INTEGRITY ASSESSMENT OF DEFECTIVE PRESSURIZED PIPELINES AND PRESSURE VESSELS

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SUMMARY: This paper is concerned with the integrity assessment of defective pressurized pipelines and pressure vessels by means of diverse methods comparatively, for two kinds of failure processes: (I) plastic collapse under internal pressure monotonically increasing in time; (II) incremental collapse (or more generally lack of shakedown, "inadaptation") under fluctuating internal pressure. The defects considered are part-through slots of various geometrical configurations. The analysis methods employed and comparatively discussed are as follows: for situation (I), empirical formulae of current industrial use, rigid-plastic limit analysis, elastic-plastic time-stepping computations by a commercial nonlinear code; for situation (II), a direct method based on classical shakedown theory and evolutive computations like for (I). The main assumptions are Mises perfect elastoplasticity and small deformations. As the main conclusion, the kinematic direct methods of limit and shakedown analysis in a finite element setting turn out to represent a cost-effective and reliable tool for integrity assessment in the present context.

KEY WORDS: Pipelines, defects, fracture, plastic collapse, incremental collapse.

INTRODUCTION

Metal structures designed to contain fluids under pressure are recurrent in various technological areas, such as the oil and gas industry, power plant engineering and chemical factories. The integrity assessment of defective structures in this broad category represent a practically important task of structural analysis and designers. In the above technological areas, defects which may jeopardize the integrity (i.e. reduce carrying capacity) of pipelines, pressure vessels and similar containment structures are mostly represented by local reductions of thickness in form of part-through slots (cavities or notches) or cracks, sometimes associated with indentation of metal wall.

In this paper focus is on part-through slots which can be generated by localized corrosion ("pitting"), fabrication defects or abrading and gauging mechanical accidents (such as impacts of external objects). The events of integrity loss considered in this study are ultimate limit state, envisaged in classical plasticity theory (Lubliner, 1992), namely: (I) plastic collapse under internal pressure monotonically increasing in time; (II) the persistent development of plastic deformations (i.e. cumulative dissipated energy unbounded in time) under fluctuating (variable repeated) internal pressure.

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The main objective pursued here is of critical and engineering-oriented nature, rather then creative and mechanics-oriented (although novel contributions are believed to be included in the proposed procedure of shakedown analysis). In fact this study aims at comparative assessment of the computational merit from an industrial standpoint (in terms of cost-effectiveness, reliability and amount of results with practical interest) of diverse methods of analysis.

The approaches to be employed and discussed for limit analysis (I) are as follows:

(a) empirical or semiempirical formulae available in literature and currently adopted in industrial environments;

(b) elastic-plastic time-stepping finite element analysis by a commercial computer code (ABAQUS) (Hibbitt *et alii*, 1995);

(c) rigid-plastic limit analysis in its computer implementation due to Liu *et alii* (1995a,b) based on the kinematic (upper bound, "unsafe") limit theorem (Lubliner, 1992) and on finite element modelling of the displacement field (with special provisions to avoid locking manifestations);

As for the evaluation of the safety margin of defective structures under variable-repeated loading, the following quite distinct methodologies have been comparatively applied:

(d) direct method of simplified shakedown analysis based on Koiter's kinematic theorem of classical plasticity (Lubliner, 1992), mathematical programming (Cohn and Maier, 1979) and 3D-FE discretization like in approach (c), which can be regarded as a special case of the present one (for vanishing amplitude of pressure fluctuations);

(e) evolutive analysis by a professional finite element code, like in (b).

Two hypotheses are crucial in all the above approaches: (i) elastic-perfectly plastic material model with von Mises yield criterion and unlimited ductility; (ii) small deformation i.e. linear kinematics. The relaxation of these restrictions will not be dealt with here but are pursued in the sequel of the present study.

Limit analysis and shakedown theory have been recently generalized much beyond classical perfect plasticity (e.g. to nonlinear-plasticity with saturation and nonassociativity, see Pycko and Maier, 1995, Stein *et alii*, 1992) but not to softening. Similarly extensions of those theories to geometric effects (Maier *et alii*, 1993) seem unlikely to represent practical propositions in the situation considered herein.

