

MODE III CRACK TIP FIELDS IN COUPLE STRESS ELASTIC MATERIALS WITH TWO CHARACTERISTIC LENGTHS

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SOMMARIO

In questo lavoro viene effettuata un'analisi asintotica dei campi di tensione e deformazione in prossimità dell'apice di una frattura in un materiale con microstruttura, in condizioni di modo III. Per rappresentare il comportamento del materiale è stato utilizzato il modello costitutivo elastico polare sviluppato da Koiter, che include due distinte lunghezze caratteristiche proprie della microstruttura del materiale. Pertanto, il modello considerato risulta in grado di simulare i rilevanti effetti di scala che si riscontrano in prossimità dell'apice di una frattura nei materiali con microstruttura, a distanze comparabili a quelle delle lunghezze caratteristiche. In particolare si osserva che la presenza di microstruttura comporta un notevole incremento della singolarità delle componenti antisimmetriche di tensione e, quindi, delle trazioni davanti all'apice, mentre le componenti simmetriche di tensione risultano limitate. Tali osservazioni si presentano in accordo con i risultati ottenuti considerando una sola lunghezza caratteristica del materiale. Nel presente lavoro viene inoltre evidenziata l'influenza esercitata dal rapporto tra le due lunghezze caratteristiche sugli andamenti dei campi asintotici in funzione della coordinata angolare e sul rilascio di energia in prossimità dell'apice.

ABSTRACT

The asymptotic fields near the tip of a stationary crack in a strain gradient elastic material under Mode III loading conditions are investigated by adopting the indeterminate theory of couple stress elasticity with two characteristic lengths. The adopted constitutive model is able to account for the underlying microstructure of the material as well as for the strong size effects arising at small scales. The effects of microstructure on Mode III crack tip fields mainly consist in a substantial increase in the singularities of the skew-symmetric stress and couple stress fields thus resulting in a significant increase of the tractions level ahead of the crack-tip. Correspondingly, the symmetric stress field turns out to be non-singular. These results agree with the findings of the asymptotic analysis performed for a single material characteristic length. However, the angular variations of the crack tip fields and the energy release rate turn out to be strongly influenced by the ratio between the two characteristic lengths.

1. INTRODUCTION

Due to the lack of a length scale, the classical theory of elasticity is not able to characterize the constitutive behavior of brittle materials at the micron scale. This lack is expected to be particularly significant for the analysis of the stress and deformation fields very near the tip of a crack, which are altered by the presence of the microstructure. Therefore, for the investigations of the crack tip fields at the micron scale it becomes necessary to adopt enhanced constitutive models, which account for the presence of microstructure. A way of doing that consists in the inclusion of one or more characteristic lengths, typically of the same order of the compositional grain size, generally few microns, for many

advanced materials like silicon nitride ceramics, which can be regarded as a class of material comparable to steel. The size and shape of the grains have a strong influence on their different mechanical properties. In particular, high strength materials exhibit a fine-grained, elongated microstructure, while materials with a high fracture toughness are more coarse grained [1]. The indeterminate theory of CS elasticity developed by Koiter [2] allows to account for the size and shape of the grains since it involves two material characteristic lengths and it can describe the mechanical behaviour of many elastic materials with microstructures, like cellular materials, fibrous composites and laminates, where moments may be transmitted through fibers, or in the cell ribs or walls. In fibrous composites, the characteristic lengths may be the on the order of the spacing between fibers [3]; in cellular solids they may be comparable to the average cell size [4,5]; in laminates they may be on the order of the lamination thickness [6].

The problem of Mode III crack propagation in couple stress elastic materials was first analyzed in [7] by considering a single characteristic length. The results obtained therein [7] indicate that the skew-symmetric stress components have $r^{3/2}$ singularity near the crack-tip, where r is the distance to the crack tip. Although this singularity is much stronger than the conventional square-root singularity, it does not violate the boundness of strain energy surrounding the crack tip and leads to a finite energy release rate. However, up to now, the effects of both characteristic lengths on the crack tip fields are almost unexplored.

