Crack Path Analysis Based on a Variational Principle

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ABSTRACT. *The present paper deals with the global criterion based on a variational principle of the crack theory to estimate the crack path in the case of brittle fracture. The variational principle of the crack path on the surface of a solid as the minimum of the certain functional was formulated. The crack path is interpreted as a generalized geodesic line on the solid surface. Crack path estimations based on a variational principle have been done for a circular cone under torsion, a plane weakened by a circular hole, a half-plane under a concentrated force.*

INTRODUCTION

There are two basic approaches in crack path analysis, namely, differential (piecewise) methods based on local fracture criteria and integral (global) method.

Differential methods are based on the determination of an angle between the initial and subsequent directions of the crack propagation. In this case, each small load increment is suggested to be accompanied by a correspondent increment of the crack length. An angle of crack propagation is evaluated by means of the known local fracture criterion that defines the line along which the crack length increases. Various fracture criteria have been currently employed for a crack path computation, namely, the maximum circumferential stress criterion, the maximum energy release rate criterion, criterion of the minimum energy consumption for fracture, criterion of the minimum strain energy density, the J-integral theory. The details of these basic approaches in crack path analysis can be found in the literature (e.g., [1, 2]).

The integral methods presume determination of equation of the crack propagation line by a single stress analysis for a solid either containing a crack or no crack. The present paper deals with the global criterion based on a variational principle of the crack theory to estimate the crack path at once [3, 4].

VARIATIONAL PRINCIPLE

The crack propagation is assumed to be represented by mechanical motion of the crack tip as a material particle with the effective mass that is moving at the crack tip [5]. The optimal crack path of such crack motion leads to the variational problem which is given by the following equation

$$\delta L = 0. \tag{1}$$

For a no planar surface the functional L is written as follows

$$L = \int_A^B \Phi(u,v)\sqrt{E + 2Fv' + G(v')^2}\, du\,. \tag{2}$$

Here, $\Phi(u,v)$ is the weight function which depends on the stresses (or strains) in the uncracked body. The crack path on the solid surface has been described by the radius-vector which is given by equation $r = r(u,v)$, where u and v are curvilinear coordinates of a point which belongs to the crack path. Coefficients *E, F, G* for first quadratic form of the solid surface are given in the form

$$E = \left(\frac{\partial r}{\partial u}\right)^2, \quad F = \left(\frac{\partial r}{\partial u}\right)\left(\frac{\partial r}{\partial v}\right), \quad G = \left(\frac{\partial r}{\partial v}\right)^2. \tag{3}$$

For the flat plate the functional L can be represented by equation

$$L = \int_A^B \Phi(x,y)\,ds\,, \quad ds = \sqrt{1 + y'^2}\, dx\,. \tag{4}$$

An extremal which should be determined from Eq. 1 is equation of the crack path. Boundary conditions from one end to the other end of the crack path can be various and depends on a formulation of the problem. Equation 1 allows interpreting the crack path as a geodesic line on the solid surface. The geodesic line is the shortest line between points (*A* and *B*) on the surface and satisfies the condition $\delta\int_A^B ds = 0$. Moreover, the crack propagates in such a way that the energy lost in creating a new crack surface has the minimal energy value. From this assumption it also follows that the crack path is a geodesic line on the surface under consideration [6].

However, the crack path can be not determined only by the geometry of a solid. Therefore, it is assumed that the length element is skewed by the stress state. A metric of the generalized geodesic line depends on the stress state in the untracked solid, namely, $ds^* = \Phi ds$, i.e. $\delta\int_A^B ds^* = 0$ [6, 7]. There are many remarkable properties of geodesic lines. A choice of the function Φ as well as the Lagrange function in integral variational principles of physics advances in solution of the problem under consideration. For brittle fracture the function Φ could be assumed to be proportional to the maximum principal stress or strain in the uncracked solid [6, 7].

