IN MODELLING THE DYNAMIC BEHAVIOUR OF CRACKED PLANE STRUCTURES

L. Nobile[1], P. Ricci[1], E. Viola[1]

[1]DISTART, University of Bologna, Viale Risorgimento 2-40136 Bologna - Italy
email: erasmo.viola@mail.ing.unibo.it

ABSTRACT

In this paper, a finite element model for cracked plane structures, under bending moment, axial and shear forces, is formulated, by employing Euler-Bernoulli and Timoshenko theories. The vibration characteristics of the structures with a single edge crack is investigated using a modified line-spring model and a cracked finite element. For the two different methods considered, the stiffness matrix for a zero-length cracked element and for a l-length cracked element having two nodes and three degrees of freedom is derived, starting from an integration of stress intensity factors. A parametric study of a transverse open crack is carried out for various crack depths and crack locations using the two different theories.

INTRODUCTION

As well known, a crack in a structure introduces a local flexibility which is a function of the crack depth. This flexibility changes the stiffness and the dynamic behaviour of the structure. Consequently their static, dynamic and stability behaviour is altered. The local flexibility of the cracked region of the structural element was put into relation with the SIFs. One of the objectives of this paper is to determine the vibration characteristics of plane structures with a single edge crack, under bending moment, axial and shear forces, by employing Euler-Bernoulli and Timoshenko theories and by using a modified line-spring model and a cracked finite element. The line-spring model has the features of having two nodes and zero length, the cracked finite element, two nodes and l-length. The stiffness matrix is derived starting from an integration of stress intensity factors. Numerical and graphical results for the conventional Euler-Bernoulli and Timoshenko plane structures, using a modified line-spring model and a cracked finite element, are presented and compared. The equation of motion of the Timosenko model includes translational and rotatory mass matrices. A parametric study of a transverse open crack has been carried out for various crack depths and crack locations.

CRACKED ELEMENT MODEL: THE STIFFNESS MATRIX

For general loading, a local stiffness matrix relates forces to displacements. In this analysis, rotational and translational crack compliance are assumed in the local flexibility matrix. So, bending, shear and axial effects is included.
When a crack is introduced to the structure, additional strain energy induced by the crack should be added to the above strain energy to give the total strain energy of the

cracked structure. Consider a crack in a beam-type structure. The work for crack formation is expressed as (Tada et al., 1973)

$$W^{(1)} = w \int_0^a \left[\frac{\left(K_I^2 + K_{II}^2\right)}{E'} + (1+v) \frac{K_{III}^2}{E} \right] da \quad (1)$$

where $E' = E/\left(1 - v^2\right)$ for plane strain, $E' = E$ for plane stress, v is the Poisson's ratio and K_I, K_{II}, K_{III} are the crack tip stress intensity factors for opening mode, sliding mode and tearing mode of crack surface displacements, respectively. Since the cracked element is subjected to three loading force system applied simultaneously, the total K is the algebric sum of K values for each system applied separately. With the actions of bending moment M, shear force T and axial force P, equation (1) can be expressed as a function of the following stress intensity factors, given by [5]: $K_{IM} = \sigma_M \sqrt{\pi a} F_{IM}$, $K_{IT} = \sigma_T \sqrt{\pi a} F_{IT}$, $K_{IP} = \sigma_P \sqrt{\pi a} F_{IP}$, $K_{IIT} = \tau \sqrt{\pi a} F_{II}$ and $F_{IM}, F_{IT}, F_{IP}, F_{II}$ are evaluated by Brown and Srawley [8] and by Tharp [3]. Substituting the above expressions into Eq. (2) and assuming

$$R_I = \int_0^a a F_I^2 da , \quad R_{II} = \int_0^a a F_{II}^2 da , \quad Q = \int_0^a a F_{IP}^2 da , \quad Z = \int_0^a a F_I F_{IP} da , \quad n = \frac{36\pi}{E' w b^2} , \quad m = \frac{\pi}{E' w b^2} \quad (2)$$

the strain energy due to the crack becomes

$$W_c = n \left(M + Tl/2\right)^2 R_I + m P^2 Q + \frac{6m}{b} \left(2MP + TPl\right) Z + m T^2 R_{II} \quad (3)$$