The structures considered here for comparative computations by the above listed methods are defective pipelines, i.e. thick cylinders with internal pressure and an external slot. The geometry of the slot is varied like in Liu *et alii* (1995a,b), in order to corroborate comparisons among approaches by means of numerical tests concerning real-life situations, and covering a meaningful set of cases.

SEMIEMPIRICAL APPROACHES FOR LIMIT PRESSURE

Some of the semiempirical formulae widely used in industrial environments to compute the limit pressure of pipes and vessels with part-through defects can be summarized in:

$$P_L = P_{LM}\Theta \qquad (1)$$

where P_{LM} is Mariotte's limit pressure and Θ represents a factor which depends on geometrical parameters, specified in Figure 1, through closed form expressions proposed by various authors. Some of these expressions for Θ, listed in Miller's review (Miller, 1988), read as follows:

(Ewing) $$\Theta = 1 - \frac{a}{t} + \frac{a/t}{\sqrt{1+1.61\left(c^2/aR\right)}} \tag{2}$$

(Chell) $$\Theta = \frac{\eta}{1-(1-\eta)\Big/\sqrt{1+1.61\rho^2(1-\eta)}} \tag{3}$$

(Battelle) $$\Theta = \frac{\eta}{1-(1-\eta)\Big/\sqrt{1+1.05\rho^2}} \tag{4}$$

where: a=t-b, $\eta = b/t$ and $\rho = c/\sqrt{Rt}$

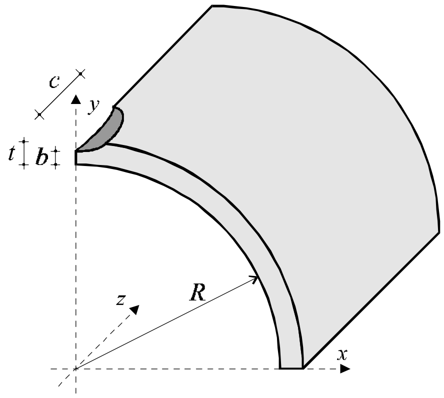


Figure 1: geometric parameters.

KINEMATIC LIMIT ANALYSIS

Consider a 3-D, perfectly-plastic body V with the boundary S. The reference surface tractions f_i (i=1,2,3) are prescribed on a part S_f of the surface, while the remaining part S_u is held in a fixed position. Let the limit state of plastic collapse under proportionally increasing tractions (body forces are ignored) be reached at the load level or “safety factor” s.

When the von Mises yield criterion is adopted, the kinematic (upper bound, “unsafe”) limit theorem of classical plasticity (see e.g. Lubliner, 1992) leads to the following mathematical programming formulation (Cohn and Maier, 1979):

$$\begin{cases} s = \min\limits_{\underline{\dot{u}}} \sqrt{\tfrac{2}{3}}\sigma_s \int\limits_V \sqrt{\dot{p}_{ij}\dot{p}_{ij}}\, dV \text{, subject to:} \\ \int\limits_{S_f} \dot{u}_i f_i dS = 1 \\ \dot{p}_{ij} = \frac{1}{2}\left(\dot{u}_{i,j} + \dot{u}_{j,i}\right) \quad \text{in } V \\ \dot{u}_{i,i} = 0 \quad \text{in } V \\ \dot{u}_i = 0 \quad \text{on } S_u \end{cases} \tag{5}$$

where σ_s is the yield stress of the material, $\dot{u}_i$ and $\dot{p}_{ij}$ (i,j=1,2,3) denote the velocity and plastic strain rate, respectively.

In the following the factor $\sqrt{\tfrac{2}{3}}\sigma_s$ is omitted to simplify the notation.

