A General Weight Function for Inclined Kinked Cracks in a Semi-Plane

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ABSTRACT. *A general method for evaluating the Stress Intensity Factors of an inclined edge kinked crack in a semiplane is presented. An analytical Weight Function with a matrix structure was derived by extending a method developed for an inclined edge crack. The effects of the principal geometrical parameters governing the problem were studied through a parametric Finite Element analysis, carried out for different reference loading conditions. The Weight Function can be used to produce efficient and accurate evaluations of the Stress Intensity Factors for cracks with initial inclination angle in the range -60° to +60° and kinked angle in the range from -90° to +90°. The agreement between the results with those obtained by accurate Finite Element solutions suggests that the proposed Weight Function can be used as a general tool for evaluating the Fracture Mechanics parameters of an inclined kinked crack.*

INTRODUCTION

Fatigue cracks, in general, can start growing along a particular direction, even at an inclination angle relative to the remote load, for instance due to mixed-mode conditions [1] or anistropic crack growth resistance properties of the material characteristics [2]. The term ‘slanted cracks’ or ‘inclined cracks’ is frequently used to describe these cracks. Furthermore, it is common practice that fatigue cracks can deviate from their original trajectory, for istance due to the variation of the loading direction during the service life [2] or during fast brittle fracture or subcritical crack growth under mixed-mode loading [3]. Such cracks have been termed as ‘deflected cracks’ or ‘kinked cracks’.

The fatigue life assessment of structural components requires not only precise crack growth estimation but also the prediction of the fatigue crack trajectory. In fact, the crack path can determine whether fatigue failure is benign or catastrophic. As reviewed by Socie and Marquis [4], the study of crack paths has received increasing attention in recent years, leading to the formulation of various models predicting critical plane for crack propagation. All the proposed approaches may be expressed as a function of the stress intensity factor (SIF) components ahead the crack tip [5].

The Weight Function (WF) method turns out to be particularly efficient for solving this kind of problems, where a lot of SIFs calculations have to be performed under general remote loading of the cracked body. In fact, being based on the general properties of the cracked body geometry, the WF method enables an efficient and direct SIF calculation for complex loading conditions, allowing to account for loading variations during fatigue cycles, and giving an efficient prediction of the crack evolution. Moreover, the WF method can be adopted for evaluating also the crack opening displacement (COD) that is a fundamental quantity for predicting possible crack closure during the loading cycle.

The problem of the oblique edge crack in an unnotched and notched semiplane was already faced by the Authors [6, 7] and the related WF was obtained for evaluating K_I and K_{II} SIF components. It was demonstrated that, when the crack is not perpendicular to the surface, a matrix structure is necessary to define the WF in order to account for the lack of symmetry. Starting from this WF, the Green Function (GF) was also determined [8], which allowed the COD evaluation by direct integration of the tractions applied to the crack faces. It was demonstrated that, by a simple analysis in the one-dimensional domain of the crack length including nominal and contact stresses, the WF and the GF give an accurate and efficient fracture mechanics solution under a completely general loading producing also partial closure. Numerical solutions of the SIFs of a kinked crack are available in the literature [9], a WF function for kinked cracks starting from an edge crack orthogonal to the surface has been derived by [10], however a WF for an inclined kinked edge crack has not been proposed yet.

In the present paper, an inclined kinked edge cracks in an elastic semi-plane is analyzed. The aim is to develop an efficient and accurate analytical WF by which the SIF components can be calculated by a simple integration for crack lengths and inclinations before and after kinking within broad ranges, covering the typical conditions of practical interest. To this purpose, an extensive parametric finite element analysis has been performed, by varying the geometrical parameters for different reference loading conditions, in order to build up a database of SIFs values. These results have been used to obtain the properties of a parametric WF. The WF was validated by comparing the SIFs with the results of FE analyses carried out for loading conditions different from those used in the database definition as well as with solutions found in the literature.