The stiffness matrix for the cracked element may be derived as follow:

$$\mathbf{k}^c = \begin{bmatrix} k_{11}^c & 0 & k_{13}^c & k_{14}^c & 0 & k_{16}^c \\ 0 & k_{22}^c & k_{23}^c & 0 & k_{25}^c & k_{26}^c \\ k_{31}^c & k_{32}^c & k_{33}^c & k_{34}^c & k_{35}^c & k_{36}^c \\ k_{41}^c & 0 & k_{43}^c & k_{44}^c & 0 & k_{46}^c \\ 0 & k_{52}^c & k_{53}^c & 0 & k_{55}^c & k_{56}^c \\ k_{61}^c & k_{62}^c & k_{63}^c & k_{64}^c & k_{65}^c & k_{66}^c \end{bmatrix} \quad (4)$$

where the elements of matrix (4) are reported in [16]. By neglecting the terms Z, Q, R_I, R_{II}, the stiffness matrix for Timoshenko beam element will be derived. Moreover, by setting $\Gamma = 0$ in resultant equations, the matrix coefficients reduce to elements of the Euler-Bernoulli beam model.

MASS MATRIX OF THE CRACKED ELEMENT

The consistent translational $\boldsymbol{m_u}$ and $\boldsymbol{m_v}$ and rotational $\boldsymbol{m_\vartheta}$ mass matrices are assumed

the same both for cracked and uncracked Timoshenko beam finite element. The above matrices can be derived by using the Kinetic energy of the beam element of length l

$$\tau = \frac{1}{2}\int_0^l \mu A \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^l \mu J \left(\frac{\partial v}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^l \mu A \left(\frac{\partial \vartheta}{\partial t}\right)^2 dx \quad (5)$$

including the effects of both translational displacements u,v and rotatory inertia, respectively. In Eq. (12), μ is the mass density of the material. The explicit expressions for the elements of translational mass matrices $\boldsymbol{m_u}$ and $\boldsymbol{m_v}$ and for the rotational mass matrix $\boldsymbol{m_\vartheta}$ are given as:

$$m_{ij}^u = \int_0^l N_{ui}\mu A N_{uj}dx, \quad m_{ij}^v = \int_0^l N_{vi}\mu A N_{vj}dx, \quad m_{ij}^\vartheta = \int_0^l N_{\vartheta i}\mu J N_{\vartheta j}dx \quad (6)$$

respectively, for $i,j = 1,2,...,6$. In the above expressions, N_u, N_v, N_ϑ are the interpolation functions for axial displacement, transverse displacement and for the rotation of the cross-section about the positive x axis. The total mass matrix of the two mode finite element is $\boldsymbol{m} = \boldsymbol{m_u} + \boldsymbol{m_v} + \boldsymbol{m_\vartheta}$. Letting $\boldsymbol{m_t} = \boldsymbol{m_u} + \boldsymbol{m_v}$ where $\boldsymbol{m_t}$ is the total translational mass matrix, $\boldsymbol{m} = \boldsymbol{m_t} + \boldsymbol{m_\vartheta}$. Carrying out the integration over the beam-length l, the well known mass matrices $\boldsymbol{m_t}$ and $\boldsymbol{m_\vartheta}$ can be derived [16].

The above matrices depend upon Γ: if the shear deformation parameter Γ is set equal to zero and the rotatory inertia mass matrix $\boldsymbol{m_\vartheta}$ is omitted, the resulting model is identical to the classical Euler-Bernoulli beam model.

LINE SPRING MODEL STIFFNESS MATRIX

The compliance expression for a cracked element may be derived according to the theory presented by Okamura et al. [1], which is based on the relationship between load and deflection. The bending moment M, the axial force P and the shearing force T can be related to the rotation ϑ, the axial extension u and the deflection v as follows

$$\lambda_{pp} = \frac{u}{P} \quad , \quad \lambda_{mm} = \frac{\vartheta}{M} \quad , \quad \lambda_{tt} = \frac{v}{T} \quad (7)$$

where $\lambda_{pp}, \lambda_{mm}, \lambda_{tt}$ are compliance expressions for bending , extension and shear, respectively. Moreover, compliances are related to the energy release rate G and the stress intensity factors by the relations

$$G_p = \frac{1-v^2}{E}K_{IP}^2 = \frac{P^2}{2}\frac{d\lambda_{pp}}{dA}, \quad G_m = \frac{1-v^2}{E}K_{IM}^2 = \frac{M^2}{2}\frac{d\lambda_{mm}}{dA}, \quad G_t = \frac{1-v^2}{E}K_{IIT}^2 = \frac{T^2}{2}\frac{d\lambda_{tt}}{dA} \quad (8)$$

given by Irwin and Kies [14]. In eqs. (8), K_{IP} and K_{IM} are the mode I stress intensity contributions caused by the axial load P and the bending moment M respectively, and

