Mode-I and Mode II elastodynamic analysis of intersonic crack propagation in an orthotropic medium

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Abstract

In the present work, the problem of an infinite orthotropic body with a semi- infinite line crack propagating at constant velocity in the intersonic regime is analyzed.

Sommario

In questa memoria si esamina il problema di una fessura rettilinea che si propaga in un materiale ortrotropo con velocità costante, in regime intersonico.

1.Introduction

The theoretical study of intersonic crack propagation, i.e. when the crack tip velocity is larger than the shear wave velocity of the material, is drawing the efforts of many investigators.
Particular attention has been devoted to study the intersonic crack propagation along bimaterial interfaces. Theoretical analyses performed by Lin et al.[1], Yu and Yang[2] and Huang et al.[3] among others, have shown that the asymptotic elastic fields for intersonic interfacial crack propagation is predominantly of a shear nature and the power of stress singularity at the crack tip is always less than one half. In addition, the above mentioned studies showed that a pure Mode-I, steady state, intersonic crack propagation is impossible because the energy release rate takes an unbounded negative value.
Recently, the theoretical investigation of intersonic crack propagation has been extended to orthotropic materials as well as to unidirectional fiber reinforced composites. Piva an Hasan[4] developed the singular asymptotic analysis of intersonic crack growth in orthotropic materials and Huang et al. [5] extended the analysis to unidirectional fiber reinforced composites, modelled as orthotropic materials.
In the present work, the problem of an infinite orthotropic body with a semi- infinite line crack propagating at constant velocity in the intersonic regime is analyzed.
The basic analysis is performed by using an approach which differs from those used in the above mentioned papers. In particular, the local elastic fields may be obtained by assuming a separated variables scheme for the displacement field which allows solving the elastic problem without the use of the complex variable technique. The asymptotic near-tip

expressions of the elastic fields may be obtained for Mode-I and Mode-II intersonic crack propagation.

2. Foundation

Consider an infinite orthotropic elastic medium with the axes of elastic symmetry coinciding with the axes of a Cartesian coordinate system $O\,(X, Y, Z)$.
By assuming plane stress conditions, the system of equations of motion governing elastodynamic problems in the X-Y plane are:

$$c_{11}\, u_{XX} + c_{66}\, u_{YY} + (c_{12} + c_{66})\, v_{XY} = \rho\, u_{tt}\ ,\quad c_{66}\, v_{XX} + c_{22}\, v_{YY} + (c_{12} + c_{66})\, u_{XY} = \rho\, v_{tt} \qquad (2.1a,b)$$

in which $u = u(X,Y,t)$, $v=v(X,Y,t)$ are the displacement components in X and Y directions respectively, t is the time, ρ is the mass density of the material and c_{ij} are the elastic coefficients. The in-plane stress-strain equations may be written as follows:

$$\sigma_{XX} = c_{11}\, u_X + c_{12}\, v_Y\ ,\quad \sigma_{YY} = c_{12}\, u_X + c_{22}\, v_Y\ ,\quad \tau_{XY} = c_{66}\,(u_Y + v_X) \qquad (2.2a,b,c)$$

By setting $x = X-ct$, $y = Y$, where c is a constant speed, eqs. (2.1a,b) become:

$$u_{xx} + \alpha\, u_{yy} + 2\beta\, v_{xy} = 0,\quad v_{xx} + \alpha_1\, v_{yy} + 2\beta_1\, u_{xy} = 0 \qquad (2.3a,b)$$

where:

$$\alpha = \frac{c_{66}}{c_{11}\left(1 - M_1^2\right)},\quad 2\beta = \frac{c_{12} + c_{66}}{c_{11}\left(1 - M_1^2\right)},\quad \alpha_1 = \frac{c_{22}}{c_{66}\left(1 - M_2^2\right)},\quad 2\beta = \frac{c_{12} + c_{66}}{c_{66}\left(1 - M_2^2\right)} \qquad (2.4a,b,c,d)$$

