

# STRUCTURES WITH RE-ENTRANT CORNERS

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## **Abstract**

When considering a structural element with a re-entrant corner, the experimental analysis shows how the failure load increases with the angle of the corner. In other words, the failure load increases with a decrease of the mass of the structure, in opposition to what we are used to observe in plasticity. To predict this behaviour, a theory, based substantially on the hypothesis of existence of a fracture quantum, is herein presented. Theoretical predictions and experimental results seem to agree satisfactorily.

## **Sommario**

*Considerando un elemento strutturale provvisto di angolo rientrante, la sperimentazione evidenzia come al crescere dell'apertura dello stesso, a cui corrisponde un decremento di materiale della struttura, il carico di collasso aumenti anche notevolmente. Avviene cioè un fatto anomalo, opposto a quanto si è soliti osservare in plasticità, ove un decremento di materiale in una struttura porta ad una riduzione della sua capacità portante. Per prevedere tale fenomeno viene proposta una teoria basata sull'ipotesi di esistenza di un quanto di frattura. Tale teoria risulta essere in buon accordo con l'evidenza sperimentale.*

## 1. Introduction

Since the pioneer paper by Williams [1] the problem of stress intensification at the vertex of re-entrant corners has not been sufficiently addressed if compared with its considerable practical importance. Shapes and sizes of notches or re-entrant corners in structural components are studied more frequently than shapes and sizes of cracks. In spite of this, fracture mechanics applied to sharp cracks by Griffith [2] and Irwin [3] and has been broadly developed in the last three decades, even if only as a special case of the more general problem of re-entrant corners.

Carpinteri [4] has generalized the expression of the brittleness number to study the transition between brittle and ductile collapses and the stress-intensity factor at the vertex of a re-entrant corner applying Buckingham's Theorem. In the same paper a shape function for generalized stress-intensity factor, assuming a combination of LFM and ultimate strength function, is defined. According to the last hypothesis and to the results of an experimental investigation, the values of stress-intensity factors varying the corner angle are reported.

In Seweryn's paper [5] a relation between the stress-intensity factor for a corner and that for a crack, obtained from the Novozhilov's [6] brittle fracture criterion, is presented. This criterion is based on the hypothesis that the fracture of solids is a discrete process: the destruction of the connection between just one pair of atoms will be a fracture *quantum*. Seweryn's equation agrees with experimental results.

Purpose of the present paper is the prediction of the failure load for a structural member with a re-entrant corner [7]. The theory presented herein is based on the fracture criterion examined by Novozhilov in [6] and on the generalized stress-intensity factor obtained by Seweryn in [5]. Thanks to the last expression it has been possible to obtain the generalized shape function defined in Carpinteri's paper [4] and therefore the failure load of the structure. The theoretical predictions agree with the experimental results satisfactorily.

## 2. Plate in tension

Considering a linear elastic plate with a boundary crack, the symmetrical stress field around the tip of the edge-crack can be written as:

$$\sigma_{ij} = K_I r^{-1/2} S_{ij}(\varphi) \quad (1)$$

where  $K_I$  is the stress-intensity factor for the Mode I,  $r$  and  $\varphi$  are the polar coordinates represented in Figure 1 and  $S_{ij}$  is a function describing the angular profile of the stress field.

For every structure it is possible to express the stress intensity factor as:

$$K_I = \sigma b^{1/2} f(a/b) \quad (2)$$

where  $\sigma$  is the nominal stress,  $b$  is a characteristic size of the structure,  $a$  is the crack length and  $f$  is a shape function depending on the geometry of structure and on the ratio  $a/b$ . The stress of failure  $\sigma_f$  is achieved when  $K_I$  is equal to its critical value  $K_{IC}$ :

$$K_{IC} = \sigma_f b^{1/2} f(a/b) \quad (3)$$

The equations presented can be generalized to the case of re-entrant corner with angle  $\gamma$ .

Williams [1] proved that, when both the notch surfaces are free, the symmetrical stress field at the notch tip is:

$$\sigma_{ij} = K_I^*(\gamma) r^{-\alpha(\gamma)} S_{ij}^{(\gamma)}(\varphi) \quad (4)$$

where the power  $\alpha$  of the stress singularity is provided by the eigen-equation:

$$(1-\alpha)\sin(2\pi-\gamma) = \sin[(1-\alpha)(2\pi-\gamma)] \quad (5)$$

and ranges between  $1/2$  (when  $\gamma=0$ ) and zero (when  $\gamma=\pi$ ).

