Fractal Modelling of Kinked Cracks and its Implications for their Fatigue Propagation in Concrete

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ABSTRACT. *Threshold condition and rate of fatigue crack growth appear to be significantly affected by the degree of deflection of cracks. In this paper, the reduction of the fatigue crack growth rate for a so-called 'periodically-kinked crack' as compared to that for a straight counterpart is quantified via the Paris-Erdogan law modified according to some simple theoretical arguments. It is shown that such a reduction increases as the value of the kinking angle increases. Then, a so-called 'continuously-kinked crack' (the kink length tends to zero) is considered and modelled as a self-similar invasive fractal curve. Using the Richardson's expression, the fractal dimension of the crack is expressed as a function of the kinking angle. It is shown that the fatigue crack growth rate in the Paris range depends not only on the above fractal dimension and in turn on the kinking angle, but also on the crack length. Some experimental results related to concrete and showing a crack size effect on the fatigue crack growth rate are analysed.*

INTRODUCTION

During fatigue propagation, cracks in both brittle and ductile materials tend to deflect as a result of far-field multiaxial stresses, microstructural inhomogeneities (such as grain boundaries and interfaces), residual stresses and so forth. Threshold condition and rate of fatigue crack growth appear to be significantly affected by the degree of deflection of cracks. This might be induced by the fact that the value of the near-tip Stress Intensity Factor (SIF) of kinked fatigue cracks can be considerably different from that of a straight crack of the same projected length.

With reference to two-dimensional elastic problems, analytical solutions for SIF of kinked cracks are available in the literature [1,2]. Some of such results have been used to gain a quantitative understanding of the relation between fatigue crack growth rate and the degree of crack deflection in the fatigue crack path (e.g. see Ref. [3]).

In comparison with the highly idealised picture of a straight crack, a kinked crack represents a first step towards the description of actual irregularities of fracture surfaces. A further step in that direction consists in using the fractal geometry, as has been shown in several publications (e.g. see Ref. [4] for a review). Successful applications of fractal geometry to size effect-related fatigue problems have recently been proposed by the present authors [5-8].

In the present paper, the reduction of the fatigue crack growth rate for a 'periodically-kinked crack' as compared to that for a straight counterpart is quantified via the Paris-

Erdogan law modified according to some simple theoretical arguments. It is shown that such a reduction increases as the value of the kinking angle increases. Then, a 'continuously-kinked crack' (i.e. the kink length tends to zero) is considered and modelled as a self-similar invasive fractal curve. The kinking angles in the crack are constant but the sequence of kinking directions is such that the fatigue crack path is 'on average' straight. Using the Richardson's expression for self-similar fractals, the fractal dimension of the crack is expressed as a function of the kinking angle. It is shown that the fatigue crack growth rate in the Paris range depends not only on the above fractal dimension and in turn on the kinking angle (this behaviour being also predicted by the periodically-kinked crack model), but also on the crack length. Finally, some experimental results related to concrete and showing a crack size effect on the fatigue crack growth rate are analysed.

SIF AND CRACK GROWTH RATE FOR A PERIODICALLY-KINKED CRACK

SIF for a kinked crack

Let us consider the linear elastic two-dimensional problem of the kinked crack in Fig. 1. The loading axis is taken to be perpendicular to the projected crack length l so that the projected straight crack would be submitted to a Mode I loading characterised by the SIF K_I (e.g. for a centrally-cracked infinite plate: $K_I = \sigma\sqrt{\pi(l/2)}$). The local Mode I and Mode II SIFs, k_I and k_{II}, at the crack tips A and C can be expressed as a function of the SIF K_I, the kinking angle ϑ and the ratio b/a [1,2]. Excluding the case of an infinitesimal kink ($b/a \to 0$), the local SIFs at the crack tip C are approximately equal to those of an inclined straight crack of projected length l forming an angle $\pi/2 - \vartheta$ with respect to the loading axis [1], namely:

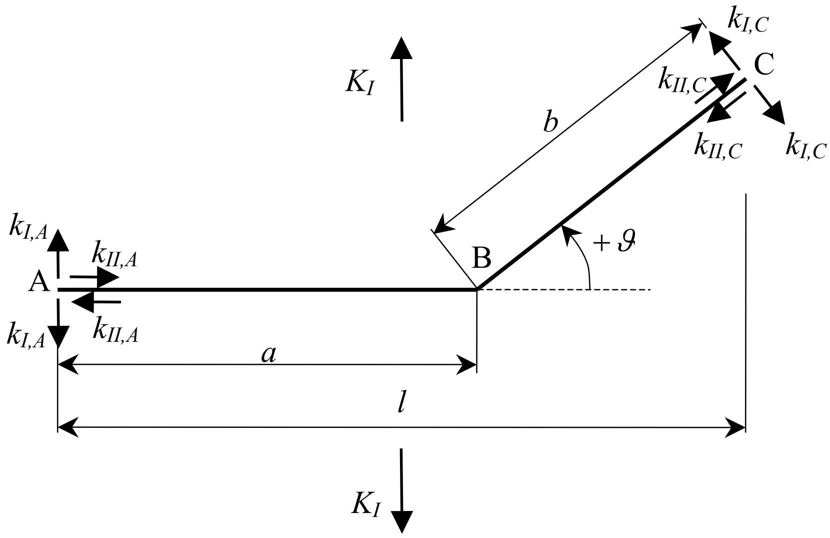

Figure 1 – Nomenclature for the kinked crack.

$$k_{I,C} = \cos^{3/2}\vartheta \cdot K_I, \quad k_{II,C} = \sin\vartheta \cdot \cos^{1/2}\vartheta \cdot K_I \tag{1}$$

On the other hand, for any value of ϑ and for $b/a < 2$, the local SIFs at the crack tip A are approximately equal to those of the projected straight crack of length l [1], namely:

$$k_{I,A} = K_I, \qquad k_{II,A} = 0 \tag{2}$$

Crack growth rate for a periodically-kinked crack

Let us assume that the kinked crack in Fig.1 nominally propagates under fatigue Mode I loading with SIF K_I (the loading axis is perpendicular to the projected crack length) following the path described in Fig. 2, from left to right (from point A to point E and so on). Such a deflected crack is here termed 'periodically-kinked crack'. The crack path is characterised by straight segments of length a and by deflected segments of length b. The degree of kinking in two successive segments is the same although the deflections occur in opposite directions so that the overall ('average') propagation direction is along the Mode I plane. The deflection behaviour is periodic with the repeated pattern described by the crack path ABC (the repeated growth distance $a+b$ is understood to be much smaller than the total projected length of the crack).

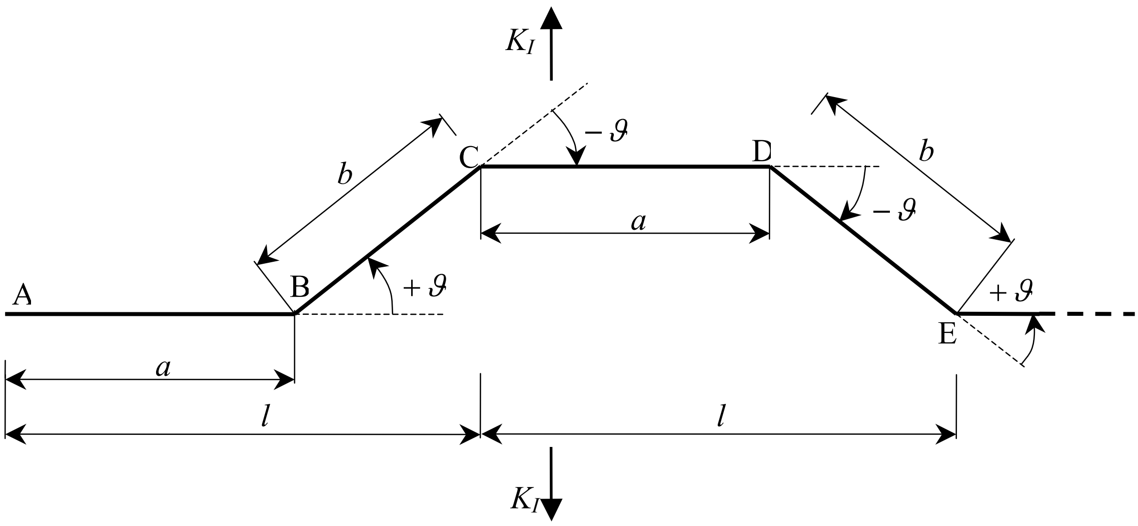

Figure 2 - Nomenclature for the periodically-kinked crack.