Discretizing eqn. (5) by means of displacement isoparametric finite elements, we obtain as an approximation:

$$\begin{cases} s \cong \min\limits_{\underline{\dot{U}}} \sum\limits_{h \in I} \rho_h |J|_h \sqrt{\underline{\dot{U}}^T \underline{G}_h \underline{\dot{U}}}\ , \text{ subject to:} \\ \underline{F}^T \underline{\dot{U}} = 1 \\ \underline{H}^T \underline{B}_h \underline{\dot{U}} = 0 \qquad \forall h \in I \end{cases} \tag{6}$$

where: $\underline{\dot{U}}$ is the vector of velocities in the unconstrained nodes and $\underline{F}$ the relevant equivalent nodal load vector corresponding to the prescribed tractions; ρ denotes Gauss integration weight, $|J|$ the Jacobian determinant, I the set of all integration points run by index h. As for the matrices in eqn. (6), $\underline{G} = \underline{B}^T \underline{B}$ and $\underline{B}$ is the strain matrix relating the strain field to nodal displacements (computed at Gauss points h), $\underline{H}$ is a constant vector.
In eqn. (6) all constraints are equalities but the nonlinear objective function is not continuously differentiable. In such finite elements formulation, based on displacement modelling, the incompressibility condition (6c) is known to generate “locking” phenomena and unsafe safety factor s. In order to avoid this spurious effect a penalty function approach is here adopted like in Liu *et alii*, (1995a,b); an alternative provision may be represented by reduced integration (see Nagtegaal *et alii*, 1974).
Enforcing the incompressibility constraint (6c) by a penalty procedure (α_h being penalty factors) and the normalization constraint (6b) by the Lagrange method (λ being the Lagrange multiplier), we obtain from eqn. (6):

$$s \cong \min_{\underline{\dot{U}},\lambda} \left\{ \sum_{h \in I} \rho_h |J|_h \sqrt{\underline{\dot{U}}^T \underline{G}_h \underline{\dot{U}}} + \sum_{h \in I} \alpha_h \rho_h |J|_h \underline{\dot{U}}^T \left(\underline{G}_V\right)_h \underline{\dot{U}} - \lambda\left(\underline{F}^T \underline{\dot{U}} - 1\right) \right\} \tag{7}$$

where $\underline{G}_V = \underline{B}^T \underline{C} \underline{B}$ and $\underline{C}$ is a constant matrix. The optimality conditions of the problem (7) leads to a set of nonlinear equations, that can be solved through the following iterative scheme according to the algorithm of Liu *et alii*, (1995a,b):

$$\begin{cases} \sum\limits_{h \in I} \dfrac{\rho_h |J|_h \underline{G}_h \underline{\dot{U}}_{k+1}}{\sqrt{\underline{\dot{U}}_k^T \underline{G}_h \underline{\dot{U}}_k}} + \sum\limits_{h \in I} \alpha_h^{k+1} \rho_h |J|_h \left(\underline{G}_V\right)_h \underline{\dot{U}}_{k+1} = \lambda_{k+1} \underline{F} \\ \underline{F}^T \underline{\dot{U}}_{k+1} = 1 \end{cases} \tag{8}$$

where k is the iteration counter.
At each iteration the rigid and plastic zones are distinguished, and the objective function and constraint conditions are modified. Namely, before proceeding with the $(k+1)$th iteration, the quantity $\underline{\dot{U}}^T \underline{G}_h \underline{\dot{U}}$ is computed at every integration point and the set I of Gauss points is subdivided into the rigid zone subset R_{k+1} and the plastic zone subset P_{k+1}, i.e.:

$$R_{k+1} = \left\{ h \in I \text{ such that } \underline{\dot{U}}_k^T \underline{G}_h \underline{\dot{U}}_k = 0 \right\},\ \ P_{k+1} = \left\{ h \in I \text{ such that } \underline{\dot{U}}_k^T \underline{G}_h \underline{\dot{U}}_k \neq 0 \right\} \tag{9}$$

Condition of zero dissipation in the Gauss points is imposed on the rigid zone:

$$\underline{\dot{U}}^T \underline{G}_h \underline{\dot{U}} = 0 \qquad h \in R_{k+1} \tag{10}$$

Enforcing these constraints by a penalty function method (β_h being new penalty factors), the optimality conditions (8) become the system of linear equations to solve:

$$\begin{cases} \sum_{h\in P_{k+1}} \rho_h |J|_h \dfrac{\underline{G}_h \underline{\dot{U}}_{k+1}}{\sqrt{\underline{\dot{U}}_k^T \underline{G}_h \underline{\dot{U}}_k}} + \sum_{h\in P_{k+1}} \alpha_h^{k+1} \rho_h |J|_h \left(\underline{G}_V\right)_h \underline{\dot{U}}_{k+1} + \sum_{h\in R_{k+1}} \beta_h^{k+1} \rho_h |J|_h \underline{G}_h \underline{\dot{U}}_{k+1} = \lambda_{k+1} \underline{F} \\ \underline{F}^T \underline{\dot{U}}_{k+1} = 1 \end{cases} \tag{11}$$

The initialization for k=0 is performed by setting $\underline{\dot{U}}_k^T \underline{G}_h \underline{\dot{U}}_k = 1$. The penalization factors α and β are suitably updated at each iteration (Liu *et alii*, 1995b).