PROBLEM DEFINITION AND WEIGHT FUNCTION FORMULATION

The geometry of the problem is schematically illustrated in Fig. 1, where a kinked inclined edge crack in a semiplane is shown. The geometrical parameters governing the problem are: the initial crack length a_0 and inclination angle α with respect to the semiplane bisector, the kinked crack length a and inclination β with respect to the initial crack direction. For an oblique crack (not symmetrical problem), the WF has a matrix structure [6-8]:

$$WF = \begin{pmatrix} h^{I\sigma} & h^{I\tau} \\ h^{II\sigma} & h^{II\tau} \end{pmatrix} \tag{1}$$

In the present case, the crack is composed by two straight lines, thereofore the SIFs components can be obtained by splitting the integration of the WF and stress components into two parts:

$$\begin{aligned} K_I = &\int_{\Gamma_1} \left[h^{I\sigma}\left(a,\alpha,a_0,\beta,\vec{\gamma}\right)\cdot\sigma_M\left(\alpha,\vec{\gamma}\right) + h^{I\tau}\left(a,\alpha,a_0,\beta,\vec{\gamma}\right)\cdot\tau_M\left(\alpha,\vec{\gamma}\right)\right] d\vec{\gamma} + \\ &+\int_{\Gamma_2} \left[h^{I\sigma}\left(a,\alpha,a_0,\beta,\vec{\gamma}\right)\cdot\sigma_K\left(\alpha,\beta,a,\vec{\gamma}\right) + h^{I\tau}\left(a,\alpha,a_0,\beta,\vec{\gamma}\right)\cdot\tau_K\left(\alpha,\beta,a,\vec{\gamma}\right)\right] d\vec{\gamma} \end{aligned} \tag{2a}$$

$$\begin{aligned} K_I = &\int_{\Gamma_1} \left[h^{II\sigma}\left(a,\alpha,a_0,\beta,\vec{\gamma}\right)\cdot\sigma_M\left(\alpha,\vec{\gamma}\right) + h^{II\tau}\left(a,\alpha,a_0,\beta,\vec{\gamma}\right)\cdot\tau_M\left(\alpha,\vec{\gamma}\right)\right] d\vec{\gamma} + \\ &+\int_{\Gamma_2} \left[h^{II\sigma}\left(a,\alpha,a_0,\beta,\vec{\gamma}\right)\cdot\sigma_K\left(\alpha,\beta,a,\vec{\gamma}\right) + h^{II\tau}\left(a,\alpha,a_0,\beta,\vec{\gamma}\right)\cdot\tau_K\left(\alpha,\beta,a,\vec{\gamma}\right)\right] d\vec{\gamma} \end{aligned} \tag{2b}$$

where $d\vec{\gamma}$ is the curvilinear coordinate along the crack path, $h^{M\mu}$ and $h^{M\mu}$ (with $M = I$ or II and $\mu = \sigma$ or τ) represent the WF components, while σ_M, τ_M and σ_K, τ_K are the normal and shear nominal stress distributions acting on the main and on the kinked crack, respectively. The nominal stress is usually defined as the stress acting along the segment of the crack location in the equivalent uncracked body, subjected to the same constraint and loading conditions of the cracked body.

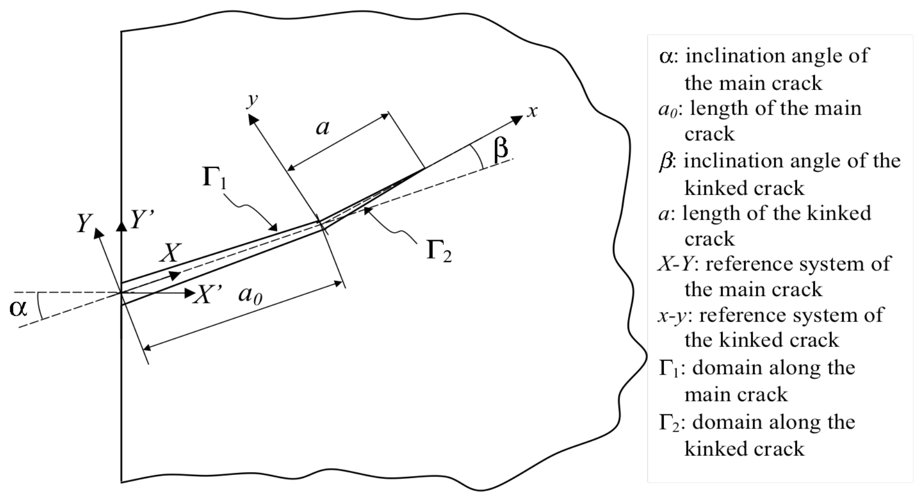

Figure 1. Schematic representation of the problem.