By assuming that, as the crack propagates following the periodic path in Fig. 2, only the latter deflection of the crack path influences the stress field near the right-hand crack tip (e.g. along the straight segment CD only the deflection C but not the deflection B has an influence) and that a is equal to b, the local SIFs at the right-hand tip can be calculated approximately according to Eqs 2 for straight (Mode I) segments (the segments AB and CD in Fig. 2) and to Eqs 1 for deflected (Mode I+II) segments (the segments BC and DE in Fig.2).

Along the deflected segments, an effective driving force can be determined by applying the coplanar strain energy release rate theory. Accordingly, the effective SIF k_{eff} is given by:

$$k_{eff} = \sqrt{k_I^2 + k_{II}^2} \tag{3a}$$

and, by using Eq. 1, we obtain:

$$k_{eff} = \sqrt{\cos\vartheta}\; K_I \tag{3b}$$

Now let us apply the Paris-Erdogan law to the periodically-kinked crack:

$$\frac{ds}{dN} = C\,\Delta\overline{K}_I^m \tag{4}$$

where ds/dN = crack growth rate for the kinked crack (s = linear-piecewise coordinate along the kinked-crack path); $\Delta\overline{K}_I$ = mean value of the Stress Intensity (SI) range for the kinked crack.

The value of $\Delta\overline{K}_I$ is represented by the weighted average of the Mode I SI range ΔK_I along the straight segments and of the effective SI range Δk_{eff} (see Eq. 3b in terms of SI ranges) along the deflected segments [3], that is

$$\Delta\overline{K}_I = \frac{a\Delta K_I + b\Delta k_{eff}}{a+b} = \Delta K_I \frac{a + b\sqrt{\cos\vartheta}}{a+b} \tag{5a}$$

and, since $a = b$, we obtain:

$$\Delta\overline{K}_I = \Delta K_I \frac{1+\sqrt{\cos\vartheta}}{2} \tag{5b}$$

Considering the fact that, when the kinked crack spans a distance $a+b$, the projected straight crack spans a distance $a + b\cos\vartheta$, the following relationship between the crack growth rate for the kinked crack (ds/dN) and that for the projected straight crack (dl/dN) holds :

$$\frac{ds}{dN} = \frac{a+b}{a+b\cos\vartheta}\frac{dl}{dN} \tag{6a}$$

and, since $a = b$, we obtain:

$$\frac{ds}{dN} = \frac{2}{1+\cos\vartheta}\frac{dl}{dN} \tag{6b}$$

By substituting Eqs 5b and 6b in Eq. 4, the following fatigue crack growth law in terms of the nominal quantities dl/dN and ΔK_I is determined:

$$\frac{dl}{dN} = \left[2^{-(1+m)}(1+\cos\vartheta)\left(1+\sqrt{\cos\vartheta}\right)^m C\right]\Delta K_I^m \tag{7}$$

The modified expression of the Paris-Erdogan law proposed in Eq. 7 takes into account the influence of the degree of kinking on the fatigue crack growth rate. In particular, it is shown that, at equal ΔK_I values, the crack growth rate dl/dN decreases as the value of the kinking angle increases.

SIF AND CRACK GROWTH RATE FOR A CONTINUOUSLY-KINKED CRACK

Self-similarity of a continuously-kinked crack

Conversely to a Euclidean curve which has the integer physical dimension L^1, a mathematical self-similar invasive fractal curve is a geometric object characterised by a non-integer dimension (the so-called 'fractal dimension D', with $1 \leq D \leq 2$), and by an invariance in its morphology at different scales of observation (the so-called "self-similarity") or, in other words, at different steps in the fractal generation procedure [9].

The model of the periodically-kinked crack shown in Fig. 2 can be extended to the fractal geometry by considering the generation procedure sketched in Fig. 3 (the crack is assumed to be fractal along its length and smooth along its tip). Accordingly, the straight segment E_0 of length $2l$ (*initiator*) is replaced by the linear-piecewise curve E_1 representing the *generator* for the fractal curve under consideration (the generator E_1 is described by the curve ABCDE in Fig. 2, with $a = b$). Then the generator and its 180° clockwise rotation are used to replace, respectively, the first and third segments and the second and fourth segments of E_1 in order to obtain the curve E_2, and so forth for the successive steps (the fractal curve is determined after an infinite number of steps). The obtained fractal curve describing the crack (here termed "continuously-kinked crack") is characterised by linear segments of length tending to zero and by a constant degree of kinking in two successive segments.

