LOCAL FIELDS FOR DYNAMIC CRACK GROWTH IN POROELASTIC FLUID-SATURATED MEDIA

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Sommario

Viene presentata una soluzione analitica dei campi asintotici di tensione e deformazione in prossimità dell'apice di una frattura che si propaga rapidamente in un materiale poroelastico saturo, in condizioni di Modo I. Si assume che il comportamento del materiale sia descritto dalla teoria di Biot della poroelasticità in presenza di effetti dinamici. L'approccio seguito prevede la trasformazione delle equazioni del moto, espresse in termini di potenziali di spostamento, in un sistema disaccoppiato del secondo ordine, di cui viene fornita una soluzione a variabili separate, valida localmente in prossimità dell'apice. I risultati ottenuti mostrano che la pressione del fluido in prossimità dell'apice presenta la stessa singolarità della tensione nel materiale della fase solida. A differenza del caso quasistatico in cui l'apice della frattura è in condizioni drenate, per propagazione dinamica della frattura il fluido interstiziale non ha il tempo necessario per defluire lontano dall'apice.

Abstract

A closed-form asymptotic solution is provided for the stress, pore pressure and displacement fields near the tip of a Mode I crack, dynamically running in elastic fluid-saturated porous solids. The Biot theory of *poroelasticity with inertia forces is assumed to govern the motion of the medium. The equations of motion, in terms of displacement potentials, has been reduced into a second order uncoupled system and solved under a scheme of separated variables. The obtained asymptotic solution reveals the pore pressure near the crack-tip displays the same square root singularity as the stress in the solid skeleton. Differently from the quasistatic case, where the crack-tip is effectively drained, for dynamic crack propagation the pore fluid has no time to diffuse away from the crack-tip.*

Introduction

In frictional materials as soils, rocks and sands that are infiltrated with ground water, the coupling of deformation with diffusion can significantly affect the mechanical response. In particular, the pore fluid interaction has been identified as a factor in fault creep and in enhancing the recovery of oil via the hydraulic fracturing process [3, 13-16]. During mechanical loading of a poroelastic medium, the load is carried partly by the porous soil skeleton and partly by the pore fluid. For constant loading and relatively low permeability of the medium, the load is initially borne by the pore fluid. With progress of time, the pore fluid pressure decreases and at the end of the consolidation process, the external loadings are borne entirely by the solid skeleton. For slow crack growth governed by quasistatic equations the effects of inertia can be neglected. In this case, in proximity of the crack-tip the pore pressure is vanishing small and the material is effectively drained [14]. It follows that, the local stress fields are similar to those for a crack in a homogeneous elastic material, except for a reduction in the amplitude due to energy dissipated by the flux of fluid towards the crack-tip [2, 3, 4]. However, for rapid crack growth, as usually occurs in hydraulic fracturing for energy resources exploration, the pore fluid has less time to diffuse away from the crack-tip, leading to a completely different mechanical scenario from that predicted by the quasistatical analyses without inertial effects. Knowledge of the stress and deformation fields near a propagating crack-tip is of importance for the understanding of fracture mechanisms. Some effort has been made to obtain asymptotic solutions for dynamic fracture problems in homogeneous elastic materials [1, 4, 12]. Nevertheless, the effects of inertia terms on the stress and pore pressure crack-tip fields in a poroelastic material are almost unexplored. Therefore, for a better understanding of fracture mechanisms in elastic fluid-saturated solids, the influence of inertia on the local crack-tip fields in the solid elastic skeleton and in the diffusing pore fluid is investigated in this work, apparently for the first time.

In particular, an asymptotic solution is provided in closed-form expressions for the stress and velocity fields near the tip of a crack dynamically running in elastic fluid-saturated porous solids, under Mode I plane strain conditions. The mechanical behavior of the saturated porous medium is described by the coupled constitutive equations derived by Biot [6-8] by using a phenomenological approach based on the addition of inertia terms to his quasi-static theory [5]. Later, Bowen obtained similar result by using mixture theory [9]. In the following, an isotropic Darcy's law is used to model the diffusion process, and both problems with permeable and impermeable crack faces are considered.