The diagonal WF components $h^{I\sigma}$and $h^{II\tau}$ represent the direct effect, i.e. the contribution on K_I produced by σ and the contribution on K_{II} produced by τ respectively, while the off-diagonal components $h^{II\sigma}$and $h^{I\tau}$ represent the coupling effect, i.e. the contribution on K_{II} produced by σ and on K_I by τ respectively. When the edge crack lays on the symmetry axis (crack inclination angle θ= 0°), the off-diagonal terms vanish and no coupling effects are expected either between shear nominal stress and K_I or between normal stress and K_{II}.

As formally indicated in Eqs. (2), the WF components depends on the *x* and *X* positions and on the parameters characterising the crack geometry. From Eqs. (2) it can be deduced that the physical dimension of the WF components is $[\text{length}]^{-1/2}$. As characteristic length the total crack length $a+a_0$ is assumed. In order to simplify the analysis, WF has been assumed to be a function the dimensionless parameter a/a_0. As a consequence, any WF component can be simplified as follows:

$$h^{M\mu}\left(\xi,a,\alpha,a_0,\beta\right)=\left(a+a_0\right)^{-1/2}\cdot f^{M\mu}\left(\frac{\xi}{a+a_0},\frac{a}{a_0},\alpha,\beta\right) \tag{3}$$

where ξ, $0\le\xi\le a+a_0$, is the curvilinear coordinate along the crack path, $f^{M\mu}$ is a dimensionless function of the dimensionless variables $\xi/\left(a+a_0\right)$ and a/a_0 with parameters the angles α and β. By adopting the formulation proposed in [6], suitable for reproducing the asymptotical properties when $x\rightarrow a$, the following expressions were assumed, in the case of $M\mu = I\sigma$ or $II\tau$:

$$f^{M\mu}\left(\frac{\xi}{a+a_0},\frac{a}{a_0},\alpha,\beta\right)=\sqrt{\frac{2}{\pi}}\cdot\left[\left(1-\frac{\xi}{a+a_0}\right)^{-1/2}+\sum_{k=1}^{n}B_k^{M\mu}\left(\alpha,\beta,\frac{a}{a_0}\right)\cdot\left(1-\frac{\xi}{a+a_0}\right)^{k-1/2}\right] \tag{4a}$$

and in the case $M\mu = II\sigma$ or $I\tau$:

$$f^{M\mu}\left(\frac{\xi}{a+a_0},\frac{a}{a_0},\alpha,\beta\right)=\sqrt{\frac{2}{\pi}}\cdot\left[\sum_{k=1}^{n}B_k^{M\mu}\left(\alpha,\beta,\frac{a}{a_0}\right)\cdot\left(1-\frac{\xi}{a+a_0}\right)^{k-1/2}\right] \tag{4b}$$

As shown in the following sections, a reasonable approximation of the scalar functions $B_k^{M\mu}$ can be obtained by a mixed analytical-numerical technique based on the results of a parametrical FE analysis carried out for at least two independent loading conditions.

FINITE ELEMENT ANALYSIS

The finite element models were developed by using the ANSYS® 11 code and built up with eight node plane strain iso-parametric elements (PLANE82). In order to simulate a virtually semi-infinite body, the global dimension of the model was set equal to $1000(a+a_0)$. Two independent loading conditions are necessary for building up the SIF database to be used in the numerical evaluation of the WF. The adopted loading conditions are shown in figure 2: normal uniform traction at infinite (fig. 2a) and pure shear at infinite (fig. 2b). The remote linearly variable normal traction (fig. 2 c) was also considered in order to validate of the WF under loading conditions different from those used for the evaluation.