The quantities $M_1 = c/c_1$ and $M_2 = c/c_s$ are the Mach numbers, where $c_l = \left(c_{11}/\rho\right)^{1/2}$ and $c_s = \left(c_{66}/\rho\right)^{1/2}$ are the longitudinal wave speed and the shear wave speed of the material, respectively. According to [4] the system of equations (2.3a,b) may be rewritten as:

$$\boldsymbol{\Phi}_x + \boldsymbol{A}\boldsymbol{\Phi}_y = \boldsymbol{0} \qquad (2.5)$$

where $\boldsymbol{\Phi}$ is a 4x1 matrix valued function defined as:

$$\boldsymbol{\Phi}^T = (\Phi_1, \Phi_2, \Phi_3, \Phi_4) = \left(u_x, u_y, v_x, v_y\right) \qquad (2.6)$$

and $\boldsymbol{A}$ is a 4x4 constant matrix, given by:

$$\boldsymbol{A} = \begin{pmatrix} 0 & \alpha & 2\beta & 0 \\ -1 & 0 & 0 & 0 \\ 2\beta_1 & 0 & 0 & \alpha_1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad (2.7)$$

The characteristic equation of (2.7) is:

$$m^4 + 2a_1 m^2 + a_2 = 0 \qquad (2.8)$$

in which:

$$2a_1 = \alpha + \alpha_1 - 4\beta\beta_1\ ,\quad a_2 = \alpha\alpha_1 \qquad (2.9)$$

In what follows the intersonic regime, $0<M_1<1$ and $M_2>1$, will be assumed so that eq.(2.8) provides the four eigenvalues $m_1=p$, $m_2=-p$, $m_3=iq$, $m_4=-iq$, with $p=\left[\left(a_1^2-a_2\right)^{1/2}-a_1\right]^{1/2}$ and $q=\left[\left(a_1^2-a_2\right)^{1/2}+a_1\right]^{1/2}$ positive numbers.

According to [4] the system (2.5) may be transformed to the following form:

$$\boldsymbol{Y}_x + \boldsymbol{B}\,\boldsymbol{Y}_y = \boldsymbol{0} \tag{2.10}$$

where:

$$\boldsymbol{F} = \boldsymbol{P}\,\boldsymbol{Y} \tag{2.11}$$

and:

$$\boldsymbol{P}=\begin{pmatrix} -\dfrac{2bp^2}{a+p^2} & -\dfrac{2bp^2}{a+p^2} & 0 & \dfrac{2bq^2}{a-q^2} \\ \dfrac{2bp}{a+p^2} & -\dfrac{2bp}{a+p^2} & \dfrac{2bq}{a-q^2} & 0 \\ -p & p & -q & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \tag{2.12a}$$

$$\boldsymbol{B}=\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}=\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & 0 & -q \\ 0 & 0 & q & 0 \end{pmatrix} \tag{2.12b}$$

Eq.(2.10) leads to the first order systems:

$$y_{1x}+p\,y_{1y}=0\ ,\ y_{2x}-p\,y_{2y}=0 \tag{2.13a,b}$$

and:

$$y_{3x}-q\,y_{4y}=0\ ,\ y_{4x}+q\,y_{3y}=0 \tag{2.14a,b}$$

It should be noted that the system (2.13) is of hyperbolic type whereas eqs. (2.14) represent a Cauchy-Reimann system.

By setting:

$$z=y_1+y_2\ ,\quad y=y_1-y_2 \tag{2.15a,b}$$

it may be shown that the above functions satisfy the following system:

$$z_x+p\,y_y=0\ ,\ y_x+p\,z_y=0 \tag{2.16a,b}$$

which leads to satisfy the same wave equation:

$$z_{xx}-p^2 z_{yy}=0\ ,\ y_{xx}-p^2 y_{yy}=0 \tag{2.17a,b}$$

Keeping in mind eqs.(2.6), (2.11), (2.12a) and (2.15a,b) the stress-strain relations (2.2) may be rewritten as:

$$\frac{\sigma_{xx}}{c_{66}}=l_1\psi_4+l_2\zeta\ ,\ \frac{\sigma_{yy}}{c_{66}}=l_3\psi_4+l_4\zeta\ ,\ \frac{\tau_{xy}}{c_{66}}=l_5\psi_3-l_6\psi \tag{2.18a,b,c}$$

where:

$$l_1=\frac{c_{12}}{c_{66}}+\frac{c_{11}}{c_{66}}\left(\frac{2\beta q^2}{\alpha-q^2}\right)\ ,\ l_2=\frac{c_{12}}{c_{66}}-\frac{c_{11}}{c_{66}}\left(\frac{2\beta p^2}{\alpha+p^2}\right),\ l_3=\frac{c_{22}}{c_{66}}+\frac{c_{12}}{c_{66}}\left(\frac{2\beta q^2}{\alpha-q^2}\right)$$

$$l_4=\frac{c_{22}}{c_{66}}-\frac{c_{12}}{c_{66}}\left(\frac{2\beta p^2}{\alpha+p^2}\right)\ ,\ l_5=q\left((\frac{2\beta}{\alpha-q^2}-1)\right),\ l_6=p\left(1-\frac{2\beta}{\alpha+p^2}\right)$$

In what follows eqs.(2.14a,b) will be integrated by referring to the system of moving polar coordinates defined by: $x=r\cos\theta$, $y=r\sin\theta$, $r>0$, $-\pi<\theta<\pi$, and assuming a separated variables representation for the displacement field, i.e.:

$$u=r^{\gamma}U(\theta)\ ,\ v=r^{\gamma}V(\theta) \tag{2.19}$$

where the exponent γ will be determined through appropriate conditions.
By using (2.6), (2.11), (2.12a) and (2.19) gives:

$$\psi_3(r,\theta)=r^{\gamma-1}h_3(\theta)\quad ,\quad \psi_4(r,\theta)=r^{\gamma-1}h_4(\theta) \tag{2.20a,b}$$

where:

$$h_3(\theta)=\frac{2k}{q}\left\{\gamma\sin\theta U(\theta)+\cos\theta U'(\theta)+\frac{2\beta}{\alpha+p^2}\left[\gamma\cos\theta V(\theta)-\sin\theta V'(\theta)\right]\right\} \tag{2.21}$$

$$h_4(\theta)=\frac{2k}{\alpha}\left\{\gamma\cos\theta\ U(\theta)-\sin\theta\ U'(\theta)+\frac{2\beta}{\alpha+p^2}\left[\gamma\sin\theta\ V(\theta)+\cos\theta\ V'(\theta)\right]\right\} \tag{2.22}$$

and $k=\dfrac{(\alpha+p^2)(\alpha-q^2)}{4\beta(p^2+q^2)}$.

3. The general solution of the Cauchy-Riemann system

The general solution to the Cauchy-Riemann system (2.14) may be obtained by introducing the following polar-coordinates transformation:

$$r_1=r[g(\theta)]^{1/2}\quad ,\quad \theta_1=tg^{-1}\left(\frac{tg\theta}{q}\right) \tag{3.1a,b}$$

with $g(\theta)=\cos^2\theta+(\sin^2\theta)/q^2$. In view of (3.1) the functions (2.20) become:

$$\gamma_3(r_1,\theta_1)=r_1^{\gamma-1}H_3(\theta_1)\ ,\ \gamma_4(r_1,\theta_1)=r_1^{\lambda-1}H_4(\theta_1) \tag{3.2a,b}$$

where:

$$H_3(\theta_1)=g[\theta(\theta_1)]^{\frac{(1-\gamma)}{2}}h_3[\theta(\theta_1)]\ ,\ H_4(\theta_1)=g[\theta(\theta_1)]^{\frac{(1-\gamma)}{2}}h_4[\theta(\theta_1)] \tag{3.3a,b}$$

By using the rule of differentiation:

$$\frac{\partial}{\partial x}=\cos q_1\frac{\partial}{\partial r_1}-\frac{\sin q_1}{r_1}\frac{\partial}{\partial q_1}\quad ,\quad \frac{\partial}{\partial x}=\frac{\sin q_1}{q}\frac{\partial}{\partial r_1}+\frac{\cos q_1}{q r_1}\frac{\partial}{\partial q_1} \tag{3.4a,b}$$