2. Governing equations

For each phase of the porous medium, the balance of momentum, neglecting body forces, gives:

$$div\,\sigma_s = \rho_s\,\ddot{\mathbf{u}}_s + \xi(\dot{\mathbf{u}}_s - \dot{\mathbf{u}}_w) \qquad n\,\nabla p + \rho_w\,\ddot{\mathbf{u}}_w = \xi(\dot{\mathbf{u}}_s - \dot{\mathbf{u}}_w) \qquad (1)$$

where ρ_s, ρ_w, $\mathbf{u}_s$ and $\mathbf{u}_w$ are the apparent mass densities and the displacement vectors of the solid and fluid components, n is the porosity, ξ is a constant material parameter proportional to the inverse of the permeability, s_s denotes the partial stress in the solid phase and p the fluid pore pressure. The latter quantities depend on the strains of the two components through the following constitutive equations:

$$\sigma_s = 2\mu \;\; \varepsilon_s + (\lambda_s \, tr \, \varepsilon_s + \lambda_{sw} \, tr \, \varepsilon_w) \, \mathbf{I}, \qquad -n\, p = \lambda_{sw} \, tr \, \varepsilon_s + \lambda_w \, tr \, \varepsilon_w, \qquad (2)$$

where $e_s = sym\ (\nabla\ \mathbf{u}\ _s)$ and $e_w = sym\ (\nabla\ \mathbf{u}\ _w)$. The four parameters λ_s, λ_{sw}, λ_w and μ define the elastic material response and can be related to the Biot parameters [9, 10]. The total stress is defined as:

$$s = s_s - n\, p\, \mathbf{I}. \qquad (3)$$

The equations of motion (1) in terms of displacements become:

$$\begin{aligned} &\mu\, \Delta \mathbf{u}_s + (\mu + \lambda_s)\, \nabla div\, \mathbf{u}_s + \lambda_{sw}\, \nabla div\, \mathbf{u}_w = \rho_s\, \ddot{\mathbf{u}}_s + \xi\, (\dot{\mathbf{u}}_s - \dot{\mathbf{u}}_w), \\ &\lambda_{sw}\, \nabla div\, \mathbf{u}_s + \lambda_w\, \nabla div\, \mathbf{u}_w = \rho_w\, \ddot{\mathbf{u}}_w - \xi\, (\dot{\mathbf{u}}_s - \dot{\mathbf{u}}_w). \end{aligned} \qquad (4)$$

The problem of a plane crack propagating at time-dependent speed $v(t)$ along a rectilinear path in an infinite medium is considered. A Cartesian coordinate system $(0, x_1, x_2, x_3)$ and a cylindrical co-ordinate system $(0, r, \vartheta, x_3)$ both centered at the crack-tip and moving with it towards the $\vartheta = 0$ direction are considered, with the x_3-axis along the straight crack front. In asymptotic analysis only the most singular terms need to be retained. Therefore, in the material derivative the time derivative term is negligible compared with the spatial derivative, namely

$$(\ )^\bullet = -\, v(t)\, (\ )_{,1} = v(t)\, [(\ )_{,\vartheta}\, r^{-1} \sin \vartheta - (\ )_{,r} \cos \vartheta]. \ (5)$$

Note that if the stress in the solid skeleton is singular such that $s_s = O(r^{\gamma})$ as r tends to zero ($\gamma < 0$), then asymptotically $div\ s_s = O(r^{\gamma - 1})$ and the constitutive relation (2_1) implies that $e_s = O(r^{\gamma})$. Therefore, it follows that $\mathbf{u}_s = O(r^{\gamma + 1})$, $\dot{\mathbf{u}}_s = O(r^{\gamma})$ and $\ddot{\mathbf{u}}_s = O(r^{\gamma - 1})$. In order to satisfy the equation (2_2) at lowest order, at least one of the conditions $p = O(r^{\gamma})$ or $e_w = O(r^{\gamma})$ must be met. Let us assume the latter as true, then $\mathbf{u}_w = O(r^{\gamma+1})$, $\dot{\mathbf{u}}_w = O(r^{\gamma})$ and $\ddot{\mathbf{u}}_w = O(r^{\gamma-1})$, so that by ($1_2$) it follows $\nabla p = O(r^{\gamma-1})$ and thus the former condition hold true also. Moreover, it is worth noting that the terms in (1) and (4) containing the velocities $\dot{\mathbf{u}}_s$ and $\dot{\mathbf{u}}_w$ give higher order contribution, and disappear in the analysis of the leading-order problem. These terms derive from the Darcy's law, and the circumstance that they asymptotically vanishes means that for rapid dynamic crack propagation the diffusion of the pore fluid does not play a role at the crack-tip.

