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ON STEADY CRACK GROWTH IN POROUS ELASTOPLASTIC MATERIALS WITH NON-ASSOCIATIVE FLOW-LAW

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SOMMARIO

Si ottiene una soluzione asintotica per i campi tensionali e deformativi all'apice di una frattura che si propaga in linea retta, a velocità costante, in condizioni di Modo I in un mezzo elastoplastico infinito. Il modello elastoplastico adottato è una variante del modello di Gurson, adatto alla descrizione del comportamento meccanico di materiali metallici duttili porosi. La variante proposta del modello di Gurson, pur non incorporando una legge di evoluzione della porosità, riduce la componente volumetrica del flusso plastico rispetto al caso della legge di normalità e definisce quindi un comportamento non-associato. Si mostra che la porosità è un fattore instabilizzante nella propagazione della frattura legato, tra l'altro, alla possibilità di comparsa di discontinuità nei campi tensionali

1. INTRODUCTION

Slow crack propagation in ductile, elastoplastic materials is a topic in fracture mechanics, which has received increasing attention in the last twenty years (Amazigo and Hutchinson, 1977; Achenbach et al., 1981; Lam and McMeeking, 1984; Aoki et al. 1987; Ponte Castañeda, 1987; Östlund and Gudmunson, 1988; Tvergaard and Needleman, 1992; Narasimhan et al., 1993; Miao and Drugan, 1995). In recent years, the authors were successful in generalizing the asymptotic solution obtained by Ponte Castañeda (1987) for J_2-flow theory, to various elastoplastic models (Bigoni and Radi, 1993; 1996; Radi and Bigoni, 1993; 1994; 1996). In particular, a constant porosity version of the Gurson (1977) model was analyzed in (Radi and Bigoni, 1994; 1996), for both isotropic and kinematic hardening. In this model, the associative flow law has been assumed. It is however a well-established experimental fact that the associative flow law overestimates the dilatancy due to plastic flow in practically all materials exhibiting pressure-sensitive yielding. In particular, this is also true for porous metals, to which the Gurson model refers (Needleman and Rice, 1978). Moreover, the non-associativity of the plastic flow is known to be connected to a number of instabilities [shear banding (Rice, 1976), loss of second order work positiveness (Maier and Hueckel, 1979), flutter instability (Bigoni, 1995)]. For crack propagation problems, non-associativity is related to the appearance of discontinuity in the stress field (Drugan and Rice, 1984; Drugan, 1995). Despite of its importance, non-associativity of the flow law was rarely investigated in asymptotic solutions of crack problems. In particular, the only contributes are that by Nemat-Nasser and Obata (1990), referred to a special kind of non-associativity resulting from a non-coaxial flow rule, and that of the authors (Radi and Bigoni, 1993). The latter work is restricted to a simple Drucker-Prager plasticity model with isotropic hardening and shows that the flow mode tensor, rather than the yield function gradient, is the dominant parameter in determining the asymptotic fields. Some of these results are confirmed in the present work, were a version of Gurson model with linear hardening, constant porosity, but different yield and plastic potential functions is proposed. In essence, the plastic potential function corresponds to a yield function with different (smaller) porosity. This simple generalization of the constant porosity version of the Gurson model employed in (Drugan and Miao, 1992; Miao and Drugan, 1993; 1995; Radi and Bigoni, 1993; 1996) respects the basic physical fact that dilatancy results inferior than that corresponding to the normality rule. However, the model is, to a some extent, artificial, in the sense that a nucleation and growth law for the porosity is not defined. On the other hand, the model remains simple enough to allow us to obtain an asymptotic solution of the near tip fields for mode I crack propagation, so that the effects related to non-associativity can be investigated. These are found to be relevant, instabilizing with respect to crack propagation, and related to the possibility of the appearance of stress jumps in the solution.