In the present work, the effects of strain rotation gradients on a stationary Mode III crack are investigated by performing an asymptotic analysis of the crack-tip fields. Two characteristic material lengths of the same order, denoted by L and L' , are taken into account. According to the results obtained in [7] for a single characteristic material length L , the skew-symmetric stress field dominates the asymptotic field, producing thus a remarkable increase of tractions level at the crack tip. The roles of both characteristic lengths are examined in detail and the influence of their ratio on the crack tip fields and energy release rate is analytically explored. The inclusion of two distinct characteristic lengths provides more realistic predictions on the tractions level ahead of the crack-tip than the classical linear elastic solution and gives more accurate results than the CS theory of elasticity with a single characteristic length, allowing the detailed mechanisms by which fracture may grow and propagate in brittle materials to be understood in more depth, up to the micron scale.

2. GOVERNING EQUATIONS

Reference is made to a Cartesian coordinate system $(0, x_1, x_2)$ centred at the crack-tip. Under anti-plane shear deformation, the indeterminate theory of CS elasticity [2] adopted in the present study provides the following kinematical compatibility conditions between the out-of-plane displacement w , rotation vector ϕ , strain tensor ϵ and deformation curvature tensor χ

$$\epsilon_{13} = w_{,1}/2 \quad \epsilon_{23} = w_{,2}/2 \quad \phi_1 = w_{,2}/2, \quad \phi_2 = -w_{,1}/2, \quad (1)$$

$$\chi_{11} = -\chi_{22} = w_{,12}/2 \quad \chi_{21} = -w_{,11}/2 \quad \chi_{12} = w_{,22}/2. \quad (2)$$

Therefore, rotations are derived from displacements and the tensor field χ turns out to be irrotational. According to the CS theory [2] the non-symmetrical Cauchy stress tensor \mathbf{t} can be decomposed into a symmetric part σ and a skew-symmetric part τ , namely $\mathbf{t} = \sigma + \tau$. In addition, the couple stress tensor μ is introduced as the work-conjugated quantity of χ^T . For the antiplane problem within the couple stress theory ϵ , σ , χ and μ are purely deviatoric tensors. The reduced surface tractions vector \mathbf{p} and couple stress tractions vector \mathbf{q} are defined respectively as

$$\mathbf{p} = \mathbf{t}^T \mathbf{n} + \frac{1}{2} \nabla \mu_{nn} \times \mathbf{n}, \quad \mathbf{q} = \mu^T \mathbf{n} - \mu_{nn} \mathbf{n}, \quad (3)$$

where \mathbf{n} denotes the outward unit normal. The conditions of quasistatic equilibrium of forces and moments write

$$\sigma_{13,1} + \sigma_{23,2} + \tau_{13,1} + \tau_{23,2} = 0, \quad \mu_{11,1} + \mu_{21,2} + 2 \tau_{23} = 0, \quad \mu_{12,1} + \mu_{22,2} - 2 \tau_{13} = 0 \quad (4)$$

Within the context of small deformations theory, the total strain ϵ and the deformation curvature χ are related to stress and couple stress through the following isotropic constitutive relations

$$\sigma = 2G \epsilon, \quad \mu = 2G L^2 (\chi^T + \eta \chi), \quad (5)$$

where G is the elastic shear modulus and $\eta = L'/L$ the dimensionless ratio between characteristic lengths introduced by Koiter [2], with $-1 < \eta < 1$. The magnitude of the material characteristic lengths depends on the microstructure and it generally is of the order of few microns for brittle ceramics. In particular, they can be related to the characteristic lengths for torsion and for bending [8].

The constitutive equations of the indeterminate CS theory do not define the skew-symmetric part $\boldsymbol{\tau}$ of the total stress tensor \mathbf{t} . However, $\boldsymbol{\tau}$ can be obtained from the equilibrium equation (4)₂. Constitutive equations (5) together with compatibility relations (1) and (2) give

$$\sigma_{13} = G w_{,1}, \quad \sigma_{23} = G w_{,2}, \quad (6)$$

$$\mu_{11} = G L^2 (1 + \eta) w_{,12}, \quad \mu_{21} = G L^2 (w_{,22} - \eta w_{,11}), \quad \mu_{12} = -G L^2 (w_{,11} - \eta w_{,22}). \quad (7)$$