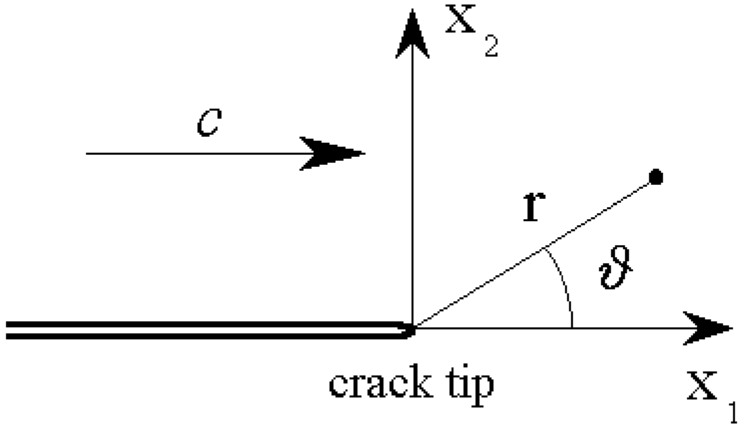


Figure 1
Cartesian and cylindrical co-ordinate systems centered at the moving crack-tip.

For a plane problem, the in-plane displacement vectors can be expressed through the Green-Lamé decomposition, by introducing the longitudinal and shear displacement potentials for the solid $\phi(x_1, x_2, t)$ and $\psi(x_1, x_2, t)$, and a single longitudinal displacement potential $\varphi(x_1, x_2, t)$, for the fluid, namely

$$u_s = \nabla\phi + \nabla\psi \times \mathbf{e}_3, \qquad u_w = \nabla\varphi, \qquad (6)$$

since the displacement vector of the fluid must be irrotational. A substitution of (6) into the equations of motion (4), by using the material derivative rule (5_1) and after a successive rearrangement, allows to reduce the problem to the following system of three second order PDEs in the unknown displacement potentials:

$$\begin{bmatrix}\phi_{,11}\\ \varphi_{,11}\end{bmatrix} + \begin{bmatrix} a & -b \\ -c & d\end{bmatrix}\begin{bmatrix}\phi_{,22}\\ \varphi_{,22}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \quad \psi_{,11} + (1 - M_s^2)^{-1}\,\psi_{,22} = 0, \qquad (7)$$

where

$a = [(2 + L_S)(L_W - M_W^2) - L_{SW}^2] / \delta$, $b = (L_{SW} M_W^2) / \delta$, (8)

$d = [(2 + L_S - M_S^2) L_W - L_{SW}^2] / \delta$, $c = (L_{SW} M_S^2) / \delta$,

together with the following definitions:

$L_S = \lambda_S / \mu$, $L_W = \lambda_W / \mu$, $L_{SW} = \lambda_{SW} / \mu$, (9)

$M_S^2 = \rho_S v^2 / \mu$, $M_W^2 = \rho_W v^2 / \mu$, $\delta = (2 + L_S - M_S^2)(L_W - M_W^2) - L_{SW}^2$.

It is worth noting that the equations (7_1), involving the longitudinal potentials ϕ and φ , are coupled. This circumstance corresponds to the explicit coupling between the dilatation of the elastic solid skeleton and the pressure p in the diffusing pore fluid. On the contrary, equation (7_2), involving the shear potential ψ for the displacement of the solid phase, is uncoupled. The same equation holds for the homogeneous problem, which has been solved by Achenbach and Bazant [1]. In order to uncouple the problem in the longitudinal potentials the matrix in (7_1) must be diagonalized. The characteristic equation provides two distinct positive eigenvalues, namely:

$$\alpha_1 = \frac{a+d}{2} - \sqrt{\left(\frac{a-d}{2}\right)^2 + b\,c}\,, \qquad \alpha_2 = \frac{a+d}{2} + \sqrt{\left(\frac{a-d}{2}\right)^2 + b\,c}\,, \qquad (10)$$

and thus the system (7_1) can be transformed to the following uncoupled two second-order PDEs:

$$h_{i,11} + \alpha_i h_{i,22} = 0, (i = 1, 2), \qquad \text{where: } \begin{bmatrix} \phi \\ \varphi \end{bmatrix} = \begin{bmatrix} d-\alpha_1 & d-\alpha_2 \\ c & c \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \qquad (11)$$

which are formally similar to (7_2).