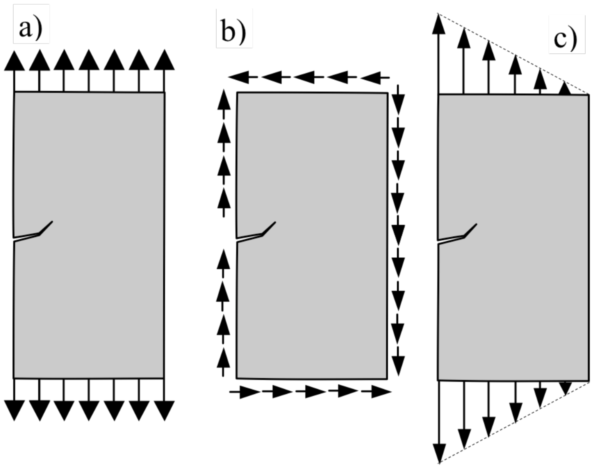

Figure 2. Reference loading conditions: a) uniform remote normal load, b) uniform shear load c) linearly variable remote normal load

Parametric FE analysis of the uncracked semi-plane was carried out for obtaining the nominal stress distribution σ and τ along the virtual crack path. For any loading conditions, the whole stress field along the virtual crack path was stored and the nominal stress components σ and τ calculated for any angle α and β by applying the rules of rotation of the stress tensor from the global reference system *X'-Y'* to the local systems *X-Y* and *x-y* as shown in Fig. 1. On this basis, in the local reference systems *X-Y* and *x-y*, the FE stresses were least-square fitted by using a linear polynomial function.

The same FE model was then modified in order to introduce an oblique kinked edge crack. The stress singularity at the crack tip was modelled by a radial arrangement of quarter-point elements, that allow for an appropriate representation of the local asymptotic displacement field ($r^{1/2}$) at the crack tip (Fig. 3).

In order to check the accuracy of the cracked FE models, a couple of reference crack configurations for which K_I and K_{II} are known were considered: the embedded Griffith crack and the edge crack normal to the external surface [11, 12]. The FE model was adapted to represent these conditions by assuming $\alpha = 0°$ and $\beta = 0°$ and by slightly modifying the boundary conditions. In the case of the Griffith crack, symmetry or anti-symmetry constraints on the free surface of the original model were introduced, according to the remote normal and shear loading. The mesh was refined up to a level that produced a relative difference lower than 0.05% between the FE and the analytical SIF values. Additional comparisons were made with reference crack configurations available in the literature: the inclined edge crack [13] and the edge crack normal to the external surface and deflected [9, 10]. In the former case, the relative difference was lower than 0.1%, in the latter a maximum relative difference of about 4% was found with respect to the solutions reported in [10] and of about 2 % in comparison with [9].

FE analyses were conducted with cracks having initial inclination α in the range $0° \leq \alpha \leq 60°$ with a step of 10°. The obtained results were extended to the interval $-60° \leq \alpha \leq 60°$, by appropriately distinguishing between symmetric and anti-symmetric loadings. The crack deflection angle β was considered in the range $-90° \leq \beta \leq 90°$ with a step of 10°. The explored values of the ratio a/a_0 ranged between 1/300 and 1/10 (8 values tested). The total number of different crack configurations studied was 1064, each one giving a couple of K_I and K_{II} values.