If Buckingham's Theorem for physical similitude and scale modeling is applied and stress and linear size are assumed as fundamental quantities [4] it is possible to write an equation analogous to eq. (2):

$$K_I^*(\gamma) = \sigma b^{\alpha(\gamma)} f^*(\gamma, a/b); \quad [K_I^*] = [F][L]^{\alpha-2} \quad (6)$$

When the angle  $\gamma$  vanishes, eq. (6) coincides with eq. (2), whereas when  $\gamma=\pi$  the stress-singularity disappears and the generalized stress intensity factor  $K_I^*$  assumes the physical dimensions of stress and becomes proportional to the nominal stress  $\sigma$ . The stress of failure  $\sigma_f$  is achieved when the  $K_I^*$  is equal to its critical value  $K_{IC}^*$ :

$$K_{IC}^*(\gamma) = \sigma_f b^{\alpha(\gamma)} f^*(\gamma, a/b) \quad (7)$$

If the angle is close to zero the corner becomes a crack and eq. (7) becomes eq. (3), where  $f$  is the following polynomial function ( $a/b < 0.6$ ):

$$f\left(\frac{a}{b}\right) = 2\left(\frac{a}{b}\right)^{1/2} - 0.4\left(\frac{a}{b}\right)^{3/2} + 18.7\left(\frac{a}{b}\right)^{5/2} - 38.5\left(\frac{a}{b}\right)^{7/2} + 53.9\left(\frac{a}{b}\right)^{9/2} \quad (8)$$

In the opposite case of angle close to  $\pi$ , eq. (7) becomes:

$$K_{IC}^*(\gamma = \pi) = K_{IC}^\pi = \sigma_u = \sigma_f g(a/b) \quad (9)$$

where the well-known function  $g$  takes into account the reduction of the resisting cross section:

$$g\left(\frac{a}{b}\right) = \frac{1}{1-a/b} \quad (10)$$

### 3. Three-point bending

A three-point bending specimen with a middle re-entrant corner is now considered. The stress-intensity factor can be expressed as:

$$K_I^*(\gamma) = \frac{Pl}{tb^{2-\alpha(\gamma)}} f^*(\gamma, a/b) \quad (11)$$

where  $P$  is the external load, and  $l$ ,  $t$  and  $b$  the length, thickness and height of the beam. Eq. (11), in the critical condition, becomes:

$$K_{IC}^*(\gamma) = \frac{P_{CR}^* l}{tb^{2-\alpha(\gamma)}} f^*(\gamma, a/b) \quad (12)$$

where  $f^*$  is the unknown generalized shape function. For a crack eq. (12) assumes the following form:

$$K_{IC} = K_{IC}^*(\gamma=0) = \frac{P_{CR} l}{tb^{3/2}} f(a/b) \quad (13)$$

where function  $f$  can be expressed as follows ( $a/b < 0.6$ ):

$$f\left(\frac{a}{b}\right) = 2.9\left(\frac{a}{b}\right)^{1/2} - 4.6\left(\frac{a}{b}\right)^{3/2} + 21.8\left(\frac{a}{b}\right)^{5/2} - 37.6\left(\frac{a}{b}\right)^{7/2} + 38.7\left(\frac{a}{b}\right)^{9/2} \quad (14)$$

If the angle becomes flat the generalized stress-intensity factor becomes:

$$K_{IC}^*(\gamma=\pi) = K_{IC}^\pi = \sigma_u = \frac{P_{CR}^\pi l}{tb^2} g(a/b) \quad (15)$$

where function  $g$  describes the reduction of the resisting cross section:

$$g\left(\frac{a}{b}\right) = \frac{3/2}{(1-a/b)^2} \quad (16)$$

### 4. Fracture quantum and generalized stress intensity-factor

In Seweryn's paper [5] a relation between the stress-intensity factor for a re-entrant corner and that for a crack is obtained from Novozhilov's [6] brittle fracture criterion.

The stress field in a cracked plate subject to tension tends to infinity at the crack tip. If it is supposed that the rupture occurs when the maximum stress becomes equal to a strength

characteristic value, the plate would collapse subject to an infinitesimal external load. In reality, the external load necessary to propagate the crack in the plate is finite, as Griffith's [2] energy criterion shows. The paradox between the tensional and the energetic approaches can be explained changing the failure criterion assumed above. The new criterion is based on the hypothesis that fracture in solids is a discrete process. The destruction of the connection between one pair of atoms is a fracture *quantum*. The crack will propagate not when the stress reaches a critical value but when its integral along a *quantum* of ligament reaches a certain threshold. Novozhilov's [6] brittle fracture criterion should be written in the following integral form:

$$\int_0^{d_0} \sigma_y(x) dx \geq \sigma_u d_0 \quad (17)$$

where  $\sigma_u$  is a strength characteristic value for the material without defects and  $d_0$  is the fracture *quantum*, a multiple of the atomic radius.