3. Asymptotic crack-tip fields

A separated variables representation of the spatial and time dependence of the displacement potentials in proximity of the crack-tip is then introduced, as common in plane elastodynamic problems for moving cracks through elastic solids [1, 11, 12], namely:

$\phi(r, \vartheta, t) = r^{\gamma+2}\, U(\vartheta)\, T(t)$, $\varphi(r, \vartheta, t) = r^{\gamma+2}\, V(\vartheta)\, T(t)$, $\psi(r, \vartheta, t) = r^{\gamma+2}\, W(\vartheta)\, T(t)$. (12)

The functions h_i can be correspondingly assumed in the separated variables form:

$$h_i(r, \vartheta, t) = r^{\gamma+2} H_i(\vartheta)\, T(t), \qquad \text{where} \quad \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} d-\alpha_1 & d-\alpha_2 \\ c & c \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \qquad (13)$$

in agreement with (11_2). The introduction of the potentials (12) into the relations (6) results in the following cylindrical components of the displacements in the solid and in the fluid phases:

$$\begin{aligned} &\tilde{u}_r^{\,s} = W' + (\gamma+2)\,U, \qquad \tilde{u}_\vartheta^{\,s} = U' - (\gamma+2)\,W, \\ &\tilde{u}_r^{\,w} = (\gamma+2)\,V, \qquad \tilde{u}_\vartheta^{\,w} = V', \end{aligned} \qquad (14)$$

where $\tilde{\mathbf{u}} = \mathbf{u} / [r^{\gamma+1}\, T(t)]$. A similar substitution into the constitutive relations (2) and (3), after some lengthy but straightforward manipulations, yields the following cylindrical components of the total stress:

$$\begin{aligned} &\tilde{\sigma}_{33} = (L_s + L_{sw})\,[U'' + (\gamma+2)^2 U] + (L_w + L_{sw})\,[V'' + (\gamma+2)^2 V], \\ &\tilde{\sigma}_{rr} = 2(\gamma+1)\,[W' + (\gamma+2)\,U] + \tilde{\sigma}_{33}, \\ &\tilde{\sigma}_{\vartheta\vartheta} = 2\,[U'' + (\gamma+2)\,U - (\gamma+1)\,W'] + \tilde{\sigma}_{33}, \\ &\tilde{\sigma}_{r\vartheta} = 2(\gamma+1)\,U' + W'' + \gamma(\gamma+2)\,W, \end{aligned} \qquad (15)$$

and to the following expression of the pore pressure:

$$\tilde{p} = -\{L_w\,[V'' + (\gamma+2)^2 V] + L_{sw}\,[U'' + (\gamma+2)^2 U]\}/\,n, \qquad (16)$$

where

$\tilde{\sigma} = \sigma / [\mu\, r^{\gamma}\, T(t)]$ and $\tilde{p} = p / [\mu\, r^{\gamma}\, T(t)]$.

After the imposition of the Mode I symmetry conditions $u_\vartheta^{s} = u_\vartheta^{w} = \sigma_{r\vartheta} = p_{,\vartheta} = 0$ at $\vartheta = 0$ and vanishing of the tractions on the crack faces $\sigma_{\vartheta\vartheta} = \sigma_{r\vartheta} = 0$ at $\vartheta = \pi$, the value $\gamma = -1/2$ is obtained, as the smallest value of the singularity allowing the strain energy density to be integrable in a neighborhood of the crack-tip. It must be observed that this result holds when the crack-tip velocity is subsonic and smaller than the Rayleigh wave velocity, in agreement with the crack-tip field singularity in a homogeneous elastic material [1, 12].

The introduction of the separated variables representation (12_3) and (13) for the functions ψ and h_i into the uncoupled second order PDEs system formed by (7_2) and (11_1) leads to the following results for the unknown angular functions W(ϑ) and $H_i(\vartheta)$:

$$\begin{aligned} W(\vartheta) &= K_0\,[2\cos\vartheta - g_0(\vartheta)]\,\sqrt{g_0(\vartheta)+\cos\vartheta} + C_0\,[2\cos\vartheta + g_0(\vartheta)]\,\sqrt{g_0(\vartheta)-\cos\vartheta}\,, \\ H_i(\vartheta) &= K_i\,[2\cos\vartheta - g_i(\vartheta)]\,\sqrt{g_i(\vartheta)+\cos\vartheta} + C_i\,[2\cos\vartheta + g_i(\vartheta)]\,\sqrt{g_i(\vartheta)-\cos\vartheta}\,, \end{aligned} \tag{17}$$

where K_0, K_i, C_0 and C_i (i = 1, 2) are constants of integration and:

$$g_i(\vartheta) = \sqrt{\cos^2\vartheta + \frac{\sin^2\vartheta}{\alpha_i}}\,, \quad g_0(\vartheta) = \sqrt{1 - M_s^2\,\sin^2\vartheta}\,. \tag{18}$$