2. MATERIAL MODEL

The simplified version of the Gurson (1977) model employed in this article is described in this section. The model is based on the Gurson yield condition for a porous solid:

$$f(\boldsymbol{\sigma}, \sigma_m) = \frac{3\,|dev\,\boldsymbol{\sigma}|^2}{2\,\sigma_m^2} + 2\,\phi\,cosh\left(\frac{tr\,\boldsymbol{\sigma}}{2\sigma_m}\right) - (1+\phi^2) = 0, \tag{2.1}$$

where ϕ is the volume fraction of voids (assumed constant), $\boldsymbol{\sigma}$ is the average macroscopic stress tensor and σ_m, the isotropic hardening parameter, defines the size of the current yield surface of the matrix material.
A non-associative plastic flow law for the macroscopic porous material is assumed

$$\dot{\boldsymbol{\varepsilon}}^p = \Lambda\,\mathbf{P}, \tag{2.2}$$

where Λ is the (non negative) plastic multiplier and $\mathbf{P}$ is defined as

$$\mathbf{P} = \frac{3}{2\sigma_m} dev\boldsymbol{\sigma} + \psi\,\frac{\phi}{2}\,sinh\left(\frac{tr\,\boldsymbol{\sigma}}{2\sigma_m}\right)\mathbf{I}, \tag{2.3}$$

where $\psi \in [0,1]$ is a non-associativity parameter. In particular, $\psi = 1$ corresponds to the

associative flow law and ψ=0 to a null volumetric plastic deformation (J_2-type flow). The yield function gradient **Q** is therefore equal to **P** when ψ = 1.
Adopting linear hardening for the macroscopic mechanical behavior, the evolution law of the hardening parameter can be obtained in the form

$$\dot{\sigma}_m = \Lambda \frac{3\,H_m}{(1-\phi)\sigma_m}\,\mathbf{P}\bullet\boldsymbol{\sigma}, \tag{2.4}$$

where H_m is the hardening modulus of the matrix material, depending on the ratio $\alpha_G = G_t/G$ of the current tangential modulus to the elastic shear modulus of the matrix material:

$$H_m = \frac{\alpha_G}{1-\alpha_G}\,G. \tag{2.5}$$

Prager consistency must be satisfied during plastic flow, whence the expression of the plastic multiplier may be derived:

$$\Lambda = \frac{\langle \mathbf{Q}\cdot\dot{\boldsymbol{\sigma}}\rangle}{H}, \tag{2.6}$$

where $\langle\,\rangle$, the Macaulay brackets, is the operator $\mathbb{R}\to\mathbb{R}^+\cup\{0\}$, $\forall x\in\mathbb{R}$, $\langle x\rangle = Sup\{x,0\}$, and

$$H = \frac{3\,H_m}{(1-\phi)\sigma_m^2}\,(\mathbf{Q}\bullet\boldsymbol{\sigma})\,(\mathbf{P}\bullet\boldsymbol{\sigma}), \tag{2.7}$$

is the macroscopic hardening modulus of the porous material.
If the elastic behavior is assumed isotropic, the elastoplastic incremental constitutive equations relating the stress rate $\dot{\boldsymbol{\sigma}}$ to the velocity of deformation $\dot{\boldsymbol{\varepsilon}}$, can finally be written in the standard form:

$$\dot{\boldsymbol{\varepsilon}} = \frac{1}{2G}\left[\dot{\boldsymbol{\sigma}} - \frac{\nu}{1+\nu}\left(tr\,\dot{\boldsymbol{\sigma}}\right)\mathbf{I}\right] + \frac{\langle \mathbf{Q}\cdot\dot{\boldsymbol{\sigma}}\rangle}{H}\,\mathbf{P}, \tag{2.8}$$

where ν is the Poisson ratio.

3. ASYMPTOTIC SOLUTION OF CRACK PROPAGATION

The problem of a crack quasi-statically and rectilinearly propagating at constant velocity c in an infinite medium, under mode I and plane strain conditions, is briefly exposed in this section [for further details the interested reader is referred to (Radi and Bigoni 1993; 1996)]. The mechanical behavior of the material is described by the incremental elasto-plastic constitutive law presented in Section 2. This framework allows incorporation of elastic unloading sectors, which may appear in the proximity of crack-tip during crack propagation. A cylindrical co-ordinate system (O, $\mathbf{e}_r$, $\mathbf{e}_\vartheta$, $\mathbf{e}_3$) moving with the crack-tip in the $\vartheta = 0$ direction is adopted, with the x_3-axis along the straight crack front. The steady-state condition yields the following time derivative rule, for any scalar, vector or second order tensor **A**:

$$\dot{\mathbf{A}} = \frac{c}{r}\left[\frac{\partial \mathbf{A}}{\partial \vartheta}\,sin\vartheta - r\,\frac{\partial \mathbf{A}}{\partial r}\,cos\vartheta\right], \tag{3.1}$$

where r and ϑ are the polar co-ordinates in the plane orthogonal to the x_3-axis. The derivatives of a tensor **A** and of its components with respect to ϑ are denoted by $\mathbf{A}'$, and by $A_{\alpha\beta,\vartheta}$, respectively. The kinematic compatibility condition between strain rates and velocities is:

$$\dot{\boldsymbol{\varepsilon}} = \tfrac{1}{2}(\nabla\mathbf{v} + \nabla\mathbf{v}^{\mathrm{T}}). \tag{3.2}$$

The plane strain condition $\dot{\varepsilon}_{33} = 0$ must be considered in addition to (3.2). The quasi-static equilibrium condition is:

$$div\,\boldsymbol{\sigma} = \mathbf{0}. \tag{3.3}$$

Compatibility (3.2), equilibrium (3.3) and the incremental constitutive eqns (2.16) and (2.17) form a system of first order PDEs which governs the problem of crack propagation. The solution is sought in a separable-variable form, by considering single-term asymptotic expansions of near crack-tip fields. In particular, the stress, velocity, and current flow stress fields are assumed in the forms:

$$\mathbf{v}(r,\vartheta) = \frac{c}{s}\left(\frac{r}{B}\right)^{s}\mathbf{w}(\vartheta),\quad \boldsymbol{\sigma}(r,\vartheta) = E\left(\frac{r}{B}\right)^{s}\mathbf{T}(\vartheta),\quad \sigma_m(r,\vartheta) = E\left(\frac{r}{B}\right)^{s}T_m(\vartheta), \tag{3.4}$$

where s is the exponent of the field singularity and B denotes a characteristic dimension of the plastic zone. Observe that s and functions **w**, **T**, and T_m are the unknowns of the problem and do not depend on the value of the crack propagation velocity c, since inertia is not accounted for. Moreover, the characteristic dimension B of the plastic zone remains undetermined, since the asymptotic problem is homogeneous. The material time rates of the fields $\boldsymbol{\sigma}$ and σ_m may be derived from representation (3.4) by using the steady-state derivative (3.1) in the form:

$$\dot{\boldsymbol{\sigma}}(r,\vartheta) = E\,\frac{c}{r}\left(\frac{r}{B}\right)^{s}\boldsymbol{\Sigma}(\vartheta),\qquad \dot{\sigma}_m(r,\vartheta) = E\,\frac{c}{r}\left(\frac{r}{B}\right)^{s}\Sigma_m(\vartheta), \tag{3.5}$$

where the functions $\boldsymbol{\Sigma}$ and Σ_m, which are independent of r, may be written, using representation (3.4), in the form:

$$\boldsymbol{\Sigma} = \mathbf{T}'\,sin\vartheta - s\,\mathbf{T}\,cos\vartheta,\qquad \Sigma_m = T_m'\,sin\vartheta - s\,T_m\,cos\vartheta. \tag{3.6}$$

When the asymptotic fields (3.4) and their rates (3.5) are introduced into the incremental constitutive relationships (2.16) and (2.17), a system of five scalar equations is obtained:

$$\left(\frac{1}{s}\,\mathbf{w}'\otimes\mathbf{e}_\vartheta + \mathbf{w}\otimes\mathbf{e}_r\right)_{\mathrm{Sym}} = (1+\nu)\,\boldsymbol{\Sigma} - \nu\,tr\boldsymbol{\Sigma}\,\mathbf{I} + \left\langle\frac{\mathbf{Q}\cdot\boldsymbol{\Sigma}}{h}\right\rangle\mathbf{P}, \tag{3.15}$$

$$T_m'\,sin\vartheta = s\,T_m\,cos\vartheta + \left\langle\frac{\mathbf{Q}\cdot\boldsymbol{\Sigma}}{\mathbf{Q}\cdot\mathbf{T}}\right\rangle T_m, \tag{3.16}$$

where $h = H/E$. The constitutive equations (3.15)-(3.16) are valid when the stress point lies on the yield surface. During elastic unloading or neutral loading, the constitutive relation (3.15) reduces to the incremental equation of linear isotropic elasticity, and equation (3.16) becomes $\dot{\sigma}_m = 0$.