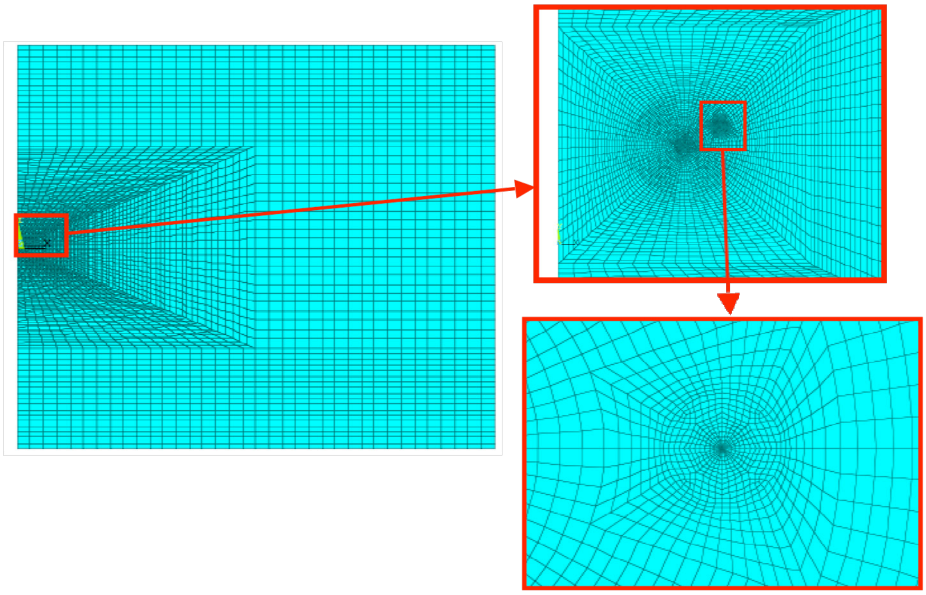

Figure 3. Finite element model of the semiplane carrying an edge inclined kinked crack.

EVALUATION OF THE WEIGHT FUNCTION

By considering the nominal stresses calculated by least-square fitting the FE stresses and the WF expressed by Eqs. (3) and (4), a linear equations system with unknowns $B_k^{M\mu}$ can be written for each SIF calculated by FE. The large number of crack configurations allows for obtaining an over-conditioned linear system in which the coefficients $B_k^{M\mu}$ are the unknowns. The Normal Equation Method can be used to solve the system thus obtaining the coefficients $B_k^{M\mu}$, which reproduce the FEM SIF values at best in the least-square sense. A reasonable compromise between the number of unknowns and the accuracy of SIF reproduction was found by assuming n = 1 in Eqs. (4). The values of the $B_k^{M\mu}$ coefficients, obtained for the different a/a_0 ratios, were then interpolated using a VI degree Chebishev polynomial series [14], function of the angles α and β. Figure 4 exemplarily reports the values and the interpolated surfaces of the four $B_1^{M\mu}$ coefficients for a/a_0 ratio equal to 0.1 and a crack length a_0 of 100 mm. The values of the $B_k^{M\mu}$ as well as Chebishev polynomial coefficients are not reported here for the sake of coincision but can be requested from the Authors. An independent verification of the WF was finally obtained by comparing the SIFs calculated with the WF with those of the FE analysis, for the (c) loading condition shown in Fig. 2.