The *k+1* solution vector $\underline{\dot{U}}_{k+1}$ yields:

$$s_{k+1} = \sum_{h\in I} \rho_h |J|_h \sqrt{\underline{\dot{U}}_{k+1}^T \underline{G}_h \underline{\dot{U}}_{k+1}} \tag{12}$$

Numerical experience shows that the above iterative process leads to the limit load multiplier s and to a collapse mechanism $\underline{\dot{U}}^s$ through a convergent sequence with monotonically decreasing s_k.

KINEMATIC SHAKEDOWN ANALYSIS

Suppose that the loads acting upon an elastic-perfectly plastic body V vary quasistatically within a given convex polyhedron G with M vertices $\underline{g}_n$ (n=1,...,M).

The main purpose is to evaluate the safety factor s against non-shakedown or inadaptation (i.e. total plastic dissipation unbounded in time because either incremental collapse or alternating plasticity). If M=1, shakedown analysis reduces to limit analysis.

The kinematic approach to shakedown analysis is centered on the notion of “kinematically admissible cycles” of plastic strain rate (Lubliner, 1992, Cohn and Maier, 1979). With reference to the above load domain, plastic flow is activated at the vertices of the domain. Using vector notation to avoid numerous indices, denoting by $\underline{\dot{p}}_n \Delta t_n$ the plastic strain subincrement at vertex n and setting $\Delta t_n = 1$ (n=1,..,M), the shakedown load factor can be determined by solving the following constrained minimization problem:

$$\begin{cases} s = \min\limits_{\underline{\dot{p}}_n} \int_V \sum_{n=1}^{M} D\left(\underline{\dot{p}}_n\right) dV \text{, subject to:} \\ \int_V \sum_{n=1}^{M} \left(\underline{\sigma}_n^e\right)^T \underline{\dot{p}}_n dV = 1 \\ \dot{p}_{Vn} = 0 \\ \Delta \underline{p} = \sum_{n=1}^{M} \underline{\dot{p}}_n = \dfrac{1}{2}\left(\nabla \Delta \underline{u} + \nabla \Delta \underline{u}^T\right) \\ \Delta \underline{u} = \underline{0} \quad \text{on } S_u \end{cases} \tag{13}$$

where: $\underline{\sigma}_n^e$ denotes the purely elastic response to load $\underline{g}_n$; $D(\underline{\dot{p}}_n)$ the plastic energy dissipation due to plastic strain $\underline{\dot{p}}_n$; $\dot{p}_{Vn}$ the volumetric plastic strain at the nth loading vertex; $\Delta \underline{p}$ is the compatible plastic strain increment and $\Delta \underline{u}$ the displacement field increment over a cycle.

With Mises yield function the plastic energy dissipation in eqn. (13) becomes $D(\underline{\dot{p}}_n) = \sqrt{\underline{\dot{p}}_n^T \underline{A} \underline{\dot{p}}_n}$ (n=1,...,M), where $\underline{A}$ is a symmetric positive-definite matrix.

After finite element modelling and numerical integration, eqns. (13) lead to the following mathematical programming problem:

$$\begin{cases} s \cong \min\limits_{\underline{\dot{p}}_n} \sum\limits_{h \in I} \sum\limits_{n=1}^{M} \rho_h |J|_h \sqrt{\underline{\dot{p}}_{nh}^T \underline{A} \underline{\dot{p}}_{nh}}\,, \text{ subject to:} \\ \sum\limits_{h \in I} \sum\limits_{n=1}^{M} \rho_h |J|_h \left(\underline{\sigma}_{nh}^e\right)^T \underline{\dot{p}}_{nh} = 1 \\ \dot{p}_{Vnh} = 0 \qquad \forall h \in I \\ \underline{p}_h = \sum\limits_{n=1}^{M} \underline{\dot{p}}_{nh} = \underline{B}_h \underline{U} \qquad \forall h \in I \end{cases} \tag{14}$$

where: ρ denotes Gauss integration weights, |J| the Jacobian determinant, I the set of all integration points run by index h; $\underline{B}$ is the strain matrix (computed at Gauss points h) and $\underline{U}$ is the nodal displacement vector.