System (3.15)-(3.16), together with the equilibrium equations:

$$\mathbf{T}'\,\mathbf{e}_\vartheta + s\,\mathbf{T}\,\mathbf{e}_r = \mathbf{0}, \tag{3.17}$$

result in seven first order ODEs of homogeneous type, for the seven unknowns components of the angular functions **w**, **T**, and T_m. The unknown exponent s may be determined as an eigenvalue of the problem, when a normalization condition for the solution is considered. Since the ODEs (3.15)-(3.17) are in implicit form, some algebraic manipulations are necessary in order to obtain the explicit forms. In particular, by considering the components rr and 33 of the constitutive eqns (3.15), and using the plane strain condition, the following system of equations may be derived:

$$(h + P_{rr} Q_{rr})\, \Sigma_{rr} - (\nu h - P_{rr} Q_{33})\, \Sigma_{33} = w_r h + (\nu h - P_{rr} Q_{\vartheta\vartheta})\, \Sigma_{\vartheta\vartheta} - 2 P_{rr} Q_{r\vartheta} \Sigma_{r\vartheta},$$

$$(\nu h - Q_{rr} P_{33})\, \Sigma_{rr} - (h + P_{33} Q_{33})\, \Sigma_{33} = 2 P_{33} Q_{r\vartheta} \Sigma_{r\vartheta} - (\nu h - P_{33} Q_{\vartheta\vartheta})\, \Sigma_{\vartheta\vartheta}. \quad (3.18)$$

Equations (3.17) may be solved for Σ_{rr} and Σ_{33}, when

$$\Delta = (1 - \nu^2)\, h + P_{rr} (Q_{rr} + \nu Q_{33}) + P_{33} (Q_{33} + \nu Q_{rr}), \quad (3.19)$$

is different from zero. Note that Δ is always positive for associative flow rule, but may vanish in the case of the non-associative flow law. When $\Delta \neq 0$, the first order ODEs system (3.15)-(3.17), which governs the near-tip stress and velocity fields may be written in the standard form:

$$\mathbf{y}'(\vartheta) = \begin{cases} \mathbf{f}_p(\vartheta, \mathbf{y}(\vartheta), s) & if \quad f(\mathbf{T}, T_m) = 0 \quad and \quad \mathbf{Q}.\boldsymbol{\Sigma} > 0, \\ \mathbf{f}_e(\vartheta, \mathbf{y}(\vartheta), s) & if \quad f(\mathbf{T}, T_m) < 0 \quad or \quad f(\mathbf{T}, T_m) = 0 \; and \; \mathbf{Q}.\boldsymbol{\Sigma} \leq 0, \end{cases} \quad (3.19)$$

where $\mathbf{y} = \{w_r, w_\vartheta, T_{r\vartheta}, T_{rr}, T_{\vartheta\vartheta}, T_{33}, T_m\}$. When the boundary conditions, corresponding to null tractions on crack surfaces and to $T_{r\vartheta}(0) = w_\vartheta(0) = 0$, are considered, system (3.19) can be numerically integrated using a Runge-Kutta procedure in which $T_{\vartheta\vartheta}(0) = 1$ is assigned to avoid the trivial solution and s and $T_{rr}(0)$ are tentatively assigned and corrected to satisfy the conditions at $\vartheta = \pi$.

It is important to remark that, when Δ vanishes, the above procedure breaks down. The condition Δ=0 corresponds to the condition of strain localization in a radial planar band (Bigoni and Hueckel, 1990) which, in turn, coincides with the possible appearance of moving shock waves (Brannon and Drugan, 1993) (see Fig. 1). In particular, jumps in the stress components σ_{33} and σ_{rr} may appear, an occurrence excluded for elastoplastic materials displaying positive hardening with associative flow law (Drugan and Rice, 1984; Drugan, 1995). Examples of stress jumps have been shown by Nemat-Nasser and Obata (1990), but these results remain controversial (Brannon and Drugan, 1993).