The Mode I boundary conditions at $\vartheta = 0$ imply that $K_0 = C_1 = C_2 = 0$, and thus the following closed form expressions for the angular functions introduced in (12) can be found from (13_2) and (17):

$$U(\vartheta) = \sum_{i=1}^{2} (d - \alpha_i)\, H_i(\vartheta), \qquad V(\vartheta) = c \sum_{i=1}^{2} H_i(\vartheta), \tag{19}$$

$$W(\vartheta) = C_0\,[2\cos\vartheta + g_0(\vartheta)]\,\sqrt{g_0(\vartheta)-\cos\vartheta}\,, \tag{20}$$

where:

$$H_i(\vartheta) = K_i\,[2\cos\vartheta - g_i(\vartheta)]\,\sqrt{g_i(\vartheta)+\cos\vartheta}\,, \tag{21}$$

In particular, the vanishing of shear stress $\sigma_{r\vartheta}$ ahead of the crack-tip at $\vartheta = 0$, which holds for Mode I loading condition, implies:

$$2\,[(\sqrt{\alpha_1} - d/\sqrt{\alpha_1})\,K_1 + (\sqrt{\alpha_2} - d/\sqrt{\alpha_2})\,K_2] - (2 - M_s^2)\,C_0 = 0. \tag{22}$$

Therefore, no more than two of the three constants C_0, K_1 and K_2 are arbitrary. In particular, for permeable crack flanks the condition $p = 0$ at $\vartheta = \pi$ is always satisfied without any additional constraint, and thus the total stress and pore pressure asymptotic fields are defined within two arbitrary constants. Differently, for impermeable crack faces an additional relation holds between the constants K_1 and K_2, consequent to the condition of vanishing pore fluid flux through the crack flanks $p_{,\vartheta} = 0$ at $\vartheta = \pi$, namely:

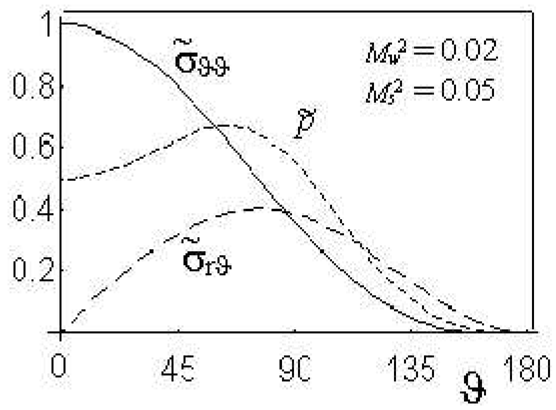
$$\alpha_2 \sqrt{\alpha_2}\ (\alpha_1 - l)(\alpha_1 - 1)\ K_1 + \alpha_1 \sqrt{\alpha_1}\ (\alpha_2 - l)(\alpha_2 - 1)\ K_2 = 0, \tag{23}$$

where $l = d + c\ \lambda_{sw} / \lambda_w$, and thus, the stress and pore pressure fields are determined within an arbitrary constant only, in agreement with the results of the asymptotic analysis for quasistatic crack growth in poroelastic materials [3].