The introduction of (6) into (3)_{2,3} yields

$$2 \tau_{13} = -G L^2 \Delta w_{,1}, \quad 2 \tau_{23} = -G L^2 \Delta w_{,2}, \quad (8)$$

where Δ denotes the Laplacian operator. Finally, The introduction of (8) and (6) into (4)₁ yields the following PDE for the unknown displacement function w :

$$2 \Delta w - L^2 \Delta \Delta w = 0, \quad (9)$$

By using (3), the conditions of traction free on the crack surfaces where $\mathbf{n} = (0, -1)$ can be written as

$$2 (\sigma_{23} + \tau_{23}) + \mu_{22,1} = 0, \quad \mu_{21} = 0, \quad \text{for } x_1 < 0, \quad x_2 = 0, \quad (10)$$

which yield the following conditions for the function w :

$$2 w_{,2} - L^2 [(2 + \eta) w_{,11} + w_{,22}]_{,2} = 0, \quad w_{,22} - \eta w_{,11} = 0, \quad \text{for } x_1 < 0, \quad x_2 = 0. \quad (11)$$

Ahead of the crack tip the skew-symmetry of the Mode III crack problem requires

$$w = 0, \quad w_{,22} - \eta w_{,11} = 0, \quad \text{for } x_1 > 0, \quad x_2 = 0. \quad (12)$$

Note that the ratio η enters the boundary conditions (11)-(12), but it does not appear into the governing equation (9). Moreover, by using (1) and (2) the strain energy density in antiplane shear deformation becomes

$$\Phi = G \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} + G L^2 (\boldsymbol{\chi} \cdot \boldsymbol{\chi} + \eta \boldsymbol{\chi} \cdot \boldsymbol{\chi}^T) = \frac{G L^2}{4} [(w_{,11} + w_{,22})^2 + 2 (1 + \eta) (w_{,12}^2 - w_{,11} w_{,22})]. \quad (13)$$

For a mode III crack in couple stress elastic materials the path-independent J -integral generalizes to

$$J = \int_{\Gamma} (\Phi \mathbf{n} - \mathbf{p} \cdot \mathbf{e}_3 \nabla w - \nabla \Phi^T \mathbf{q}) \cdot \mathbf{e}_1 \, ds, \quad (14)$$

where Γ is an arbitrary contour surrounding the crack tip. Note that J can be proved to coincide with the energy release rate [7].

3. MODE III ASYMPTOTIC CRACK-TIP FIELDS

An asymptotic analysis of the crack tip fields is performed in the present section by assuming the displacement w in separate variable form, namely $w(r, \theta) = r^p W(\theta)$, where r and θ are polar coordinates centered at the crack tip and the exponent p defines the radial dependence of the displacement w as $r \rightarrow 0$. By using the derivative rules for an arbitrary function $f = f(x_1, x_2) = f(r, \theta)$:

$$f_{,1} = f_{,r} \cos \theta - f_{,\theta} \sin \theta / r, \quad f_{,2} = f_{,r} \sin \theta + f_{,\theta} \cos \theta / r, \quad (15)$$

and keeping the most singular terms only, the governing equation (9) and boundary conditions (11) and (12) write:

$$W^{IV}(\theta) + 2(\rho^2 - 2\rho + 2)W''(\theta) + \rho^2(\rho - 2)^2W(\theta) = 0, \quad (16)$$

$$W(0) = 0, \quad W''(\pi) + \rho(1 + \eta - \eta\rho)W(\pi) = 0, \quad (17)$$

$$W''(0) = 0, \quad W'''(\pi) + [\rho^2 + (1 + \eta)(\rho^2 - 3\rho + 2)]W'(\pi) = 0. \quad (18)$$

