An implicit gradient stress failure condition

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ABSTRACT. *This contribution focuses on gradient formulations for the prediction of the static failure load of V-notched and cracked components made of brittle materials. A weighted average of the local equivalent stress, called non-local equivalent stress, is first considered and subsequently approximated by a gradient expansion up to the spatial second-order derivative. A distinguishing characteristic of the present approach is that the non-local equivalent stress is calculated by solving a differential equation of implicit type. The numerical solution for a V-notch in the presence of Neumann's boundary conditions is presented. Moreover, the analytical solutions of an one-dimensional case is proposed. Finally, the static failure loads predicted by to the present formulation through the finite element technique are compared with the experimentally determined ones.*

1. INTRODUCTION

Accurate prediction of the static failure initiation of brittle components in the presence of high stress concentrations is of great interest in mechanics. In this case, classical continuum models cannot be used. For instance, volume averages of the local stress scalar representative of the adopted failure criterion have been using in fracture and fatigue mechanics (e.g. Seweryn and Mroz, 1995[1]; Taylor, 1999[2]). However, these averages-based approaches possess an empirical origin, while lacking of an explicit theoretical basis. In particular, the interaction of points placed within the volume of integration is independent of the distance between the points. Moreover, the choice of the volume of integration has to be stated a priori, as a function of the mode of loading. On the other hand, these averages-based approaches can be seen as simple versions of the so-called non-local approaches developed in elasticity starting from the 60's by Kröner[3], Edelen[4], Eringen[5] and based on the assumption that the principle of local action does not hold in the presence of stress- or strain-singularities. Later, Bazant,[6] de Borst[7] and several others [8],developed a simpler class of non-local models for the finite element analysis of strain localization exhibited by structural elements made of softening materials. The purpose was to overcome pathological mesh-dependency of finite element models for softening materials. Classical finite element techniques converge indeed to a physically meaningless solution, where the strain field tends to localize into the smallest finite element. That is in contrast with the experimental observations that failure is reached by strain localization in a process zone with finite size. Moreover, the computed load-displacement response in the post-elastic phase tends to the elastic unloading as the mesh size decreases, leading to mesh-dependency. That motivated the adoption of enhanced formulations such as, for instance, formulations of non-local type. Basically, a non-local field is defined as the volume average of the local field weighted by a suitable weight function, $a(x, y)$. The weight function reaches its maximum at the actual point x, and decays to zero at increasing distances. Such non-local models are usually called of integral type. After Taylor series expansions of the local field and its subsequent weighted integration, gradient-approximations of the non-local field can be obtained, of both explicit and implicit gradient types[9].

In this contribution, an implicit gradient type stress-failure criterion is proposed for the analysis of the static failure of V-shaped notches under mode I loading assuming a linear elastic material. A stress scalar of non-local type, called the non-local equivalent stress, is defined by an implicit gradient approach. In particular, the principal stress is assumed as local equivalent stress for the sake of an example. A distinguishing characteristic of the present approach is that the non-local equivalent stress is calculated by solving a differential equation where the second-order spatial derivative (Laplacian) of the non-local equivalent stress appears. More details will be given in Section 2. The analytical solution is proposed in the case of Neumann-type boundary conditions for a one-dimensional stress-field singular at the notch tip.

2. NON-LOCAL MODELS

Some preliminaries notions on non-local formulations for the analysis of quasi-brittle materials are reported. Next, the present formulation is presented in more detail.
Let us consider a body with volume V and a local stress scalar, called the local equivalent stress z . Its non-local integral type counterpart at a point x in V , called non-local equivalent stress, writes

$$\bar{\zeta}(x) = \frac{1}{V_r(x)} \int_V \alpha(x,y)\zeta(y)\,dy \qquad \text{in } V \tag{1}$$