The objective function in eqn. (14), like in (6), is convex but not differentiable ("nonsmooth") in the non-plastic zones (called "rigid") where $\underline{\dot{p}}_{nh}$ vanishes. For the solution of the problem (14) a strategy and an iterative algorithm, similar to the ones employed earlier for limit analysis, are used herein with suitable adjustments. The iteration scheme can be concisely specified as follows:

$$\begin{cases} s \cong \min\limits_{\underline{\dot{p}}_n^{k+1}} \sum\limits_{h \in P_k^{k+1}} \sum\limits_{n=1}^{M} \rho_i \dfrac{(\underline{\dot{p}}_{nh}^{k+1})^T \underline{A} \underline{\dot{p}}_{nh}^{k+1}}{\sqrt{(\underline{\dot{p}}_{nh}^{k})^T \underline{A} \underline{\dot{p}}_{nh}^{k}}}\,, \text{ subject to:} \\ \sum\limits_{h \in I} \sum\limits_{n=1}^{M} \rho_i (\underline{\sigma}_{nh}^e)^T \underline{\dot{p}}_{nh}^{k+1} = 1 \\ \dot{p}_{Vnh}^{k+1} = 0 \qquad \forall h \in P_n^{k+1} \\ \sum\limits_{n=1}^{M} \underline{\dot{p}}_{nh}^{k+1} = \underline{B}_h \underline{U}^{k+1} \qquad \forall h \in I \\ \underline{\dot{p}}_{nh}^{k+1} = \underline{0} \qquad \forall h \in R_n^{k+1}, \quad n = 1, \dots, M \end{cases} \tag{15}$$

in which

$$R_n^{k+1} = \left\{ h \in I \text{ such that } \underline{\dot{p}}_{nh}^{k} = 0 \right\}, \quad P_n^{k+1} = \left\{ h \in I \text{ such that } \underline{\dot{p}}_{nh}^{k} \neq 0 \right\} \tag{16}$$

where $\underline{\dot{p}}_{nh}^{k}$ is the previous approximation to $\underline{\dot{p}}_{nh}$, R_n^{k+1} and P_n^{k+1} are the rigid and plastic subset, respectively, of the Gauss point set I (*n=1,...,M*). The imposition of rigid constrains (16) is intended to make the objective function in eqn. (15a) differentiable.

Like for limit analysis, problem (15) is solved by enforcing the constraints through Lagrange multiplier and penalty function methods, applying the optimality conditions to the augmented objective function and solving a linear equations system in the unknowns $\underline{U}^{k+1}$ and $\underline{\dot{p}}_{nh}^{k+1}$ (*n=1,...,M*).

Numerical experience shows that the above iterative process leads to the shakedown load factor *s* and to an inadaptation mechanism $\underline{\dot{U}}^s$ through a convergent sequence with monotonically decreasing objective function.

APPLICATIONS

The effects of various shapes and sizes of slots on the pressure carrying capacity of thick-wall pipes are evaluated by the direct methods outlined above and, where applicable, by the semiempirical formulae described earlier. The defects considered include part-through spherical, ellipsoidal and rectangular slots like in Liu *et alii* (1995a,b). The numerical results are compared with those of the elastic-plastic time-stepping 3-D finite element analysis by the commercial code ABAQUS (Hibbitt *et alii*, 1995).

The radius ratio (i.e. the external to internal radius) of the pipe is ψ=1.20. The cylinder thickness t=20mm and the yield stress σ_s of material is 200Mpa. The Young's modulus is E=2.1x10^5Mpa and the Poisson's ratio is η=0.3. The cylindrical shell is discretized by 3-D eight-node isoparametric finite element. The finite element meshes adopted are shown in Figure 2.