In order to obtain a nontrivial solution, the homogeneous boundary value problem (16)-(18) admits the following values of the exponent: $\rho = 1/2, 3/2, 5/2 \dots$. The boundness of the flux of energy toward the crack tip (14) require that $2(\rho - 2) + 1 \geq 0$ and thus the first admissible value for the exponent ρ is $3/2$, leading to the following expression for the out-of-plane displacement

$$w(r, \theta) = B r^{3/2} [(5/3 - \eta) \sin(3\theta/2) - (1 + \eta) \sin(\theta/2)], \quad (19)$$

where B is an amplitude factor for the lowest order asymptotic crack-tip fields. Therefore the slope of the sliding displacement behind the crack tip vanishes at $r = 0$, so that the crack tip is sharp and not blunted. According to (1), (6)-(8), the corresponding rotation, stress and couple stress fields become

$$\varphi_1 = \frac{B}{2} \sqrt{r} [1 - 3\eta + (1 + \eta) \cos\theta] \cos(\theta/2), \quad \varphi_2 = \frac{B}{2} \sqrt{r} [-3 + \eta + (1 + \eta) \cos\theta] \sin(\theta/2), \quad (20)$$

$$\sigma_{13} = BG \sqrt{r} [3 - \eta - (1 + \eta) \cos\theta] \sin(\theta/2), \quad \sigma_{23} = BG \sqrt{r} [1 - 3\eta + (1 + \eta) \cos\theta] \cos(\theta/2), \quad (21)$$

$$\tau_{13} = -\frac{BGL^2}{2 r^{3/2}} (1 + \eta) \sin(3\theta/2), \quad \tau_{23} = \frac{BGL^2}{2 r^{3/2}} (1 + \eta) \cos(3\theta/2), \quad (22)$$

$$\mu_{11} = \frac{BGL^2}{2 \sqrt{r}} (1 + \eta) [2(1 - \eta) + (1 + \eta) (\cos\theta - \cos 2\theta)] \cos(\theta/2),$$

$$\mu_{21} = \frac{BGL^2}{2 \sqrt{r}} (1 + \eta)^2 (1 - 2 \cos \theta) \sin\theta \cos(\theta/2), \quad (23)$$

$$\mu_{12} = \frac{BGL^2}{2 \sqrt{r}} (1 + \eta) [4(1 - \eta) - (1 + \eta) (\cos\theta + \cos 2\theta)] \sin(\theta/2).$$

Note that the out-of-plane displacement ahead of the crack tip at small values of θ , namely for $\theta \ll 1$, and the sliding displacement on the crack surface, at $\theta = \pi$, turn out to be opposite in sign, being

$$w(r, \theta) / w(r, \pi) = -0.75 (1 - \eta) \theta < 0, \quad \text{for } \theta \ll 1. \quad (24)$$

This feature is peculiar of fracture process in couple stress materials. Similar results also occur for Mode III crack in CS elasticity with a single material characteristic length [7] as well as for CS elastic-plastic material behavior [9]. This unusual aspect seems to be due to the presence of microstructures (compositional grains). During crack growth the separation process between two material particles at the crack tip can be divided into two steps. In the first step the particles rotate with respect to each other. Only in the second step they move apart. The local rotation of grains and particles currently at the crack-tip produces opposite displacements ahead and behind the crack-tip under Mode III loading condition thus originating a *scissors* effect.

As r goes to zero, the symmetric stress, skew symmetric stress and couple stress fields (21)-(23) behave as $r^{1/2}$, $r^{-3/2}$ and $r^{-1/2}$, respectively. Therefore, for $r < L$ the skew symmetric stress field gives the most singular contribution near to the crack tip. Conversely, for $r > L$ the couple stress and skew symmetric stress fields become negligible with respect to the symmetric stress field, in agreement with the classical LFM theory. It must be remarked that skew symmetric stresses do not contribute to the strain-energy density Φ , which is instead dominated by the weakly singular couple-stress field, so that the flux of energy toward the crack-tip remains finite for $-1 < \eta < 1$. It follows that the tractions level

ahead of the crack tip increases with respect to the classical square root stress singularity given by the LEFM theory, due to the contribution of the skew-symmetric stress components. However, the generalized tractions occur with the opposite sign with respect to the classical mode III solution.