In Eq.(1), the symbol $V_r(x)$ denotes the reference volume and is calculated as

$$V_r(x) = \int_V \alpha(x,y)dy \qquad \text{in } V$$

so that a homogeneous local equivalent stress and its non-local version coincide. Perhaps the most used weight function $a(x,y)$ is the Gauss function

$$\alpha(s) = e^{-k^2 s^2 / l^2} \tag{2}$$

where s denotes the distance from the point x, and $s = \|x-y\|$, the positive parameter l represents an intrinsic characteristic length of the model. The weight function $a(x,y)$ attains its maximum at the point x and decays to zero for increasing distances from x, vanishing when $s \approx (2-3)\,l$. After developing in Taylor series the argument of the integral and its subsequent integration over a symmetric domain, the non local field can be approximated as

$$\bar{\zeta}(x) \cong \zeta(x) + c\nabla^2\zeta(x) \qquad \text{in } V \tag{3}$$

In Eq.(2), the positive real constant c represents the square of a diffusive length. As shown by Bazant and Pijaudier-Cabot (1987)[10], the parameter c can be related to the size of the integration domain and to the intrinsic characteristic length l of the starting weight-function. For instance, by assuming the Gauss weight function (2), let us impose the condition that the integral of the weight-function $a(x,y)$ over an infinite body be equal to the volume of the hyper-sphere of radius $l/2$ (the line of length l in the one-dimensional case, the area of circle $pl^2/4$ in the two-dimensional case, the volume of the hyper-sphere $pl^3/6$ in the three-dimensional case). In particular, one gets:

- in the one-dimensional case, assuming $k = \sqrt{\pi}$, $c = l^2/4\pi$,
- in the two-dimensional case, assuming $k = 2$, $c = l^2/16$,
- in the three-dimensional case, assuming $k = \sqrt[3]{6\sqrt{\pi}}$, $c = l^2/(4\sqrt[3]{36\pi})$.

In Eq.(4), the Laplacian term is applied to the local field z which is assumed to be known. The non-local equivalent stress $\bar{z}$ is obtained by solving a differential equation of explicit type. Furthermore, by differentiating two times both sides of Eq. (3), we get

$$\nabla^2 \bar{\zeta}(x) \cong \nabla^2 \zeta(x) + c\nabla^4 \zeta(x) \qquad (4)$$

By replacing Eq.(4) in Eq.(3), and neglecting the fourth-order derivative of the local equivalent stress, the approximation

$$\bar{\zeta}(x) \cong \zeta(x) + c\nabla^2 \bar{\zeta}(x) \quad \text{in} \quad V \qquad (5)$$

is obtained. In Eq. (5), the Laplacian term is applied to the non-local equivalent stress, thus the latter can be obtained by solving a differential equation of implicit type. Because Eq.(5) does not contain the Laplacian of the local variable, regularity of lower order on the local variable is required. Neumann's boundary conditions of the type $\nabla \bar{z} \cdot \mathbf{n} = 0$ are usually imposed, where **n** denotes the normal to the surface of V (de Borst and Mühlhaus[7]; 1992; Peerlings et al., 1996[9]). This boundary condition can be alternatively derived from variational statements (Benvenuti[11] et al., 2004). One advantage of the implicit gradient model with respect to integral models is that the former can be more easily implemented in a finite element code, because it involves quantities which have to be evaluated at the actual point.

3. ONE-DIMENSIONAL ANALYTICAL SOLUTION

In order to verify the results of this approach, let us consider a one-dimensional singular stress field:

$$\zeta(x) = \frac{K_0}{\sqrt{x}} \qquad (6)$$

in the domain $x \in (0, R]$ (see figure 1). In this particular case, the differential equation (5) becomes:

$$c\frac{d^2 \bar{\zeta}(x)}{dx^2} - \bar{\zeta}(x) = -\zeta(x) \qquad (7)$$

In this paper the problem is assumed to obey the Neumann's boundary condition:

$$\left.\frac{d\bar{\zeta}(x)}{dx}\right|_{\substack{x=0\\x=R}} = 0 \qquad (8)$$

The general solution of eq. (7) is searched as the series expansion:

$$\bar{\zeta}(x) = \sum_n c_n \cos\frac{n\pi x}{R} \qquad (9)$$

By substituting eq. (9) into eq. (7) and by using the Galerkin method, we obtain:

$$c_n = 4\ R\ n\ \pi\ K_0\ \frac{\sqrt{R} + \sum_{k=0}^{\infty} R^{0.5} (-1)^{k+1} \dfrac{\pi^{2k+2}\ n^{2k+2}}{(4k+5)(2k+2)!}}{c\ n^2 \pi^2 \left(\cos(n\pi)\sin(n\pi) + n\pi\right) + R^2 \left(n\pi + \cos(n\pi)\sin(n\pi)\right)} \qquad (10)$$

for n≠0 and

$$c_0 = \frac{2\,K_0}{\sqrt{R}} + \frac{2\,K_0}{\sqrt{R}} \sum_{k=0}^{\infty} (-1)^{K+1} \frac{\pi^{2K+2}}{(4k+5)(2k+2)!} \tag{11}$$

for n=0. Figure 2 shows the trend of the analytical solution (7). It can be shown that the limit for x tending to zero of the solution (7) is its maximum value:

$$\bar{\zeta}_{max}(x=0) = \sum_{n=0}^{\infty} c_n \tag{12}$$

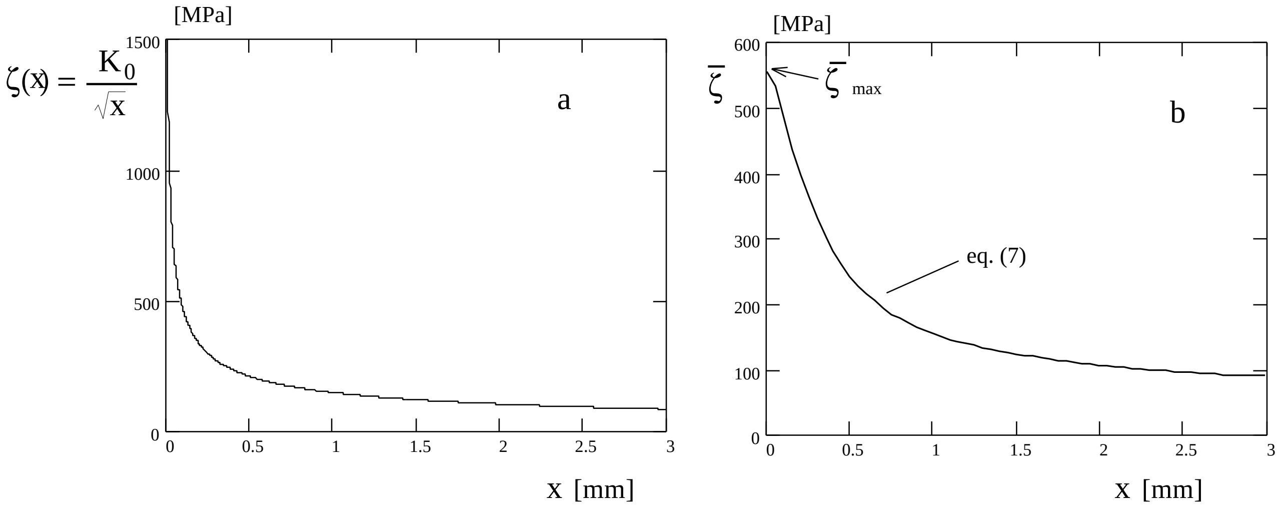


Figure 1: a) one-dimensional stress field K_0=150 MPa mm $^{0.5}$; b) analytical solution (7), c=0.04 mm^2