K_{IIT} is the mode II stress intensity caused by the shear force T. E is the Young's modulus, v equals the Poisson's ratio for plane strain and is zero for plane stress, and dA is an infinitesimal increment of crack area equal to bda, where b is the beam depth and a is the crack length. The stiffness matrix of a line-spring is obtained by the following relation :

$$\lambda_{pp} = \frac{2(1-v^2)}{E}\int_0^A \left(\frac{K_{IP}}{P}\right)^2 dA \, , \quad \lambda_{mm} = \frac{2(1-v^2)}{E}\int_0^A \left(\frac{K_{IM}}{M}\right)^2 dA \quad (9)$$

$$\lambda_{tt} = \frac{2(1-v^2)}{E}\int_0^A \left(\frac{K_{II}}{T}\right)^2 dA \, , \quad \lambda_{mp} = \frac{2(1-v^2)}{E}\int_0^A \left(\frac{K_{IP}}{P}\right)\left(\frac{K_{IM}}{M}\right) dA$$

where λ_{mp} is the compliance for the coupling of bending and extension. The stiffness matrix k_f referring to a cracked element of a beam with a rectilinear axis is derived from the stiffness matrix for a curved cracked element reported in [4] as follows:

$$k_f = \begin{bmatrix} \lambda_{mm}/D & 0 & -\lambda_{mp}/D & -\lambda_{mm}/D & 0 & \lambda_{mp}/D \\ 0 & 1/\lambda_{tt} & 0 & 0 & -1/\lambda_{tt} & 0 \\ -\lambda_{mp}/D & 0 & \lambda_{pp}/D & \lambda_{mp}/D & 0 & -\lambda_{pp}/D \\ -\lambda_{mm}/D & 0 & \lambda_{mp}/D & \lambda_{mm}/D & 0 & -\lambda_{mp}/D \\ 0 & -1/\lambda_{tt} & 0 & 0 & 1/\lambda_{tt} & 0 \\ \lambda_{mp}/D & 0 & -\lambda_{pp}/D & -\lambda_{mp}/D & 0 & \lambda_{pp}/D \end{bmatrix} \quad (10)$$

where $D = \lambda_{pp}\lambda_{mm} - \lambda_{mp}^2$ and $\lambda_{mt} = 0$, $\lambda_{pt} = 0$ are compliances for the coupling of bending and shearing, extension and shearing respectively.

DIFFERENTIAL EQUATION OF MOTION

By applying the standard finite element method, the differential motion equation for free vibration of the cracked structure is derived by $M\ddot{q} + (K + K_f)q = 0$, where M and K are the global consistent mass matrix and the stiffness matrix for the entire structure without cracks, respectively, and K_f is the stiffness matrix for the line spring model or for the cracked finite element expressed in the extended form, in order to incorporate the line-spring stiffness matrix k_f or the cracked finite element stiffness matrix k_c carried out above into the assembly procedure of the global stiffness matrix for the entire structure. After imposing the appropriate end conditions, if the global nodal displacement vector q is assumed to be harmonic in time with circular frequency ω, as $q = q^* \exp(i\omega t)$, the equation becomes an eigenvalue problem of the standard

form $\left(K - \omega^2 M \right) q^* = 0$, where q^* is a vector of displacement amplitudes of vibration. The solution of the above eigenvalue problem yields the natural frequencies and the corresponding mode shapes of the cracked structure, which depend on the crack position, the crack size, the geometric dimensions of the structure, the boundary conditions and mechanical parameters of the material.

NUMERICAL ANALYSIS AND RESULTS

Consider a Timoshenko cantilever beam, by allowing for the effects of transverse shear and rotatory inertia. When the shear deformation parameter Γ is set equal to zero and the rotatory inertia mass matrix is omitted, the resulting model is identical to the classical Euler-Bernoulli beam. The beam element consists of two nodes, i and $i+1$; each node has the degrees of freedom of transverse displacement v^e and bending rotation φ^e.

Calculation in these examples are carried out for the following beam data: length $L=1$ m, Young's modulus $E=3.1 \times 10^5$ MPa, Poisson's ratio $\nu = 1/3$ (or $G/E = 3/8$), mass density $\rho = 2500$ kg/m^3, shear coefficient $\chi = 1.5$. Two different cross-sections of the beam are considered: 0.05 x 0.05 m and 0.05 x 0.10 m. The numerical results are expressed in terms of the following dimensionless parameters:

$f_n^* = f_n / f_{n0}$ is the frequency ratio, where f_n is the *nth* computed natural frequency of the cracked structure and f_{n0} is the *nth* exact natural frequency of the corresponding uncracked structure, $d = s/L$ is the dimensionless crack position parameter and $\xi = a/b$ the crack depth ratio.