Substituting the stress field around the vertex of the corner (4) into eq. (17), we can rewrite the condition for brittle propagation as:

$$K_I^*(\gamma) \geq [1 - \alpha(\gamma)] (2\pi d_0)^{\alpha(\gamma)} \sigma_u \quad (18)$$

where the right part of the inequality represents the critical value of the stress-intensity factor:

$$K_{IC}^*(\gamma) = [1 - \alpha(\gamma)] (2\pi d_0)^{\alpha(\gamma)} \sigma_u \quad (19)$$

Evaluating eq. (19) for a crack we obtain  $d_0$  (it is interesting to emphasize how the *quantum*  $d_0$  coincides with Irwin's estimate of the plastic zone diameter):

$$d_0 = \frac{2 K_{IC}^2}{\pi \sigma_u^2} \quad (20)$$

Substituting  $d_0$  into eq. (19) we can find the fundamental equation presented by Seweryn in [5]:

$$K_{IC}^*(\gamma) = (1 - \alpha(\gamma)) \sigma_u \left( \frac{2 K_{IC}}{\sigma_u} \right)^{2\alpha(\gamma)} \quad (21)$$

## 5. Generalized brittleness number, shape function and failure load

The embrittlement of the structural response produced by the decrease in fracture toughness and/or by the increase in strength  $\sigma_u$  and/or in the size  $b$ , can be described in a unitary and synthetic manner via the variation in the following dimensionless number [4]:

$$s^*(\gamma) = \frac{K_{IC}^*(\gamma)}{\sigma_u b^{\alpha(\gamma)}} \quad (22)$$

Taking into account eq. (21), relation (22) may be reformulated in a generalized form:

$$s^*(\gamma) = (1 - \alpha(\gamma))(2s)^{2\alpha(\gamma)} \quad (23)$$

In the opposite cases of crack or corner angle close to  $\pi$  (flat angle) we respectively obtain:

$$s = s^*(\gamma = 0) = \frac{K_{IC}}{\sigma_u \sqrt{b}} \quad (24a)$$

$$s^*(\gamma = \pi) = 1 \quad (24b)$$

Considering a three point bending specimen and substituting eq. (22) into eq. (12) we obtain the dimensionless failure load as a function of the generalized brittleness number and of the shape function:

$$\frac{P_{CR}^* l}{tb^2 \sigma_u} = \frac{s^*(\gamma)}{f^*(\gamma, a/b)} \quad (25)$$

This kind of intensification collapse, when  $K_I^*(\gamma) = K_{IC}^*(\gamma)$ , is always intermediate between brittle [ $K_I = K_{IC}$ ] and ductile [ $\sigma = \sigma_u$ ] collapses.

Substituting eq. (24a) into eq. (13), eq. (25) may be evaluated for a crack:

$$\frac{P_{CR} l}{tb^2 \sigma_u} = \frac{s}{f(a/b)} \quad (26)$$

as well as substituting eq. (23b) into eq. (15), eq. (25) can be evaluated for a flat angle:

$$\frac{P_{CR}^\pi l}{tb^2 \sigma_u} = \frac{1}{g(a/b)} \quad (27)$$

For a structural element with a crack of a given relative depth, the transition between brittle and ductile collapse arises when the two failure loads (26) and (27) are equal for:

$$s_0 = \frac{f(a/b)}{g(a/b)} \quad (28)$$

If the angle is different from zero, the crack becomes a re-entrant corner and the transition arises when the two failure loads (25) and (27) are equal for:

$$s_0^*(\gamma) = \frac{f^*(\gamma, a/b)}{g(a/b)} \quad (29)$$

The value of the brittleness number for which the corresponding generalized fracture curve (25) is tangential to the curve of ductile collapse (27) represents its characteristic value; for higher values of  $s^*$  the ductile collapse precedes the generalized brittle collapse for any relative corner depth.