The solution of the homogeneous asymptotic problem can be determined up to the amplitude factor B , which depends on far-field loading and specimen geometry. The constant B can be estimated by matching the asymptotic solution with the far-field conditions by using the path independent integral (14), in agreement with the classical LEFM approach. By choosing a circular contour around the crack tip and letting its radius tends to vanish, after the introduction of the asymptotic fields (19)-(23) in the integral (14), one obtains

$$J = (1 + \eta)(3 - \eta) \pi B^2 G L^2, \quad (26)$$

Therefore, the J -integral tends to vanish as the ratio η approaches the limit value -1 . If the crack is subject to a remotely imposed classical K_{III} fields, related to the J -integral by $K_{III}^2 = 2 G J$, then the following relation between the amplitude constant B and the stress intensity factor K_{III} can be obtained

$$B = -\frac{K_{III}}{GL^2 \sqrt{2\pi(1+\eta)(3-\eta)}}. \quad (27)$$

Note from Fig. 1 that the amplitude factor B monotonically decreases with the ratio η and it tends to infinity as η approaches the limit value -1 . The negative sign in (27) has been chosen since the results (19), (22) and (24) reveal that the displacement w and shear stress τ_{23} ahead of the crack tip occur with the negative sign, unlike the classical Mode III crack-tip fields in non-polar materials. In this case, the shear stress ahead of the crack tip at $\theta = 0$ switches sign from the remotely imposed K_{III} fields for large radial distance, namely for $r \gg L$, where $\tau_{23} = K_{III} / (2\pi r)^{1/2}$ to the negative shear stress τ_{23} and the corresponding generalized traction p_2 near to the crack tip

$$\tau_{23} = -K_{III} L \sqrt{\frac{1+\eta}{8\pi r^3(3-\eta)}}, \quad p_3 = -K_{III} L \sqrt{\frac{(1+\eta)(3-\eta)}{32\pi r^3}}, \quad \text{for } r < L. \quad (28)$$

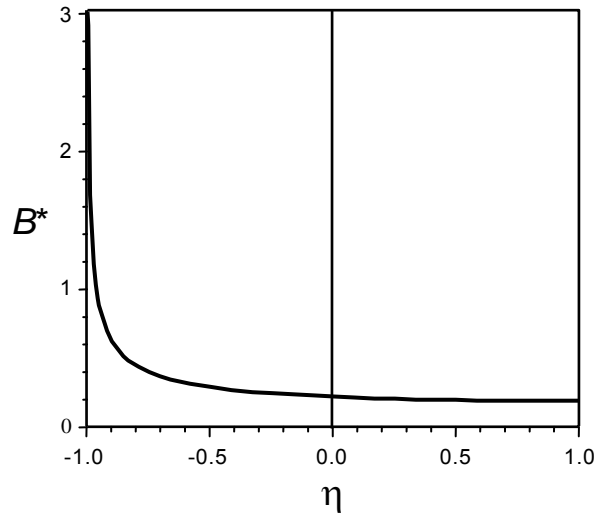


Fig. 1 – Variation of the amplitude factor $B^* = BG\ell^2/K_{III}$ with the ratio η .

4. RESULTS

The ratio η has a strong influence on the angular distribution of the crack tip fields. In particular, the angular variation of the out-of-plane displacement w , plotted in Fig. 2a for different values of the ratio η , shows that the displacements ahead and behind the crack tip are opposite in sign and their values are larger as η tends to the limit value -1 . However, the displacement w behaves almost monotonically as η tends to the opposite limit value 1 , more similarly to the classical LEFM solution.