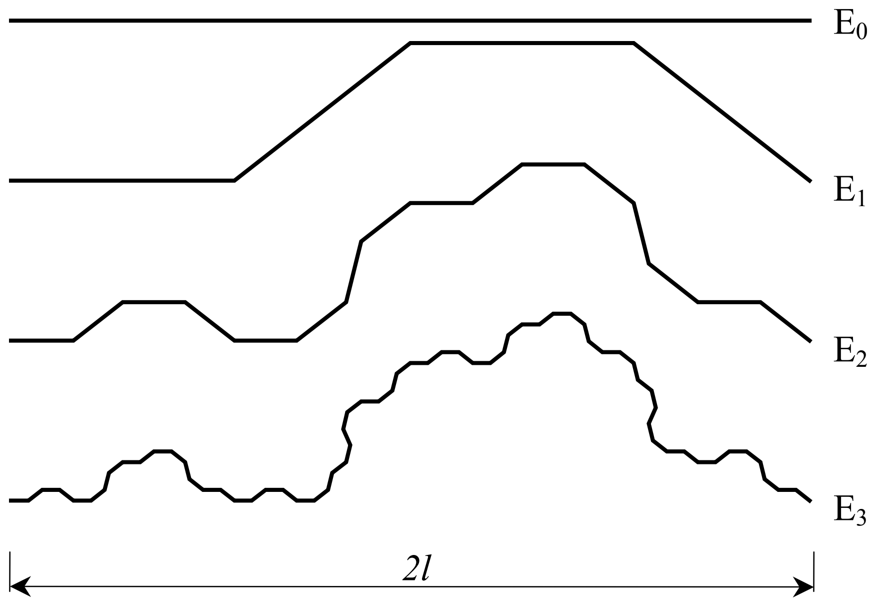

Figure 3 – Fractal generation procedure for the continuously-kinked crack.

At the n-th scale of observation, the measured crack length, e.g. using the 'yardstick' method, is equal to s_n, so that the measured length s_0 of the fractal crack at the 0-th scale of observation is equal to its projected length $2l$. For *mathematical fractals*, the following fundamental relationship (Richardson's expression, see Ref. [9]) holds at any scale of observation:

$$N_n \cdot \varepsilon_n^D = \text{constant} \tag{8}$$

where ε_n = length of the linear 'yardstick' at the n-th scale of observation of the fractal; N_n = number of linear 'yardsticks' of length ε_n ($s_n = N_n \cdot \varepsilon_n$ is the measured length of the fractal at the n-th scale of observation). Obviously, for *natural fractals*, Equation 8 holds only within a limited range of scale, with the lower bound generally associated to the characteristic size of the material microstructure and the upper bound associated to the finite size of the structural component.

Considering the 0th step and the 1st step in the generation of the continuously-kinked crack in Fig. 3, we can obtain (according to Eq. 8) :

$$1 \cdot s_0^D = 4 \cdot a^D \tag{9}$$

where $s_0 = 2l = a(2 + 2\cos\vartheta)$, and hence the fractal dimension D is given by:

$$D = \frac{\ln 4}{\ln(2 + 2\cos\vartheta)} \tag{10}$$

Note that D is equal to the unity (Euclidean curve) for $\vartheta = 0°$ (straight crack), whereas D is equal to 2 (Euclidean surface) for the limit case of $\vartheta = 90°$.

SIF for a continuously-kinked crack

From a reconsideration of the energetic approach of Griffith, it has been demonstrated that the SIF for a fractal crack is represented by the following renormalized quantity K_I^* (e.g. see Ref. [6])

$$K_I^* = K_I \; l^{\frac{1-D}{2}} \tag{11}$$

The physical dimensions of K_I^* are dependent on the fractal dimension *D*, and are equal to $F \cdot L^{-\frac{2+D}{2}}$. Note that a fractal extension of a kinked crack similar to that here proposed was presented in Ref. [10], but the related SIF was defined within the framework of Linear Elastic Fracture Mechanics (e.g. the physical dimensions of SIF were the classical ones, i.e. $F \cdot L^{-\frac{3}{2}}$).

Crack growth rate for a continuously-kinked crack

According to Ref. [6], the following modified Paris-Erdogan law can be used to describe the fatigue crack growth for the continuously-kinked (fractal) crack:

$$\frac{dl^*}{dN} = C^*\left(\Delta K_I^*\right)^m \tag{12}$$

where l^* is the renormalized crack length having physical dimensions L^D, while the material parameter C^* has the following dimensions: $F^{-1} \cdot L^{\frac{2+3D}{2}}$.