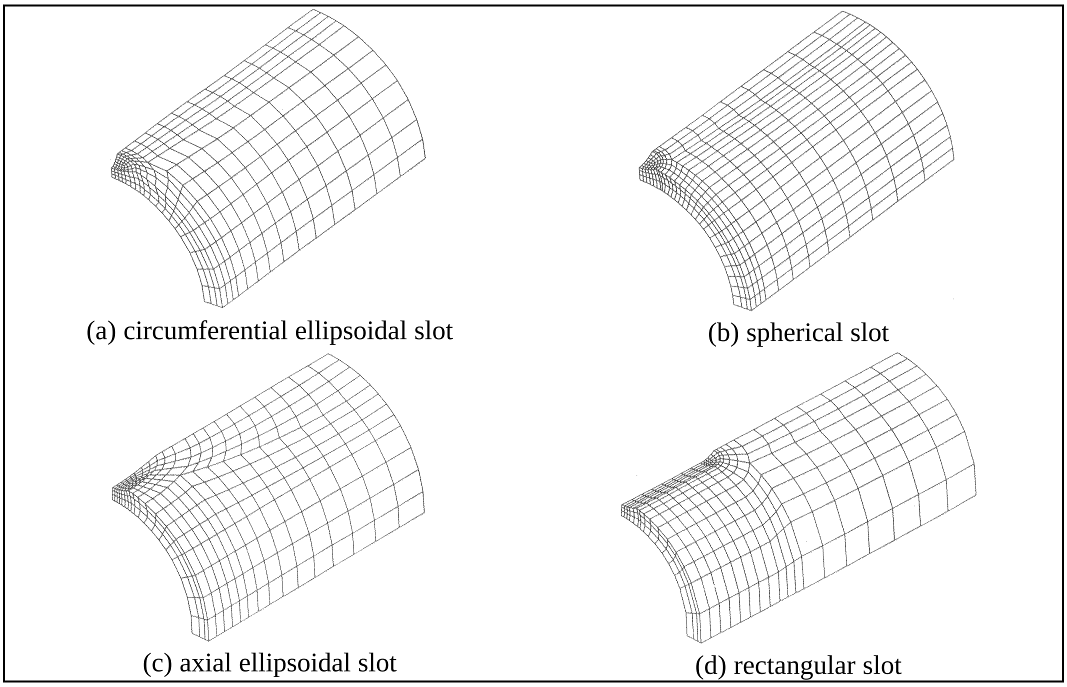


Figure 2: meshes of considered pipes.

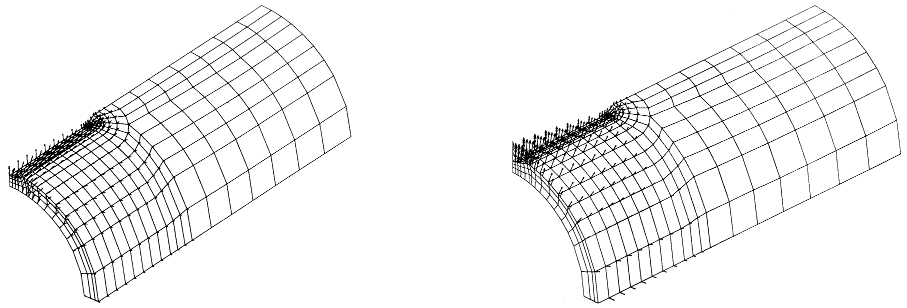
Table 1 gathers results of limit analyses performed by assuming that the pressure monotonically increases in time up to the failure, these results are compared in the table to those provided by evolutive analysis and semiempirical formulae.

In Table 1 excellent agreement can be noticed between limit pressures at plastic collapse provided by the two drastically different numerical methods resting on solid theoretical basis. The semiempirical formulae provide rather erratic, usually very conservative estimates of the carrying capacity of the defective pipes, at least in the configuration range considered in the parametric studies performed so far.

Table 1: the comparison of limit loads by different methods.