Eqs.(2.14) lead to the following system in the unknown $H_3(q_1)$ and $H_4(q_1)$:

$$\begin{pmatrix} \sin q_1 & \cos q_1 \\ -\cos q_1 & \sin q_1 \end{pmatrix}\begin{pmatrix} H_3'(q_1) \\ H_4'(q_1) \end{pmatrix}=(g-1)\begin{pmatrix} \cos q_1 & -\sin q_1 \\ \sin q_1 & \cos q_1 \end{pmatrix}\begin{pmatrix} H_3(q_1) \\ H_4(q_1) \end{pmatrix} \tag{3.5}$$

which may be reduced to the system with constant coefficients:

$$\begin{pmatrix} H_3'(q_1) \\ H_4'(q_1) \end{pmatrix}=\begin{pmatrix} 0 & -(g-1) \\ (g-1) & 0 \end{pmatrix}\begin{pmatrix} H_3(q_1) \\ H_4(q_1) \end{pmatrix} \tag{3.6}$$

By noting that the matrix: $\boldsymbol{A}=\begin{pmatrix} 0 & -(g-1) \\ (g-1) & 0 \end{pmatrix}$ has complex coniugates eigenvalues $l_1=i|g-1|$, $l_2=\overline{l}$, one obtains the general solution to (3.6) in the following form:

$$H_3(q_1)=c_1\cos(g-1)q_1+c_2\sin|g-1|q_1\quad H_4(q_1)=e(g)\left[c_1\sin|g-1|q_1-c_2\cos(g-1)q_1\right] \tag{3.7a,b}$$

where c_1 and c_2 are arbitrary constants, and $e(g)=\operatorname{sgn}(g-1)$.
Thence, the required general solution to eqs. (2.14) is:

$$y_3=r_1^{g-1}\left[c_1\cos(g-1)q_1(q)+c_2\sin|g-1|q_1(q)\right] \tag{3.8}$$

$$y_4=e r_1^{g-1}\left[c_1\sin|g-1|q_1(q)-c_2\cos(g-1)q_1(q)\right] \tag{3.9}$$

in which $q_1(q)$ is defined by (3.1b).

4. Statement of problem–Particular solutions for Mode I and Mode II crack propagation

Consider the elastodynamic problem of a traction-free semi-infinite crack situated along the fixed X-axis and propagating with a constant velocity c, such that $c_s<c<c_l$ (Fig.1).

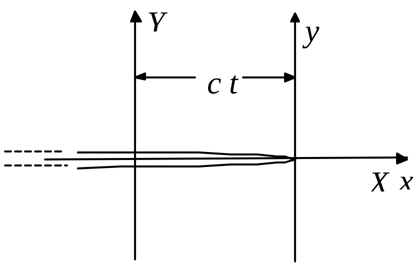

Fig.1

By using (2.16), the simmetry conditions:

$$u(x,y)=u(x,-y)\quad ,\quad v(x,y)=-v(x,-y) \tag{4.1a,b}$$

required for Mode I fracture, lead to the following conditions ahead of the crack tip:

$$U'(0)=V(0)=0 \tag{4.2}$$

In the same way, from the skew-symmetry conditions:

$$u(x,y)=-u(x,y)\ ,\ v(x,y)=v(x,-y) \tag{4.3a,b}$$

valid for Mode II fracture, one obtains:

$$U(0)=V'(0)=0 \tag{4.4}$$

Applyiing conditions (4.2) to eqs.(2.20) gives:

$$y_3(x,0)=0,\ y_4(x,0)=(x)^{g-1}\frac{2k}{a}\left[gU(0)+\frac{2b}{a+p^2}V'(0)\right],\ x>0 \qquad (4.5a,b)$$

In order for the solution (3.8) to satisfy condition (4.5a) it has to be valid that $c_1=0$.
Thence, the solution to the Cauchy-Riemann system (2.14) for Mode I fracture reduces to:

$$y_3^I=c_2r_1^{g-1}sin|g-1|q_1(q)\ ,\ \ y_4^I=-c_2e(g)r_1^{g-1}\cos(g-1)q_1(q) \qquad (4.6a,b)$$

Conditions (4.4) applied to (2.20) gives:

$$y_4(x,0)=0\ ,\ y_3(x,0)=(x)^{g-1}\frac{2k}{q}\left[U'(0)+\frac{2bg}{a+p^2}V(0)\right],\ x>0 \qquad (4.7a,b)$$

therefore, for Mode II of fracture, the following solution is obtained:

$$y_3^{\bar{u}}=c_1r_1^{g-1}\cos(g-1)q_1(q)\ ,\ y_4^{\bar{u}}=c_1r_1^{g-1}sin(g-1)q_1(q) \qquad (4.8a,b)$$