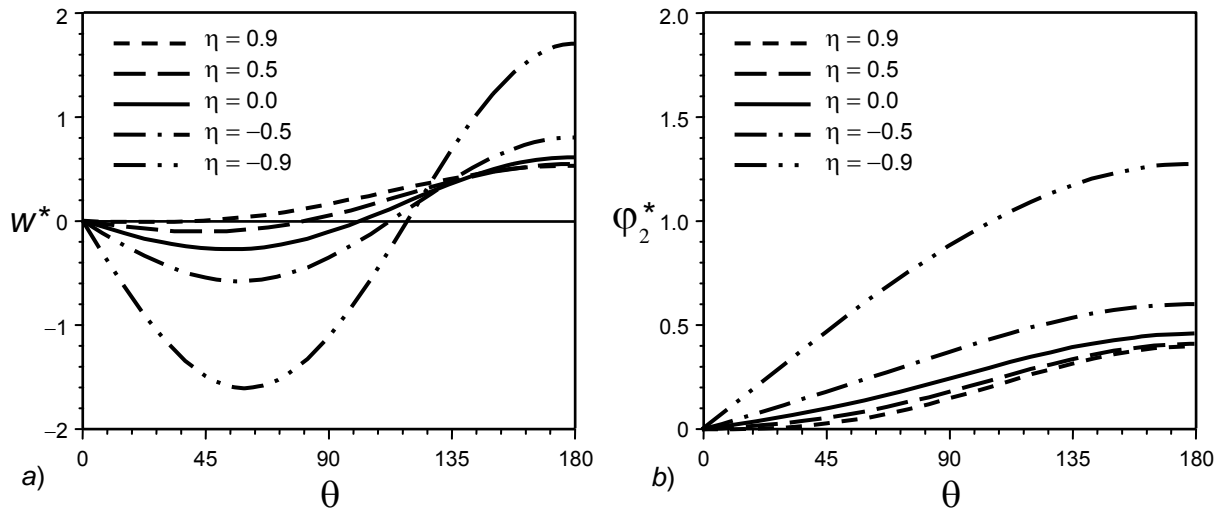


Fig. 2 – Angular variation of (a) displacement $w^* = w GLr^{-3/2}/K_{III}$ and (b) rotation $\phi_2^* = \phi_2 GLr^{-1/2}/K_{III}$.

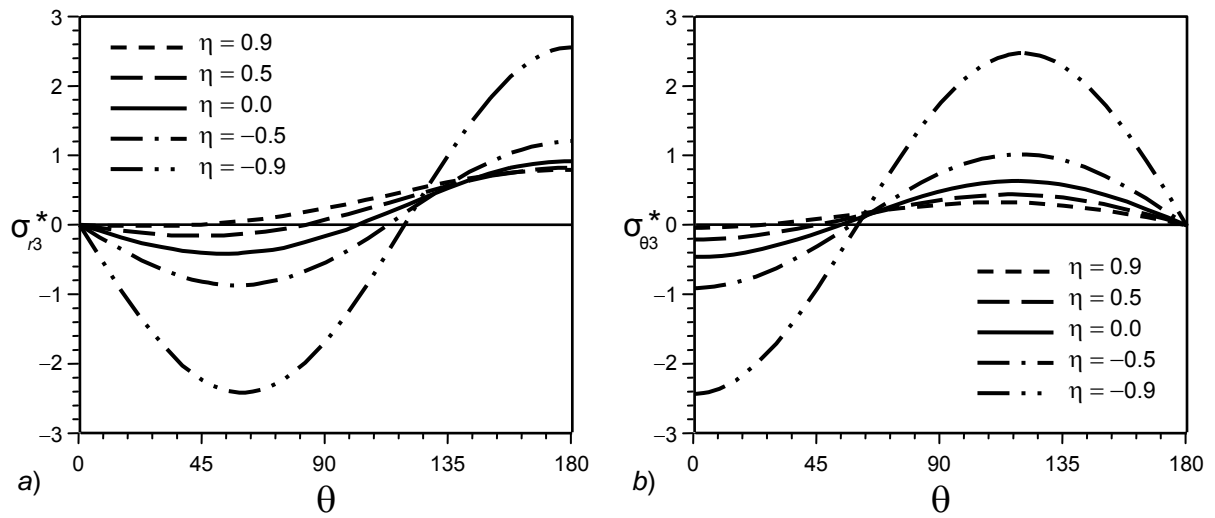


Fig. 3 – Angular variation of symmetric stresses $\sigma_{\alpha 3}^* = \sigma_{\alpha 3} L r^{-1/2}/K_{III}$, for $\alpha = r, \theta$.