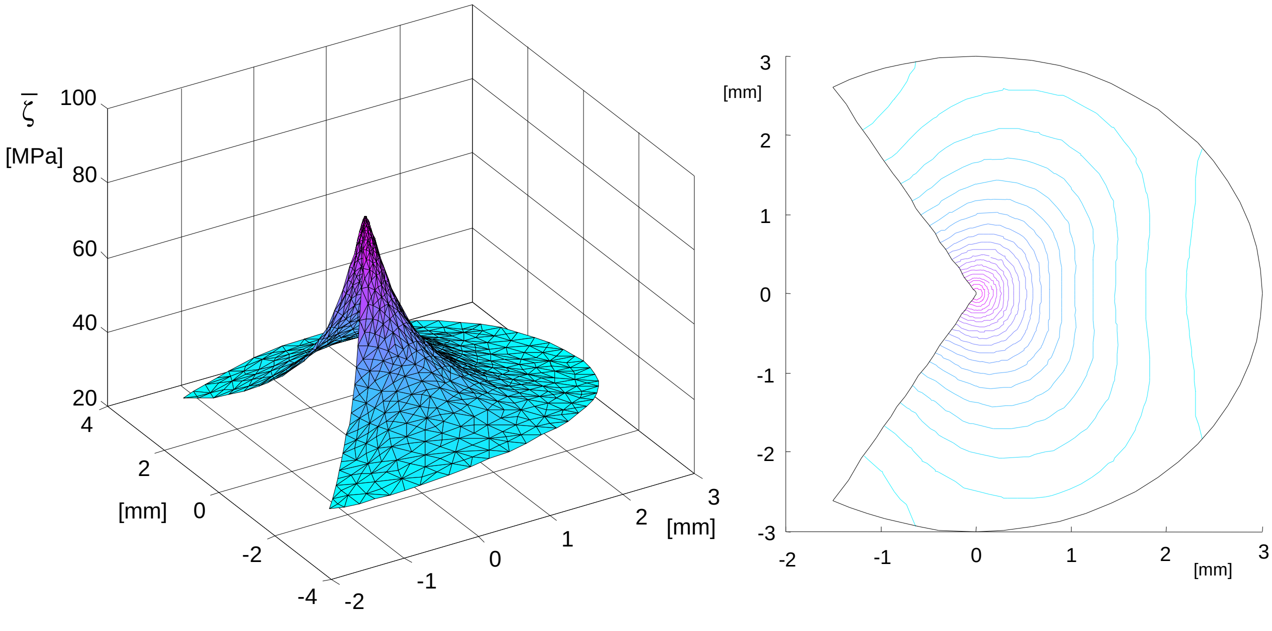
4. NUMERICAL SOLUTION

In order to obtain the solution of eq. (5) in a two-dimensional domain with geometric discontinuity (a sharp open corner), the Finite Element technique is adopted. The local equivalent stress is calculated by assuming the analytical Williams'[12] stress field and by imposing Neumann's boundary conditions. For simplicity, in this contribution, we report only the results obtained by imposing the local equivalent stress equal to the maximum principal stress. The *c* parameter, related to the diffusive length, is obtained in the case of a crack under a mode I loading. In this case failure occurs when the maximum non-local equivalent stress $\bar{\zeta}$ reaches the tensile strength of the material. Consequently, the knowledge of the two static parameters Fracture Toughness K_{IC} and ultimate strength σ_{ts} make it possible evaluating *c*.
Figure 2 shows a typical solution of the non-local equivalent obtained in a circular domain for an opening angle of 120° degrees.

5. COMPARISON BETWEEN NUMERICAL AND EXPERIMENTAL RESULTS

In order to estimate the failure load of V-shaped notches specimens, in the finite element solution, the load is increased until the maximum equivalent non-local stress reaches the ultimate strength σ_{ts}. It is worth noting that the non-local equivalent stress has been obtained over the entire spatial domain without assuming a priori the position of the critical points.