The crack path can be calculated from the Euler-Lagrange equation for the corresponding functional

$$\frac{\partial H}{\partial v} - \frac{d}{du}\frac{\partial H}{\partial v'} = 0, \tag{5}$$

where $H = \Phi(u, v)\sqrt{E + 2Fv' + G(v')^2}$.

It should be noted that the end of a crack path can be another crack or a free surface of a solid. From the minimum of the functional (Eqs 1 and 5) and the transversally condition

$$\frac{\partial T}{\partial v}\left(H - v'\frac{\partial H}{\partial v'}\right) - \frac{\partial T}{\partial u}\frac{\partial H}{\partial v'} = 0, \tag{6}$$

it follows that the geodesic line should be normal to the free surface of a solid

$$\frac{\partial T / \partial u}{\partial T / \partial v} = \frac{E + Fv'}{F + Gv'}. \tag{7}$$

Here, $T(u, v) = 0$ is the equation of the line on which the crack path is ended. Thus, the crack must propagate at right angles to other cracks or free surfaces of a solid [6].

To demonstrate the variational principle in the crack path problem, the following crack path estimations have been considered.

RESULTS OF CRACK PATH ANALYSIS

Crack path estimations based on a variational principle have been done for a circular cone under torsion, a plane weakened by a circular hole, a half-plane under a concentrated force.

A circular cone

A circular cone is loaded by torsion M at the vertex. In this case, the maximum principal stress is written as follows

$$\sigma_1 = \frac{2M}{\pi u^3 \sin^3 \theta}, \tag{8}$$

where u is the radial distance from the vertex to the point under consideration, 2θ is the angle at the cone vertex. Here, the weight function Φ is assumed to be the maximum principal stress σ_1. The functional L from Eq. 2 leads to the following equation

$$L = \frac{2M}{\pi \sin^3 \theta} \int_{u_1}^{u_2} \frac{\sqrt{1 + u^2 (v')^2 \sin^2 \theta}}{u^3} du , \tag{9}$$

where v is the angle between two generatrixs on the cone surface. The Euler-Lagrange equation takes the following form

$$\frac{v' \sin^2 \theta}{u \left(1 + u^2 (v')^2 \sin^2 \theta\right)^{1/2}} = C . \tag{10}$$

The solution of this equation leads to the crack path

$$u = u_0 \sqrt{\sin(2v \sin \theta)} , \tag{11}$$

where u_0 is a constant.

It can be seen that Eq. 11 allows describing the crack path in the double circular cone caused by a torque (Fig. 1).

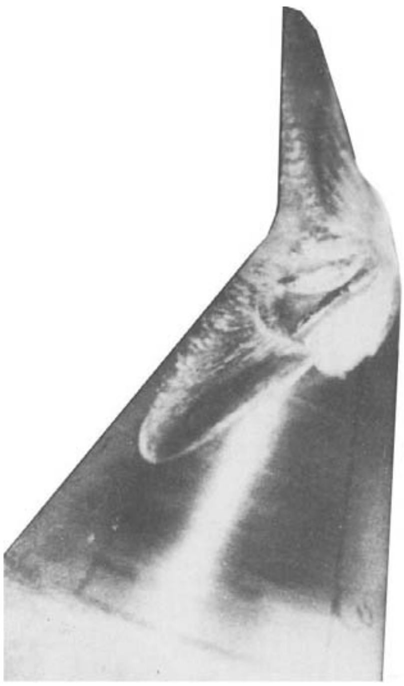

Figure 1. Fractured surface of the double Plexiglas circular cone under torsion.

A circular hole in a plate

An infinite plate contains a circular hole which is subjected to internal pressure p. The maximum principal stress is given by the following equation

$$\Phi(x, y) = \sigma_1 = \frac{pa^2}{x^2 + y^2}, \tag{12}$$

where a is the radius of the hole. The hole contour is described by equation $x^2 + y^2 = a^2$. The minimum of the functional from Eqs 1 and 4 in the form

$$\delta \int_{x_1}^{x_2} \frac{pa^2}{x^2 + y^2} \sqrt{1 + (y')^2}\, dx = 0 \tag{13}$$

leads to the Euler-Lagrange equation

$$\frac{y''(x^2 + y^2)}{1 + (y')^2} - 2xy' + 2y = 0. \tag{14}$$

The particular solution of this equation is $y = kx$, where k is a constant. It means that cracking should occur along the radial lines. This phenomenon is observed in the test that is illustrated in Fig. 2.