5. The wave-type solutions

The general solutions to wave equations (2.18a,b) are:

$$z(x,y)=f\left(x+\frac{y}{p}\right)+f_1\left(x-\frac{y}{p}\right),\ y(x,y)=g\left(x+\frac{y}{p}\right)+g_1\left(x-\frac{y}{p}\right) \qquad (5.1a,b)$$

The two functions $f_1(x-y/p)$ and $g_1(x-y/p)$ represent signals travelling to the right of the crack tip which is physically meaningless as pointed out also in previous papers (see for example [4,5]). Therefore it will be stated that $f_1=g_1=0$ so that :

$$z(x,y)=f\left(x+\frac{y}{p}\right),\ y(x,y)=g\left(x+\frac{y}{p}\right) \qquad (5.2a,b)$$

For Mode I of fracture the stress-simmetry condition:

$$t_{xy}(x,0\pm)\ ,\ |x|<¥ \qquad (5.3)$$

and the traction-free condition on the crack faces:

$$s_{yy}(x,0\pm)\ ,\ |x|<0 \qquad (5.4)$$

applied to (2.17c,b) lead respectively to:

$$y^I(x,0\pm)=\frac{l_5}{l_6}y_3^I(x,0\pm)=\mp\frac{l_5}{l_6}c_2e(g)(-x)^{g-1}singp, x<0 \qquad (5.5a)$$

$$z^I(x,0\pm)=-\frac{l_3}{l_4}y_4^I(x,0\pm)=-\frac{l_3}{l_4}c_2e(g)(-x)^{g-1}cosgp, x<0 \qquad (5.5b)$$

where eqs.(4.6a,b) have been used. The Cauchy data (5.5) allow to obtain the solutions (5.2a) and (5.2b) in the following form:

$$\zeta^{I}(x,y) = -c_2 \frac{\lambda_3}{\lambda_4} \varepsilon(\gamma) \cos\gamma\pi \left(-x - \frac{|y|}{p} \right)^{\gamma-1} H\left(-x - \frac{|y|}{p} \right) \tag{5.6a}$$

$$\psi^{I}(x,y) = -c_2 \frac{\lambda_5}{\lambda_6} \varepsilon(\gamma) \sin\gamma\pi \, \mathrm{sgn}(y) \left(-x - \frac{|y|}{p} \right)^{\gamma-1} H\left(-x - \frac{|y|}{p} \right) \tag{5.6b}$$

where $H(\ )$ is the Heaviside step function. For Mode II of fracture the stress-symmetry condition:

$$\sigma_{yy}(x,0\pm)=0 \ , \ |x|<\infty \tag{5.7}$$

and the traction-free condition on the crack faces:

$$\tau_{xy}(x,0\pm)=0 \ , \ x<0 \tag{5.8}$$

give:

$$\psi^{II}(x,0\pm) = \frac{\lambda_5}{\lambda_6} \psi_3^{II}(x,0\pm) = -\frac{\lambda_5}{\lambda_6} c_1 (-x)^{\gamma-1} \cos\gamma\pi, \quad x<0 \tag{5.9a}$$

$$\zeta^{II}(x,0\pm) = -\frac{\lambda_3}{\lambda_4} \psi_4^{II}(x,0\pm) = \pm\frac{\lambda_3}{\lambda_4} c_1 (-x)^{\gamma-1} \sin\gamma\pi, \quad x<0 \tag{5.9b}$$

Thence, for Mode II of fracture the required solutions are:

$$\zeta^{II}(x,y) = c_1 \frac{\lambda_3}{\lambda_4} \sin\gamma\pi \, \mathrm{sgn}(y) \left(-x - \frac{|y|}{p} \right)^{\gamma-1} H\left(-x - \frac{|y|}{p} \right) \tag{5.10a}$$

$$\psi^{II}(x,y) = -c_1 \frac{\lambda_5}{\lambda_6} \cos\gamma\pi \left(-x - \frac{|y|}{p} \right)^{\gamma-1} H\left(-x - \frac{|y|}{p} \right) \tag{5.10b}$$

Both solutions (5.6) and (5.10) are defined in the Mach cone, $|y|<-px$ and $x<0$, and the half-lines $|y|=-px$, $x<0$ represent the front of a shock wave following the propagating crack tip. In Fig.2 is represented the angle $\Delta=\pi-tg^{-1}(p)$ between the shock front and the upper crack face as a function of the Mach number M_2, for Graphite Epoxy, whose relevant material parameters are: C_{11}/C_{66}=27,385, C_{22}/C_{66}=2,239, C_{12}/C_{66}=0,716.