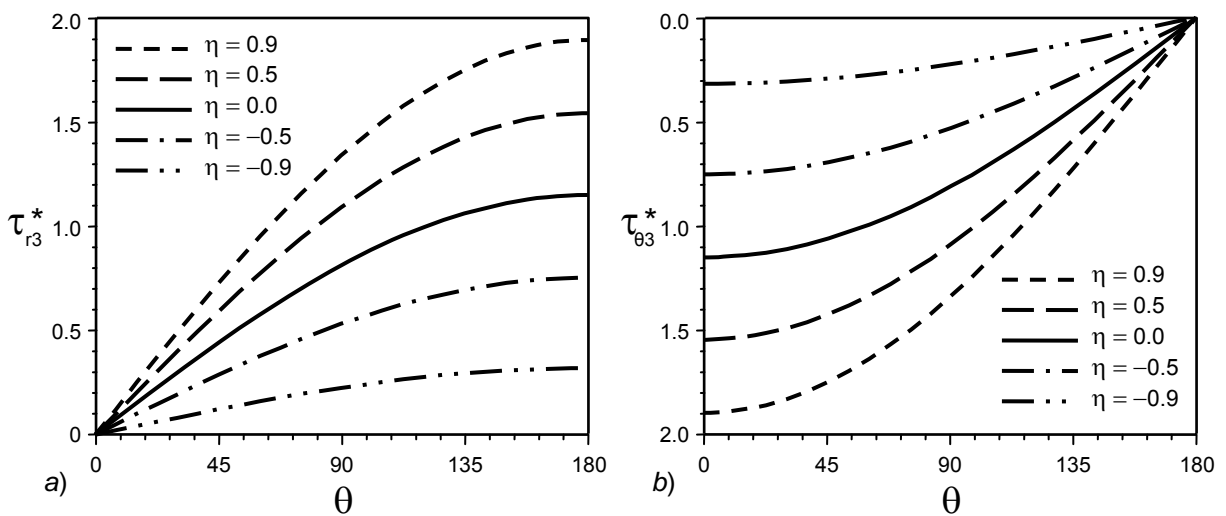


Fig. 4 – Angular variation of skew-symmetric stresses $\tau_{\alpha 3}^* = \tau_{\alpha 3} r^{3/2}/(LK_{III})$, for $\alpha = r, \theta$.