The first three frequencies of a cantilever beam discretized into 20 finite elements, are shown in Figures 1(a,b,c,d,e,f) for two dimensionless crack depths and different crack positions, for the Euler-Bernoulli and Timoshenko beams. From Figs. 1 (a,b,c,d,e,f), it appears that the effect of crack depths on the frequencies increases when deeper cracks are considered and that the effect of the crack position on the frequencies increases when closer cracks to the fixed end are considered. In the case of the Euler-Bernoulli beam, the frequency parameters for the beam are higher than those for the Timoshenko beam. Moreover, the difference between the Euler-Bernoulli and Timoshenko frequencies, for the cross-section equals to 0.05 x 0.10 m, is higher than those for the cross-section equals to 0.05 x 0.05 m. In Figs. 1, the frequencies are calculated for a cracked beam with a single cracked finite element. Figs.2 (a,b,c,d,e,f) and Figs.3 (a,b,c,d,e,f), give a comparison between the finite cracked element and the line-spring model for Bernoulli and Timoshenko cantilever beams, respectively, for the first three dimensionless frequency ratios f_n^*, as a function of two different dimensionless crack depths, $\xi = a/b = 0.25$ and $\xi = a/b = 0.5$, and two beam cross-sections equals to 0.05 x 0.05 m and 0.05 x 0.10 m, respectively. For $\xi = a/b = 0.25$, the line-spring results in terms of the first three frequency ratios f_n^*, are only slightly different than those of the cracked finite element, but this difference becomes bigger when $\xi = a/b = 0.5$. Moreover, when the element length increases, the difference of results between the line-spring and the cracked finite element becomes smaller, as shown in Figs. 4 (a,b,c,d,e,f).
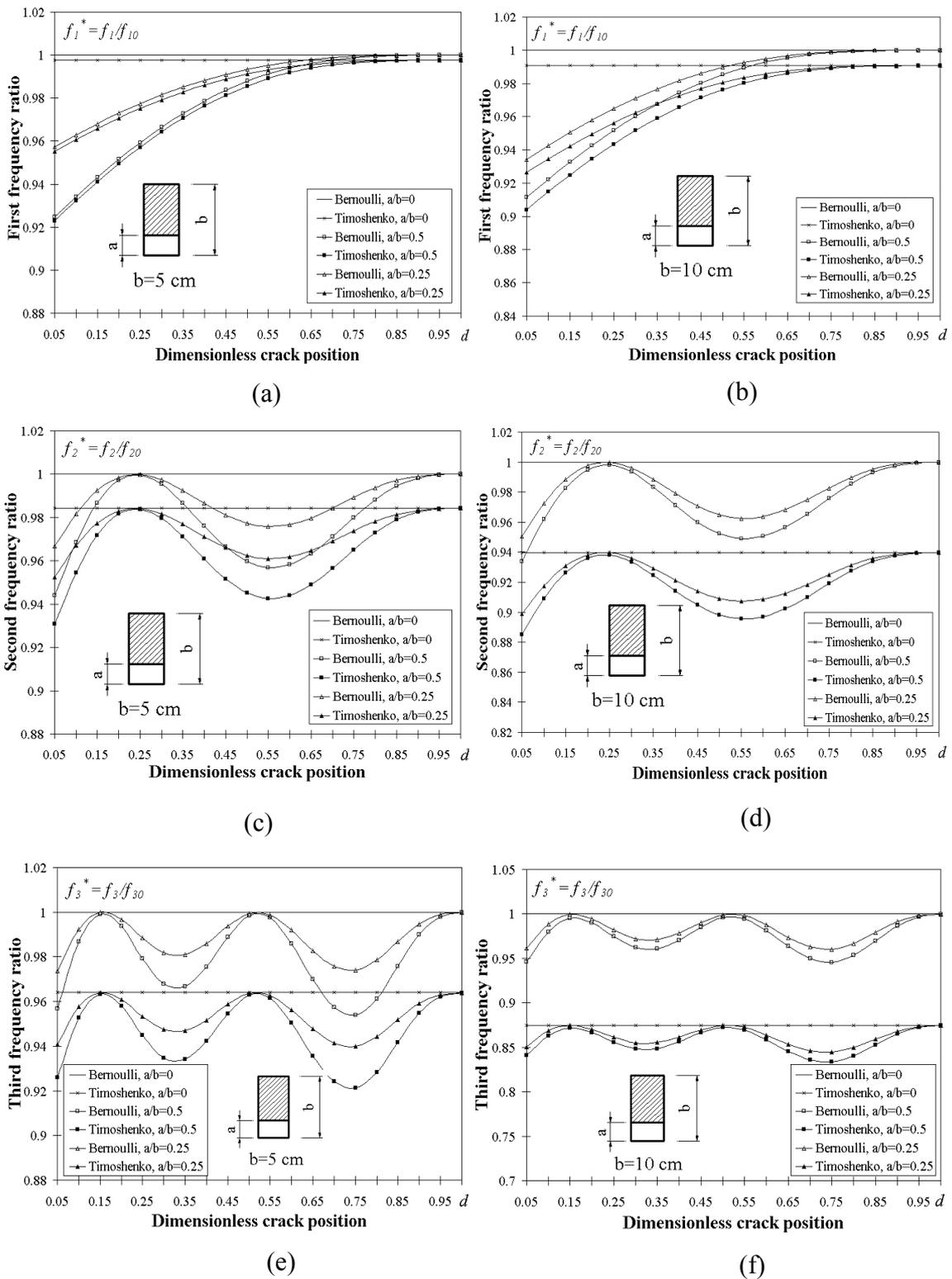
Fig.1. First three dimensionless frequencies f_n^* for the Bernoulli and Timoshenko cantilever beams as a function of the dimensionless crack location d.