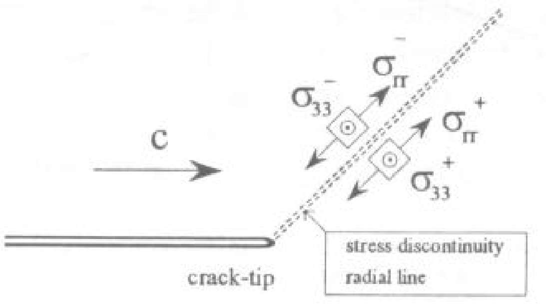

Fig. 1. Moving stress discontinuity at crack-tip

4. RESULTS

Results of integration of system (3.19), when $\Delta \neq 0$, are shown in this section. Values of the singularity (s), elastic unloading (ϑ_1) and reloading (ϑ_2) angles are reported in Tables I and II for different values of porosity (ϕ) and non-associativity coefficient ψ. The tables, as well as all other results, are restricted, for conciseness, to $\alpha_G = 0.001$, $\nu = 1/3$, $\phi = 0.01$, and $\phi = 0.05$.

The singularity coefficient s and the elastic unloading and plastic reloading angles ϑ_1 and ϑ_2 are reported in Fig. 2, as functions of ψ. Note that the curves in Fig. 2 terminates when Δ=0 has occurred.

The angular distribution of components of **T**, are plotted in Figs.3 and 4, for different values of ψ. It can be concluded that an increase in non-associativity yields:

- an increase in the strength of the singularity,
- a flatten of the curves,
- a reduction of the elastic sector in the crack wake.

Table I. Values of s, ϑ_1, ϑ_2 for Φ=0.01

ψ	s	ϑ_1	ϑ_2
1.00	-0.03224	132.885	142.724
0.80	-0.03746	133.795	140.868
0.60	-0.04394	134.574	139.466
0.40	-0.05262	135.339	138.517
0.30	-0.05849	135.769	138.262
0.20	-0.06630	136.285	138.241

Table II. Values of s, ϑ_1, ϑ_2 for Φ=0.05

ψ	s	ϑ_1	ϑ_2
1.00	-0.02746	97.471	178.510
0.90	-0.03198	99.410	176.937
0.80	-0.03671	102.848	174.393
0.70	-0.04140	108.683	170.208
0.68	-0.04230	110.163	169.084
0.65	-0.04362	112.532	167.186

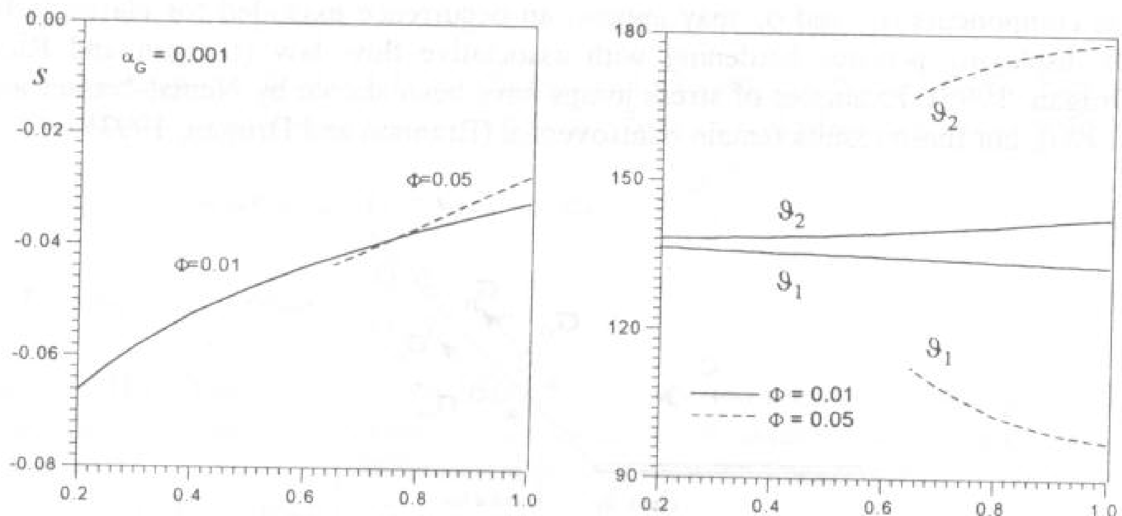

Fig. 2. Stress singularity s, elastic unloading ϑ_1 and plastic reloading ϑ_2 angles as functions of non-associativity parameter Ψ.