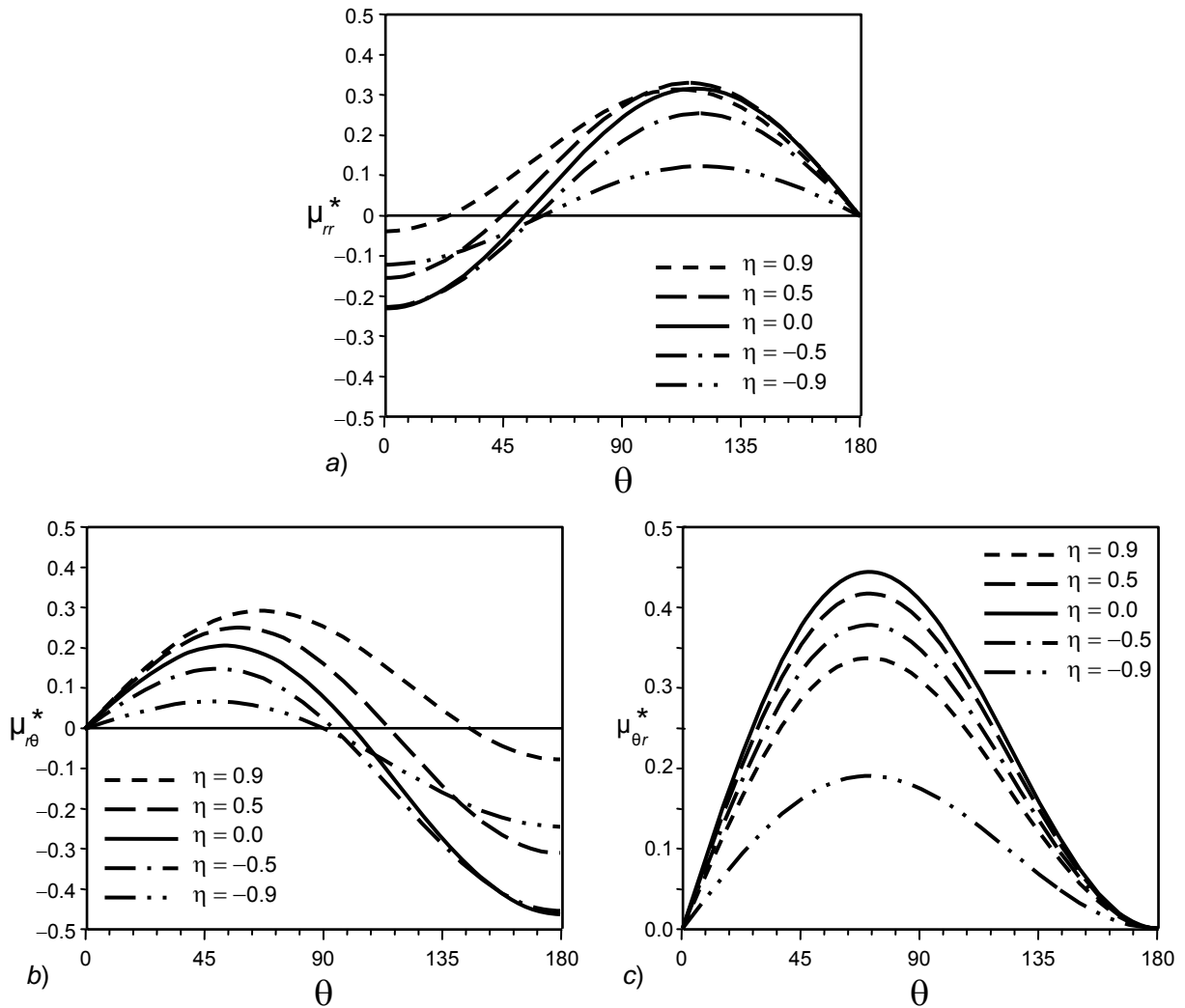


Fig. 5 – Angular variation of couple stresses $\mu_{\alpha\beta}^* = \mu_{\alpha\beta} r^{1/2}/(LK_{III})$, for $\alpha, \beta = r, \theta$, being $\mu_{\theta\theta} = -\mu_{rr}$.

The change of sign ahead of the crack tip denotes the occurring of a significant rotation of the material particles currently at the tip of the crack, much more accentuated for negative values of the ratio η , as it can be observed in Fig. 2b.

Figures 3-5 display the asymptotic angular distributions of the cylindrical components of symmetric stress, skew symmetric stress and couple stress fields, for different values of the ratio η . All the functions are plotted in non dimensional form. Note that the curves for $\eta = 0$, namely $L' = 0$, recover the results obtained for a single material length [7].

The angular variations of both symmetric and skew-symmetric stress components plotted in Figs. 3-4 agree with the inversion of the displacement field ahead of the crack-tip. Indeed, both shear stress $\sigma_{\theta 3}$ and $\tau_{\theta 3}$ are negative ahead of the crack tip, unlike the corresponding shear stress in the classical Mode III solution obtained by using the LEFM theory. This switch in the shear direction agrees with the findings in [7,9] for a single characteristic length.

As the characteristic lengths ratio η is reduced from 0.9 to -0.9 the skew-symmetric stress components, plotted in Fig. 3, reduce and correspondingly the symmetric stress components, plotted in Fig. 4, increase.

5. CONCLUSIONS

The structure of asymptotic field near a mode III crack tip in couple stress elastic materials with two distinct characteristic lengths has been investigated in the present work. The obtained results show that both near tip symmetric stresses and couple stresses contribute to the crack tip energy release rate, and the sum of their contribution equals the remote energy release rate. Moreover, the use of the CS theory of elasticity developed by Koiter [2] for the analysis of the stress field near the tip of a propagating Mode III crack gives accurate predictions on the increase of the tractions level ahead of

the crack-tip occurring at very small distances from it, comparable with the size of the compositional grains. Indeed, the obtained asymptotic solution holds up to a distance to the crack tip much smaller than the lower radius of validity of the classical LEFM theory. Therefore, the present approach provides a means to link scales in fracture mechanics, namely from atomistic through microscale to macroscopic fracture, allowing to understand the detailed mechanisms by which fracture may grow and propagate in brittle materials with complex microstructure, up to the micron scale.

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