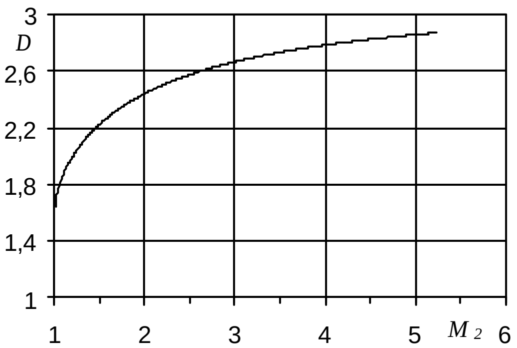

Fig.2

6. Power of stress singularity

The exponent γ is determined by substituting (5.6) and (5.10) into eqs.(2.16). One obtains:

$$\gamma_I = \Gamma_I + N_1 \tag{6.1}$$

for Mode I, and:

$$\gamma_{II} = \Gamma_{II} + N_2 \tag{6.2}$$

for Mode II, where:

$$\Gamma_I = -\frac{1}{\pi} tg^{-1} \frac{\lambda_3\lambda_6}{\lambda_4\lambda_5}, \quad \Gamma_{II} = \frac{1}{\pi} tg^{-1} \frac{\lambda_4\lambda_5}{\lambda_3\lambda_6} \tag{6.3a,b}$$

and N_1, N_2 are arbitrary integers.
In Fig.3, the quantity G_I is represented vs. M_2 for Graphite Epoxy. It should be noted that:

$$0<G_I<1/2 \tag{6.4}$$

As a consequence, follows that:

$$G_{II}=G_I-1/2 \quad , \; -1/2<G_{II}<0 \tag{6.5}$$

Thence, boundedness of displacement at the crack tip requires to chose $N_1=0$ in (6.1) and $N_2=1$ in (6.2). Therefore, the order of stress singularity is :

$$m_I = g_I - 1 = -\frac{1}{p} tg^{-1} \frac{l_3 l_6}{l_4 l_5} - 1, \; -1 < m_I < -\frac{1}{2} \tag{6.6}$$

for Mode I of fracture, and:

$$m_{II} = g_{II} - 1 = \frac{1}{p} tg^{-1} \frac{l_4 l_5}{l_3 l_6}, \; -\frac{1}{2} < m_{II} < 0 \tag{6.7}$$

for Mode II of fracture.
In Figs. 4-5 the quantities m_I and m_{II} are represented as functions of M_2 for Graphite Epoxy.

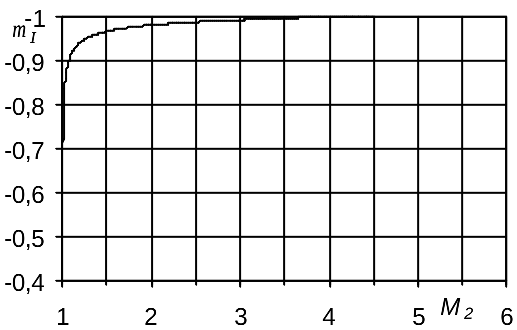

Fig.4

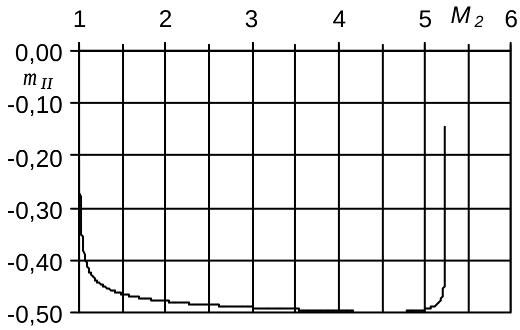

Fig.5

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