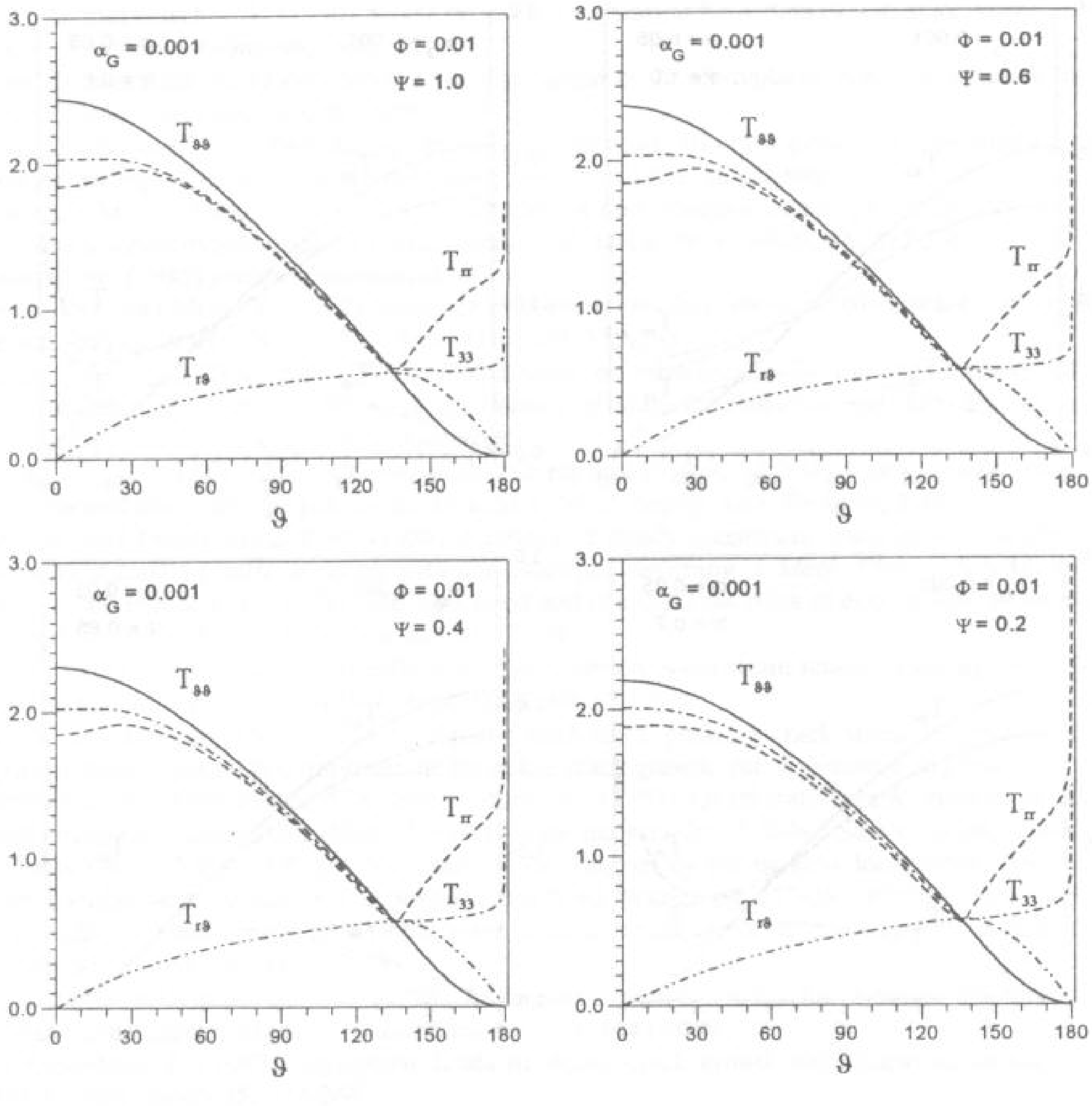

Fig. 3. Stress functions for Φ=0.01 and different values of non-associativity parameter Ψ.

5. CONCLUSIONS

The asymptotic crack tip fields in steady Mode I propagation in an elastoplastic material obeying a simplified version of the Gurson model with constant porosity, isotropic hardening and *non-associative flow law* have been determined. The results show increase in the strength of the singularity related to increase of the non-associativity. Compared to the associative case, this should be related to an instability in crack propagation. Moreover, for high values of non-associativity, the condition of *strain localization* in planar, radial bands is shown to be satisfied. This yields a break down of the numerical scheme of integration related to ill posedness of the equations. From the physical point of view, it is believed that discontinuous stress fields could appear even if they cannot be detected in the present framework.