The predictions of the failure loads of two series of brittle specimen made in PMMA material are reported in Figures 3-4. A sound agreement is achieved between the experimental failure loads and the numerical predictions.



a) b)

Figure 2: typical finite element solution of eq. (5) in a circular domain of radius R=3 mm (c=0.03 mm^2) with an opening angle 2α=120° under symmetric load conditions: a) mesh and 3D surface solution; b): integration domain and contour levels of the solution

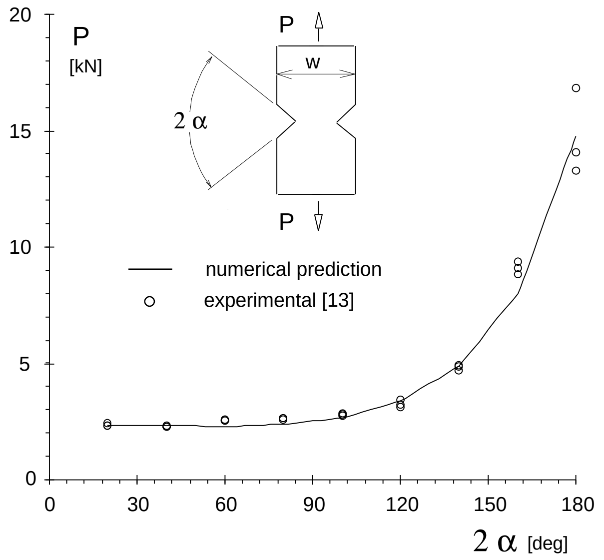


Figure 3: comparison between the experimental and the FEM-predicted failure loads of V-shaped notches under traction[13] (w=109 mm, thickness 4 mm, notch depth 27 mm, K_{IC}=58.9 MPa $mm^{0.5}$, σ_{ts} = 105 MPa, , c= 0.0030 mm^2)

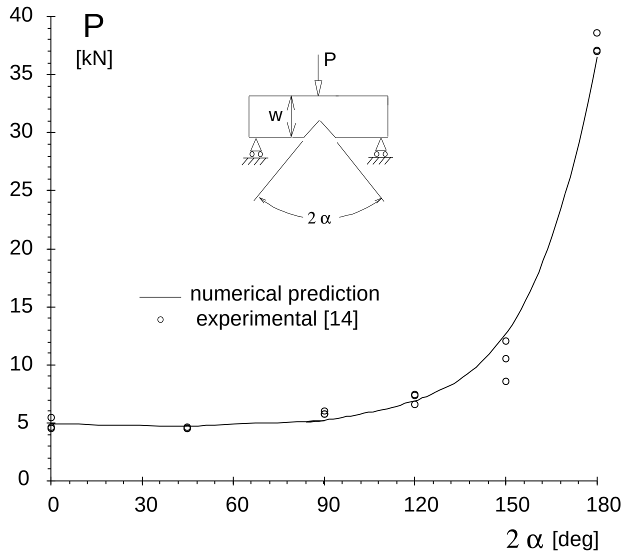


Figure 4: comparison between the experimental and the FEM-predicted failure loads of V-shaped notches under bending[14] (w=50 mm, thickness 50 mm, notch depth 27 mm, beam length 190 mm, K_{IC}=61.3 MPa $mm^{0.5}$, σ_{ts} = 130 MPa, c= 0.00143 mm^2)

6. CONCLUSIONS

A non-local equivalent stress has been obtained by an implicit gradient approach. While the existing approaches evaluate the non-local equivalent stress at predefined points, here, the field of the non-local equivalent stress has been obtained over the entire spatial domain without assuming a priori the position of the critical points.

The strength evaluated through the proposed non-local criterion is compared with the experimental tensile strength. In particular, a sound agreement has been achieved between the present predictions and the experimental failure loads of brittle specimens made in PMMA material. In this contribution, the maximum principal stress definition was assumed as local equivalent stress. The influence of other definitions of local equivalent stress will be discussed in forthcoming papers. Moreover, the analytical solution determined for a one-dimensional singular stress has been presented.

7. REFERENCES

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