	ABAQUS	Kinematic Approach	Ewing	Chell	Battelle
pipe without slot	42.1	42.2			
circum. Ellipsoidal slot (Figure 2a)	41.7	41.3	38.56	39.23	39.02
spherical slot (Figure 2b)	40.7	40.5	36.10	38.91	38.13
axial ellipsoidal slot (Figure 2c)	40.1	39.0	27.67	29.39	28.33
rectangular slot (Figure 2d)	34.4	34.0	28.06	30.12	29.02

The plastic collapse mechanism obtained by the direct ("simplified") method is compared with that by ABAQUS in Figure 3. The mechanism obtained by ABAQUS is relative to the last increment of pressure before the structure reaches the limit state. Good agreement can be observed between the collapse mechanisms provided by the two kinds of approaches.



(a) rectangular slot (ABAQUS) (b) rectangular slot (Kinematic approach)

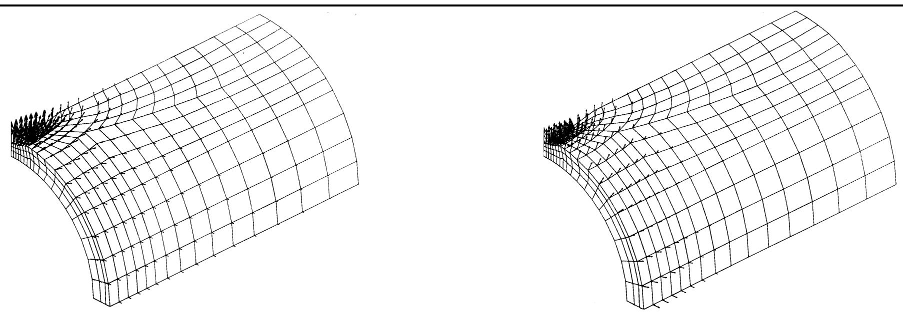
Figure 3: the comparison of collapse mechanism.

Secondly, shakedown (adaptation) analyses are presented and discussed following a pattern similar to the preceding one, but assuming pressure fluctuating in time from zero to a maximum. The comparison of computed results is given in Table 2 and Figure 4.

Table 2: the comparison of shakedown limit pressure by two kinds of approaches.

	ABAQUS	Kinematic Approach
pipe without slot	42.1	42.2
circum. Ellipsoidal slot (Figure 2a)	41.2	41.1
spherical slot (Figure 2b)	40.1	40.4
axial ellipsoidal slot (Figure 2c)	36.0	35.7
rectangular slot (Figure 2d)	32.7	31.9

To obtain the limit pressure of inadaptation by ABAQUS, it was necessary to monitor the total dissipated plastic energy in each loading cycle. A pressure amplitude marks the limit of inadaptation if it separates a set of amplitudes for which, after some initial cycles, the structure does not dissipate plastic energy, from a set for which the total plastic energy does not cease to increase as the loading history proceeds.
Table 2 and Figure 4 show that the shakedown pressure limits and the mechanisms of incremental collapse are in good agreement between the two kinds of approaches. The mechanisms of incremental collapse obtained by ABAQUS are built performing the difference between the residual displacement field after two distinct cycle loadings. The shakedown limit turns out to be quite lower than the plastic collapse limit for some defect configurations. The direct shakedown analysis method by a kinematic approach is found more economical and reliable than marching solutions achieved by commercial code.



(a) axial ellipsoidal slot (ABAQUS) (b) axial ellipsoidal slot (Kinematic approach)

Figure 4: the comparison of incremental collapse mechanism.

CONCLUSIONS

The integrity assessment of pressurized defective pipelines with part-through slots has been carried out both by an elastic-plastic time-stepping finite element analysis and by direct methods of limit and shakedown analysis based on the kinematic limit theorem and Koiter's kinematic theorem, respectively. Numerical applications have shown that the proposed direct (or "simplified") methods, in a finite element setting, represent a cost-effective, numerically stable and reliable tool for integrity assessments of the structures considered herein.
Therefore, in the engineering practice, limit and shakedown analysis might supplement, or replace, both convenient semiempirical formulae and laborious step-by-step evolutive elastic-plastic computations for the failure analysis and integrity assessment of flawed gas-ducts and pressure vessels, especially in the presence of variable-repeated internal pressure fluctuating according to an unpredictable time history.

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