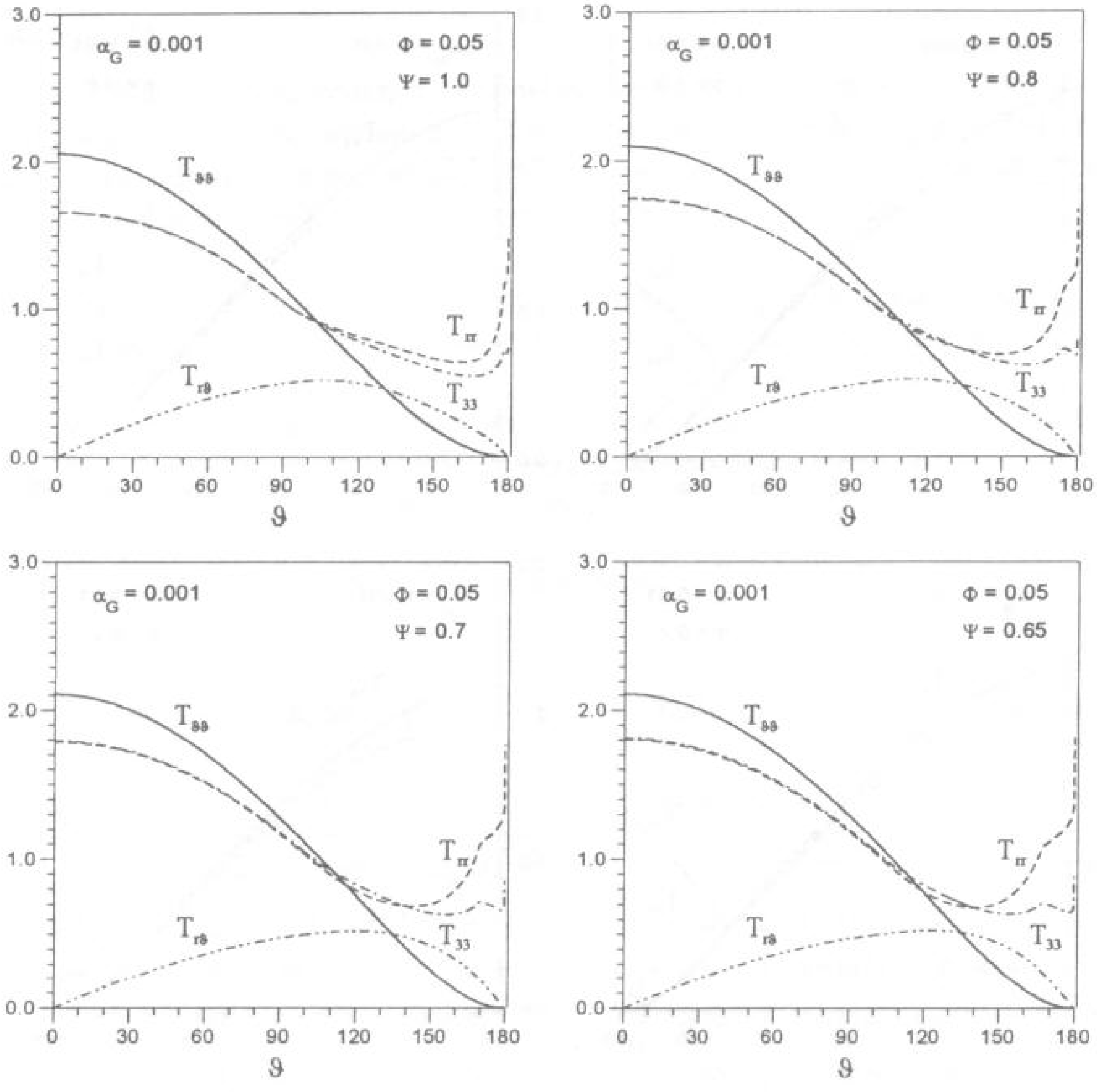

Fig. 4. Stress functions for Φ=0.05 and different values of non-associativity parameter Ψ.

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