1898) *Z. Vereines Deutscher Ing*. 42, 797-807.
2. Muskhelishvili, N. I. (1953) *Some basic problems of mathematical theory of elasticity*. Groningen: Nordhoff.

3. Neuber, H. (1985) *Kerbspannungslehre*. Berlin, Heidelberg, New York: Springer
4. Savin, G.N. (1961) *Stress concentration around holes*. New York: Pergamon Press.
5. Broek, D. (1974) *Elementary engineering fracture mechanics*. In: Chapt. 14: Plates with holes, 4th edn., pp. 361-370. Groningen: Nordhoff
6. Green, A. E. (1948) *Trans. R. Soc. London*, Ser. A. 240, 561
7. Sternberg, E.; Sadowsky, M. A. (1949) *J Appl Mech* 16, 27-38
8. Alblas, J. B. (1957) *Theorie van de driedimensionale spanningstoestand in een doorboovde plaat*. Dissertation. Technische Hogeschool Delft, H.J. Pairs, Amsterdam.
9. Reiss, E. L. (1963) *J Soc Ind Appl Math* 11, 840.
10. Kotousov, A., Wang, C.H. (2002) *Int J of Solids and Struct* 39, 4311-4326.
11. Kantorovich, L.V. and Krylov, V.L. (1964) *Approximate methods of higher analysis*, Groninger: Nordhoff.
12. Folias, E. S. and Wang, J.J. (1990) *Comp Mech* 6, 379-391.
13. Yu, P., Guo, W., She, C. and Zhao, J. (2008) *Int J Fatigue* 30, 165–171.
14. Li, Z., Guo, W. and Kuang, Z. (2000) *Int J Solids Struct* 37, 7617-7631.
15. Kane, T.R., Mindlin, R.D. (1956) *J Appl Mech* 23, 277–283.
16. Williams, M.L., (1952) *J Appl Mech* 74, 526 – 528.
17. Bogy, D.B. (1971) *J Appl Mech* 38, 377 – 386.
18. Lazzarin P., Tovo R. (1998) *Fatigue Fract Engng Mater Struct* 21, 1089-1104.
19. Nowell, D., Dini, D., Hills, D. A. (2006) *Eng Fracture Mech* 73, 207-222.
20. Kotousov, A. (2007) *Int J Adhesion and Adhesives* 27, 647-652.
21. Benthem J. P. (1963): *Quarterly J Mech and Appl Math,* XVI, 413-429.
22. Heyder, M., Kolk, K., Kuhn, G. (2005) *Eng Fracture Mech* 72, 2095-2105.
23. Kotousov, A. (2005) *Int J Fracture* 132, L29–L36.
24. Harding, S. and Kotousov, A. (2009) *Int J Solids Struct* (submitted)
25. Harding, S., Kotousov, A., Lazzarin, P. and Berto, F. *Int J Fract*, (submitted)
26. Codrington, J, Kotousov, A. and Ho, S.Y. (2008) *J Mechs Mat Struct* 3, 261-270
27. She, C. and Guo, W. (2007) *Int. J. Solids Struct* 44, 3021–3034.
28. Nakamura, T. and Parks, D. M. (1988) *J. Appl. Mech* 55, 805–813.
29. Humbert, L., Valle, V. and Cottron, M. (2000) *Int J Solids Struct* 37, 5493–5504
30. Pfaff, R. D., Washabaugh, P. D. and Knauss, W. G. (1994) *Int J Solids Struct* 32, 939–955.
31. Kotousov, A. (2007) *Int J Solids Struct* 44, 8259 – 8273.
32. Nakamura, T., Parks, D. M. (1989) *Int J Solids Struct* 25, 1411-1426.
33. Filippi S., Lazzarin P., Tovo R. (2002) *Int J Solids Struct* 39, 4543-4565.
34. Reedy, Jr E.D. and Guess, T.R. (1993) *Int. J. Solids Struct* 30, 2929-2936
35. Akisanya, A.R. (1997) *J Strain Analysis* 32, 301-311.
36. Giannakopoulos, A.E., Lindley, T.C. and Suresh, S. (1998) *Acta Mater* 46, 2855–2868.
37. Lazzarin, P. and Livieri, P. (2001) *Int J Fatigue* 23, 225-232
38. Lazzarin, P. and Berto, F. (2005) *Int J Fract* 135, 161-185.
39. Gómez, F.J., Elices, M., Berto, F. and Lazzarin P. (2007) *Int J Fract* 145, 29-45.
40. Weibull, W. (1949) *Proc Royal IT*, 27
41. Bazant, Z. and Chen, E. P. (1997) *Appl Mech Rev*, 50, pp. 593-627