Substituting eqs. (28) and (29) into eq. (23) we obtain the generalized shape function for the re-entrant corner:

$$f^*(\gamma, a/b) = (1 - \alpha(\gamma)) g(a/b) \left( 2 \frac{f(a/b)}{g(a/b)} \right)^{2\alpha(\gamma)} \quad (30)$$

The stress intensity-factor (for a crack) and the ultimate tensile strength of the material of the element in object, can be obtained as functions of the failure loads in the cases of angle equal to zero (13) and flat angle (15). Substituting the generalized stress-intensity factor (21) and the shape function (30) into eq. (12) we can predict the failure load for a member with a re-entrant corner:

$$\frac{P_{CR}^*}{P_{CR}^\pi} = \left( \frac{P_{CR}}{P_{CR}^\pi} \right)^{2\alpha(\gamma)} \quad (31)$$

Equations (30) and (31) are true also for different schemes such as the plate in tension already described, where the failure loads are equal to the failure stresses multiplied by a characteristic area.

## 6. Experimental assessment and conclusions

Equations (21), (23) and (30) permit to obtain the generalized stress-intensity factor, the brittleness number and the shape function for a structure with a re-entrant corner. All these generalized quantities  $G^*$  can be written in a unitary manner with reference to their known values for an angle equal to zero,  $G$ , or for a flat angle,  $G^\pi$ :

$$\frac{G^*(\gamma)}{G^\pi} = 2^{2\alpha(\gamma)} (1 - \alpha(\gamma)) \left( \frac{G}{G^\pi} \right)^{2\alpha(\gamma)} = \beta(\alpha(\gamma)) \left( \frac{G}{G^\pi} \right)^{2\alpha(\gamma)} \quad (32)$$

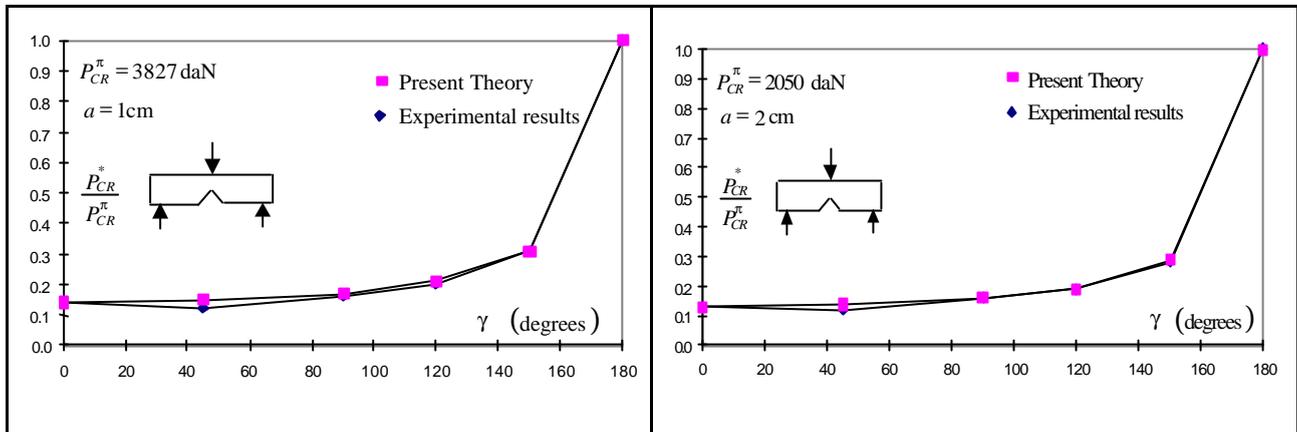
where coefficient  $\beta$  is about constant and equal to one; its maximum divergence from one is only 6%. If we put  $\beta=1$  in eq. (32) we can obtain the following simplified equation:

$$\frac{G^*(\gamma)}{G^\pi} = \left( \frac{G}{G^\pi} \right)^{2\alpha(\gamma)} \quad (33)$$

This equation permits to describe also the generalized failure load (31) and can be defined as the fundamental equation to generalize any quantity for a re-entrant corner. The theory is applicable to different schemes and also with re-entrant corners not subjected only to Mode I; actually the generalization (21) is also true for Mode II and Mode III considering the ultimate shearing stress.

The theory presented has been validated experimentally. Three point bending specimens of PMMA with two different relative depths of the re-entrant corner ( $a=1,2$  cm,  $b=5$  cm,  $t=5$  cm,  $l=19$

cm) and six different angles, for a total of twelve specimens, have been tested. The results of their failure loads are reported in Carpinteri's paper [4]. From equation (31) we can obtain the corresponding theoretical predictions. The comparison between theoretical and experimental results is shown in Figure 1. The results show basically a relevant agreement between the theoretical and the experimental approaches.



*Figure 1*

*Experimental results and theoretical predictions for the failure loads of a three point bending specimen with re-entrant corner of relative depth 0.2 and 0.4.*

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