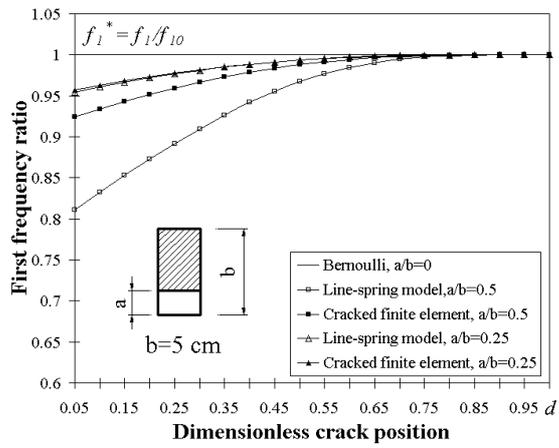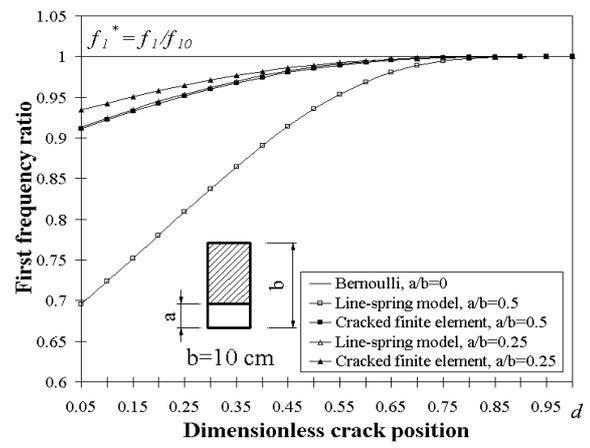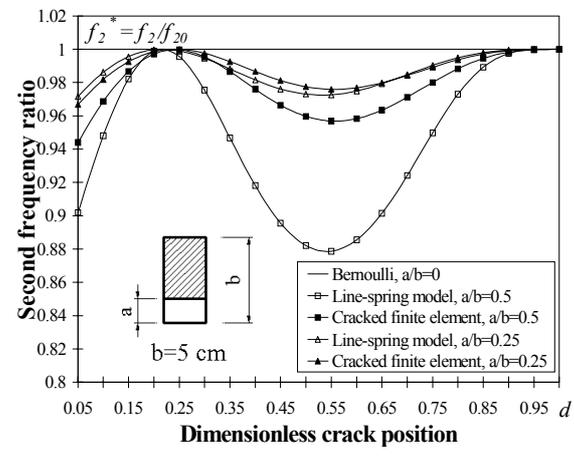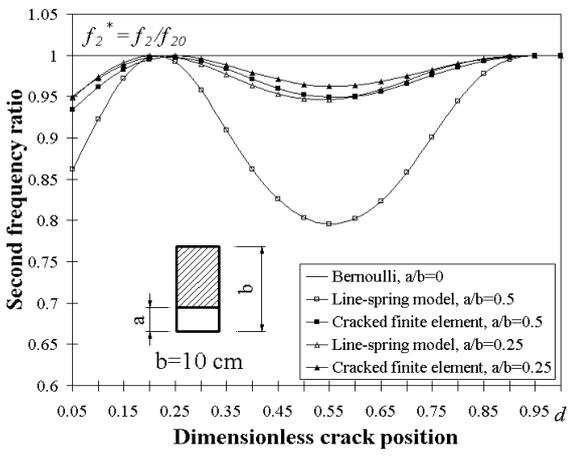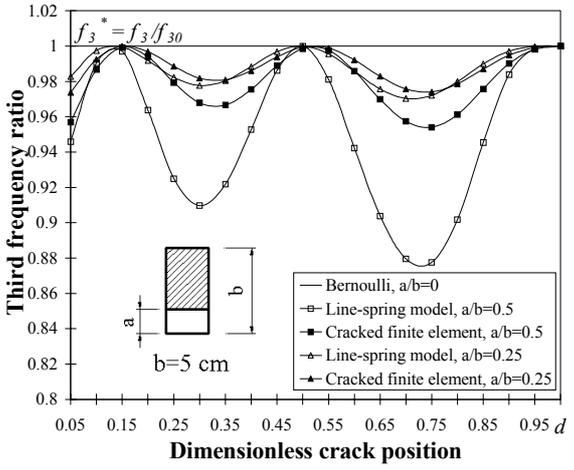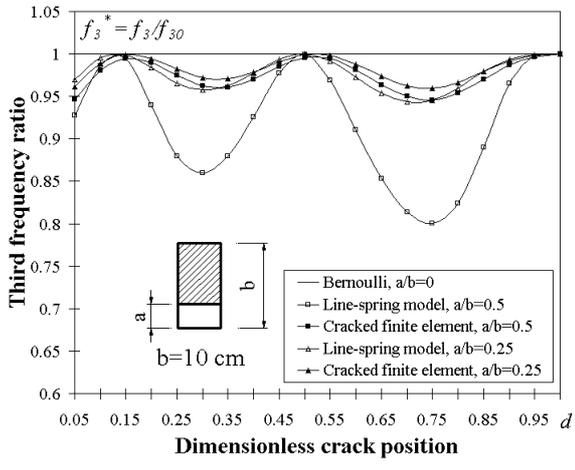
Fig.2. First three dimensionless frequencies f_n^* *for the Bernoulli cantilever beam using the line-spring model and a cracked finite element.*
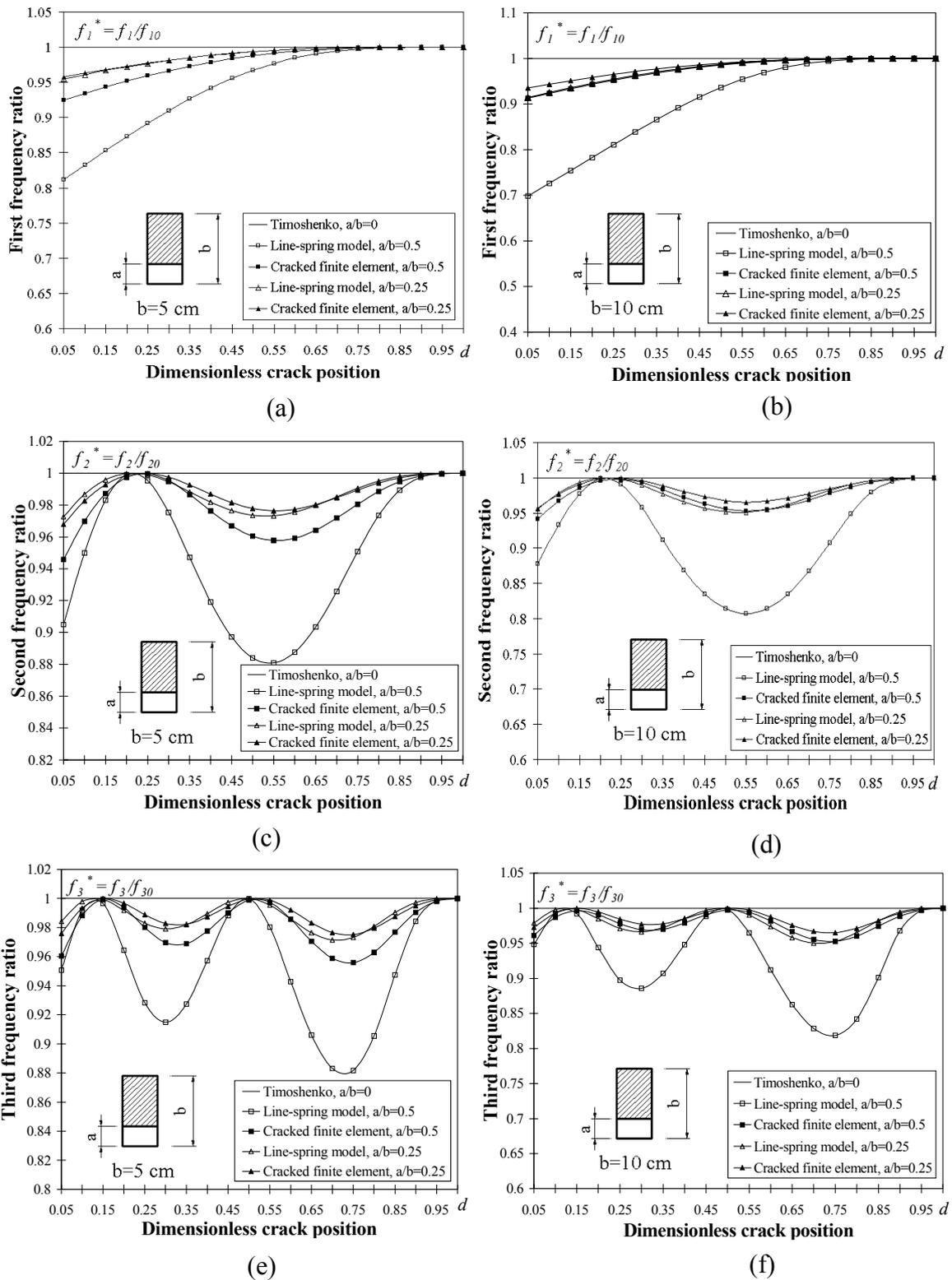
Fig.3. First three dimensionless frequencies f_n^* for the Timoshenko cantilever beam using the line-spring model and a cracked finite element.
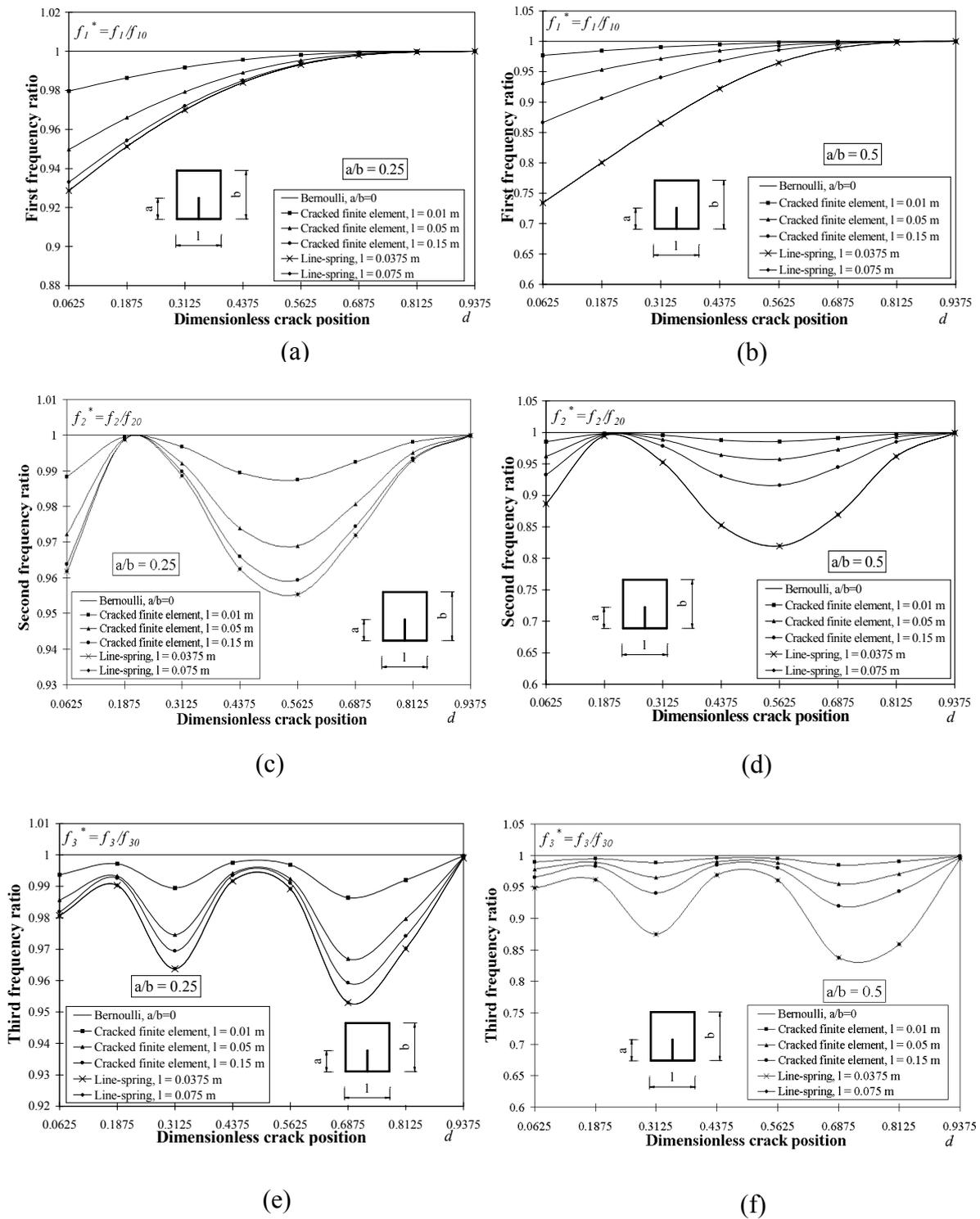
Fig.4.First three dimensionless frequencies f_n^* for the Bernoulli cantilever beam as a function of the length of the cracked element.