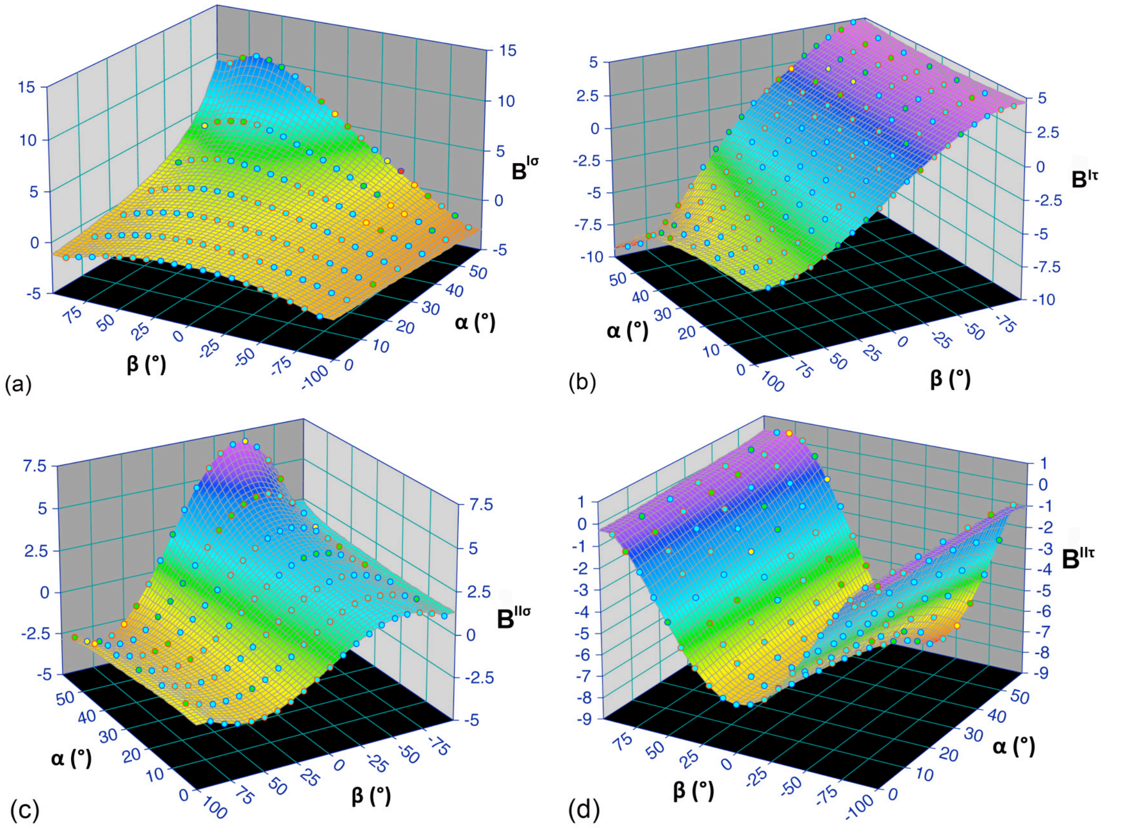

Figure 4. Plot of the coefficients $B_1^{M\mu}$ of the WF for a/a_0 ratio equal to 0.1 and a crack length a_0 of 100 mm. The point data were interpolated with Chebishev polynomial series. (a) $B_1^{I\sigma}$, (b) $B_1^{I\tau}$, (c) $B_1^{II\sigma}$, (d) $B_1^{II\tau}$.

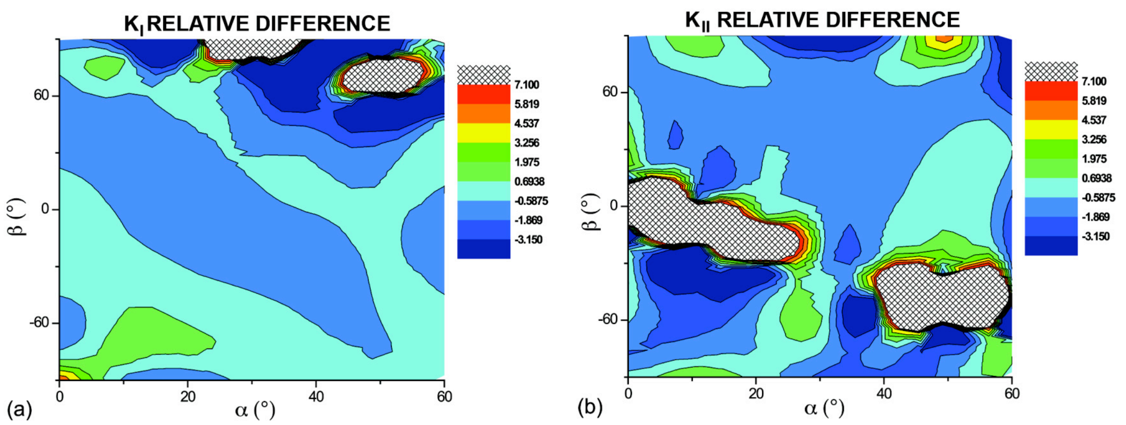

Figure 5. Relative differences between FE and WF solutions of K_I and K_{II} for the loading case (c).

Fig. 5 and b display the relative differences between FE and WF solutions of K_I and K_{II} according to the following expression:

$$Diff(K_I) = \frac{|K_{IWF} - K_{IFE}|}{K_{IFE}} \cdot 100 \qquad Diff(K_{II}) = \frac{|K_{IIWF} - K_{IIFE}|}{K_{IIFE}} \cdot 100 \qquad (5)$$

The zones marked in grey in Fig. 5, where the relative difference exceeds 7%, correspond to SIF value lower than one 100^{th} of the maximum SIF value. In conclusion, it should be emphasized that, for the edge kinked crack, the obtained WF produces SIF values with an average error lower than 1%.

CONCLUSIONS

An analytical WF with a matrix structure was proposed for determining the Stress Intensity Factors of an edge kinked crack in a semiplane. A parametric FE analysis was performed to build up a database of K_I and K_{II} for a relatively broad range of the geometrical parameters governing the problem. The calculated K_I and K_{II} values were used for obtaining the WF by means of a least-square fitting procedure. The obtained WF reproduced the FE results with typical relative differences in the order of 1%.

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