Since $l^* = l^D$, the derivation chain rule yields a relationship between the renormalized fatigue crack growth dl^*/dN and its nominal counterpart dl/dN, namely

$$\frac{dl^*}{dN} = \frac{dl^*}{dl}\frac{dl}{dN} = Dl^{D-1}\frac{dl}{dN} \tag{13}$$

By substituting Eqs 11 and 13 in Eq. 12, the following fatigue crack growth law in terms of the nominal quantities dl/dN and ΔK_I can be obtained :

$$\frac{dl}{dN} = \left[\frac{C^*}{D} l^{(1-D)\left(1+\frac{m}{2}\right)}\right]\Delta K_I^m = \left[\frac{C^*}{\frac{\ln 4}{\ln(2+2\cos\vartheta)}} l^{\left(1-\frac{\ln 4}{\ln(2+2\cos\vartheta)}\right)\left(1+\frac{m}{2}\right)}\right]\Delta K_I^m \tag{14}$$

Note that, conversely to the fatigue crack growth law in Eq. 7 for periodically-kinked cracks, Equation 14 for continuously-kinked cracks explicitly depends on the crack length l and, hence, it accounts for crack size effects on the fatigue crack growth rate.

COMPARISON WITH EXPERIMENT AND DISCUSSION

It is instructive to consider here some experimental fatigue crack growth results exhibiting crack size effects, since such results might be regarded as a counter-example for the validity of the crack growth law in Eq. 7. The experimental data are related to fatigue crack propagation in three-point bend high-strength plain-concrete specimens [11]. One series of three two-dimensional geometrically similar cracked beams (A, B and C) with height equal to h_A = 38 mm, h_B = 108 mm and h_C = 304 mm, initial length of the crack of 0.16h (l_A = 6.3 mm, l_B = 18.0 mm and l_C = 50.4 mm), span of 2.5h and thickness of 38 mm was tested. The maximum size of the aggregate was equal to 9.5 mm and the mean compression strength was equal to 90.3 MPa.

The nominal values of the crack growth rate against SI range [11] are reported as a bilogarithmic plot in Fig. 4 (17, 16 and 12 experimental points for beams A, B and C, respectively). The value of the fractal dimension D can be calculated by applying Eq. 14 to the data in Fig. 4 through a best-fit procedure (see Ref. [6] for details). It turns out that the fractal dimension D is equal to 1.27 (being the slope of the Paris-Erdogan law m = 8.2) and hence, by applying Eq. 10, the kinking angle ϑ results to be equal to 54°. The value of ϑ is deemed to be correlated to the material microstructure, but further work is needed in order to determine quantitative relationships.

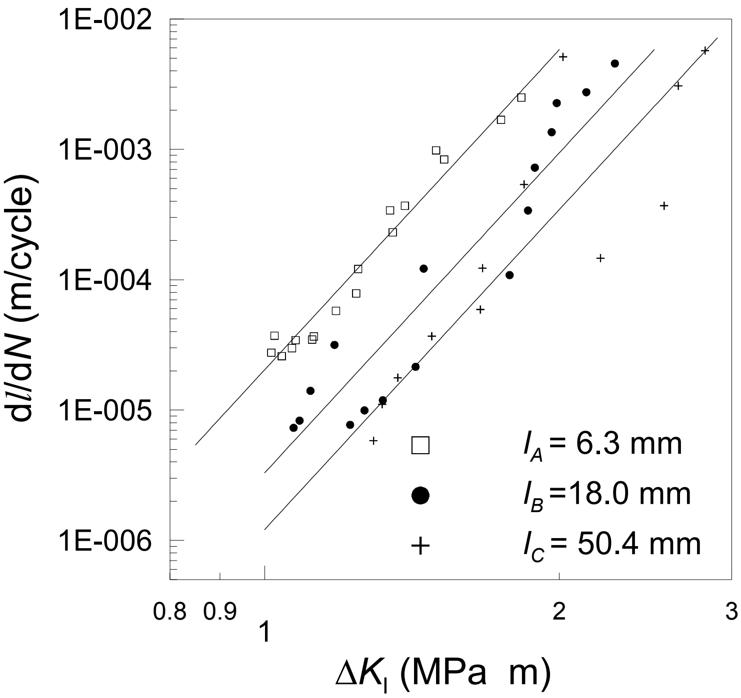

Figure 4 - Crack growth $\mathrm{d}l/\mathrm{d}N - \Delta K_I$ data for different values of initial crack length [11].

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