Calculation in Figs. 4(a,b,c,d,e,f), are carried out for the following Bernoulli beam data: length L =1.20 m, Young's modulus E =3.1 x 10^5 MPa, Poisson's ratio $v = 1/3$ (or G/E =3/8), mass density ρ = 2500 kg/m³, shear coefficient χ = 1.5. The cross-section of the beam is equal to 0.05 x 0.10 m and two different dimensionless crack depths, $\xi = a/b = 0.25$ and $\xi = a/b = 0.5$, are considered. The beam was divided into 120, 24 and 8 elements, respectively, for the cracked finite element model, and into 32 and 16 elements, respectively, for the line-spring model: in this way the crack position is the same for the two different models.

Specifically, as the cracked finite element frequencies f_n^* getting smaller, when the beam elements become bigger, the line-spring frequencies f_n^* for the first three frequency ratios, are nearly constant with the length of the element, as shown in the above Figures 4, for the first three frequency ratios f_n^* of a Bernoulli beam: it appears that the line-spring models is only slightly affect by the element length.

REFERENCES

[1] Okamura H., Watanabe K. and Takano T.*, "Deformation and strength of cracked member under bending moment and axial forces"*, Engineering Fracture Mechanics 1975, **7** 531-539.
[2] Yokoyama T., Chen M.C., "*Vibration analysis of edge-cracked beams using a line-spring model*", Engineering Fracture Mechanics 1998, **59** 403-409.
[3] Tharp T.H., "*A finite element for edge-cracked beam columns*", Internation journal for Numerical Methods in Engineering 1987, **24** 1941-1950.
[4] Nobile L., Ricci P., Viola E., "*Matrice di rigidezza di un elemento fessurato con tutti i termini di accoppiamento*", Atti del XV Convegno Nazionale del Gruppo Italiano Frattura 2000, 317-326.
[5] Kosmata J.B., "*An improved two-nodes finite element for stability and natural frequencies of axial loaded Timoshenko beams*", Computers and Structures 1995, **57** (1) 141-149.
[6] Yokoyama T., "*Vibration of Timoshenko columns beam on two-parameter elastic foundations*". Earthquake Engineering Structural Dynamics 1991, **20** 355-370.
[7] Nobile L., "*Mixed mode crack initiation and direction in beams with edge crack*", Theoretical and Applied Fracture Mechanics 2000, **33** 107-116.
[8] Brown W. F. and Srawley J. E., "*Plane Strain Crack Toughness Testing of High Strength Metallic Materials*", ASTM Special Technical Publication 513, American Society for Testing and Materials, Philadelphia, 1966.
[9] Miyazaki N. "*An application of line-spring model to a transient analysis of the dynamic stress intensity factor*", International Journal Fracture 1989; **39** R77-R80.
[10] Broek D. Elementary Engineering Fracture Mechanics, Delf University of Technology, Noorhoff International Publishing Leiden, 1974.
[11] Viola E., "*Valutazione della forma e dimensione della zona plastica al tip di un crack*", Tecnica Italiana Gennaio-Febbraio 1980; **1** 45-56.
[12] Viola E, Nobile L. "*Formulation of a cracked beam element for structural analysis*", Journal Engineering Mechanics- ASCE 2002, **128** 220-230.
[13] Viola E, Federici L, Nobile L. "*Detection od crack location using cracked beam element method for structural analysis*", Theoretical and Applied Fracture Mechanics 2001, **36** 23-35
[14] Irwin G R, Kies J A. "*Critical energy rate analysis of fracture strength of large welded structures*", Welding Journal 1954; **33** 193-198.
[15] Nobile L.,Viola E, "*Cracked beam formulation for structural analysis*" , SMiRT 16 –Washington, DC-USA 12-17 August 2001.
[16] Ricci P., Viola E., "*The dynamic behaviour of cracked plane structures*", Nota tecnica n. 80-DISTART – Scienza delle Costruzioni, Facoltà di Ingegneria di Bologna, Maggio 2002.