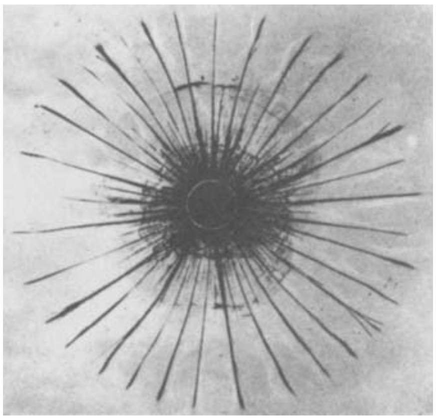

Figure 2. The Plexiglas plate with radial cracks as a result of bullet penetration.

A half-plane under a concentrated force

Let us consider a half-plane or a dihedral angle whose boundary or vertex are subjected to normal or symmetrical concentrated force P. It is suggested that the function Φ is proportional to the maximum strain ε_1, i.e.

$$\Phi(x, y) = \varepsilon_1 = \nu \frac{2P}{\pi E} \frac{x}{x^2 + y^2} . \tag{15}$$

Here, ν is Poisson's ratio, E is the elastic modulus. The solution of the variational problem in the form

$$\delta \int_{x_1}^{x_2} \frac{2\nu P}{\pi E} \frac{x}{x^2 + y^2} \sqrt{1 + (y')^2}\, dx = 0 \tag{16}$$

leads to the following equation

$$\frac{y''\left(x^2 + y^2\right)x}{1 + (y')^2} + y'\left(y^2 - x^2\right) + 2xy = 0 . \tag{17}$$

One solution of Eq. 17 gives $y = 0$, i.e. the crack propagates along the straight line corresponding to the direction of the applied force. It can be seen that the crack grows along the straight line in the glass wedge from the point of compressive force application at upper end of the wedge (Fig. 3). Moreover, the crack approachs a boundary of the wedge at right angle to free surface according to Eq. 7.

Another solution, namely, $x^2 + y^2 = R^2$ is true for tensile force applied to the upper end of the wedge. In this case, the crack path has the form of a circular arc that is illustrated in Fig. 4. It should be noted that a small notch was made before the test to initiate the crack.

Figure 3. The crack path in the glass wedge under concentrated compressive force applied at the upper end of the wedge.

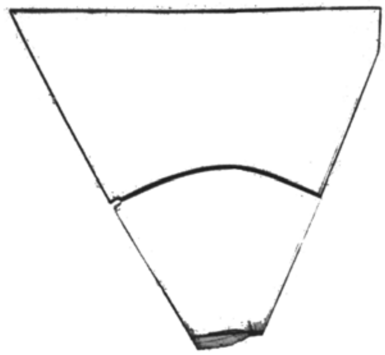

Figure 4. The crack path in the Plexiglas wedge under tension loading.

CONCLUSIONS

The variational principle of the crack path on the surface as the minimum of the certain functional was formulated. The variational principle allows interpreting the crack path as a generalized geodesic line on the solid surface. The functional includes the weight function which depends on the stress state in the uncracked solid and sought functions which describe the crack path. For example, for brittle cracking the weight function could be assumed to be proportional to the maximum principal stress or strain in the uncracked solid.

From the minimum of the functional and the transversally condition, it follows that the geodesic line should be normal to other cracks or the free surface of a solid.

To demonstrate the variational principle in the crack path problem, the crack path estimations have been done for a circular cone under torsion, a plane weakened by a circular hole, a half-plane under a concentrated force.

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