Results and conclusions

The obtained analytical results ($15_{3,4}$) and (16) for the total stress components $\tilde{\sigma}_{\vartheta\vartheta}$, $\tilde{\sigma}_{r\vartheta}$ and for the pore pressure $\tilde{p}$ have been graphically represented in Fig. 2. The cylindrical components of the solid and fluid displacements (14) have been reported in Fig. 3. These results refer to a crack dynamically propagating in a fluid saturated Berea sandstone, characterized by the following material parameters: $L_s = 1.53$, $L_w = 0.082$, $L_{sw} = 0.244$, $n = 0.19$, $\rho_s / \rho_w = 2.5$, for $M_w^2 = 0.02$ and thus $M_s^2 = M_w^2\ \rho_s / \rho_w = 0.05$. The case of impermeable crack faces is considered, so that the local crack-tip fields are defined within an arbitrary amplitude constant. The normalization condition $\tilde{\sigma}_{\vartheta\vartheta} = 1$ at $\vartheta = 0$ is assumed. The solution reveals that the singular term of the pore pressure vanishes at $\vartheta = \pi$, for both conditions of permeable and impermeable crack faces, in agreement with the circumstance that only a finite value of the pore pressure can physically be applied to the crack surfaces by hydraulic fracturing process. In Figs. 4 and 5 the total stress, pore pressure and displacement crack-tip fields are reported for a higher value of the crack-tip velocity, corresponding to $M_w^2 = 0.04$ and $M_s^2 = 0.10$, in the same fluid saturated Berea sandstone. From Fig. 3, it may be observed that at the increasing of the crack-tip velocity the level of the pore pressure becomes more pronounced with respect to the total stress. The results recover the case of dynamic crack propagation in homogeneous elastic materials for $\rho_w = \lambda_{sw} = 0$, and thus $M_w = b = c = 0$, $\alpha_1 = d = 1$ and $\alpha_2 = a$, so that $V = 0$ and the stress fields in the solid skeleton coincide with those obtained by Achenbach and Bazant [1].

Finally, it must be remarked that the obtained solution reveals a significant change of the pore pressure contribution, with respect to the quasistatical problem without the effects of inertia. In the latter case, particularly, the pore pressure vanishes in proximity of the crack-tip, where the material is effectively drained and behaves in a softer manner [3]. Conversely, for rapid dynamic crack propagation, the pore pressure displays the same square root singularity as the stress in the solid skeleton.



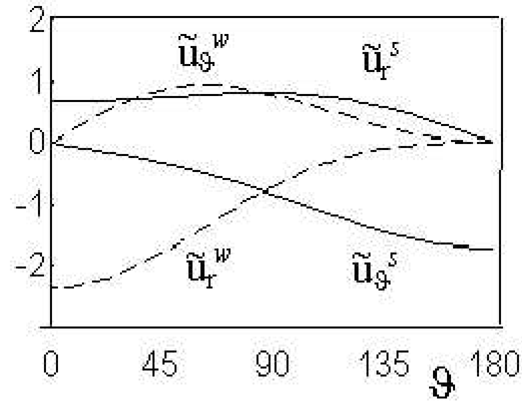
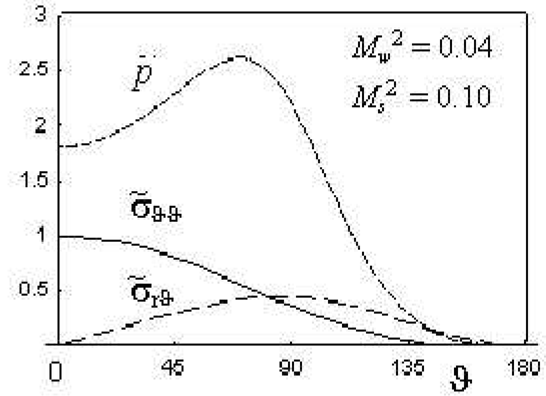


Figure 2
Angular variation of pore pressure p and total stress components $\sigma_{\vartheta\vartheta}$ *and* $\sigma_{r\vartheta}$ *near the crack-tip, under Mode I plane strain loading conditions.*

Figure 3
Angular variation of displacements of the solid $\mathbf{u}^s$ *and the fluid* $\mathbf{u}^w$ *phases near the crack-tip, under Mode I plane strain loading conditions.*



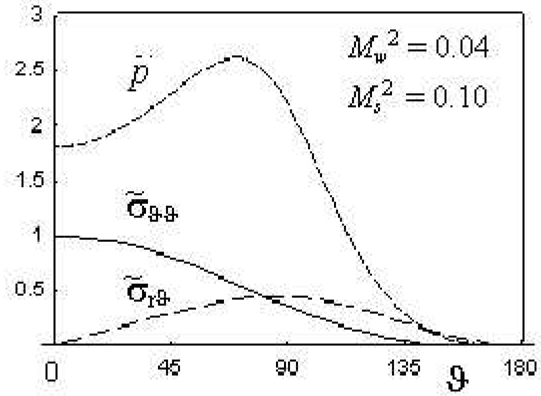


Figure 4
As Figure 2 except that $M_W^2 = 0.04$ *and* $M_S^2 = 0.10$.

Figure 5
As Figure 3 except that $M_W^2 = 0.04$ *and* $M_S^2 = 0.10$.

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