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CRACK PROPAGATION IN BRITTLE, ELASTIC SOLIDS WITH DEFECTS

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SOMMARIO

Si sviluppa una soluzione asintotica per la propagazione di una frattura in un materiale elasto-fragile contenente difetti. La fragilità del materiale è tale da giustificare l'adozione del criterio di Sih per la propagazione (K_{II}=0). La soluzione, seppure approssimata nel senso delle tecniche perturbative, è esprimibile in forma chiusa. I difetti sono rappresentati da cavità di forma circolare o ellittica e/o inclusioni elastiche circolari. Il confronto con risultati sperimentali relativi a gres porcellanato, smalto e compositi Allumina/Zirconia risulta molto soddisfacente.

1. INTRODUCTION

The analysis of failure mechanisms of brittle, composite (defect-containing, porous, particulate/fiber-reinforced) materials has design implications in a broad range of contexts. Examples of these materials are: structural and traditional ceramics, which may contain flaws or pores, fibrous biological materials, porous rocks (e.g. tuff, pumice-stone, blast furnace slag), porous high-strength metals and ceramic or metal composites. In other materials, like concrete or certain rocks (e.g. chalk with flints), stiff inclusions co-exist in a soft matrix with pores and microcracks.
It is obvious that fracture propagation is affected by the presence of inhomogeneities. These modify the crack trajectory and, consequently, influence the toughness of the material. In particular, the toughening effect of a diluted, spherical porosity in a brittle matrix remains still controversial (see, e.g., Duan et al., 1995, Claussen, 1976). On one

hand, pores may act as stress concentrators and initiate strain localisation and subsequent microcrack bridging between cavities. On the other hand, pores deviate the crack path from linearity and, when the crack tip intersects a cavity, this may produce a stress release. From the latter point of view pores yield a shielding effect on crack propagation.

The above discussion elucidates the theoretical and practical relevance of developing analytical models capable of describing fracture mechanisms of brittle, elastic materials containing voids or defects. This is the focus of works by Rubinstein (1986), Rose (1986) and of the present article. An asymptotic solution is presented for the determination of the trajectory of a crack growing in an elastic-brittle isotropic material, under plane strain (or stress) conditions. With the term "brittle" we mean a material in which the fracture propagates according to the Sih (1974) criterion, namely, the crack grows so that at its tip $K_{II}=0$ (implying that the energy release rate is locally a maximum). Two perturbed solutions are employed, one of which concerns the modification of the near-tip fields due to a perturbation from rectilinearity in the crack trajectory. In the other perturbed solution, the defects are introduced and the modification on the near-tip field evaluated. The former analysis is similar to some extent to the solution obtained by Cotterell and Rice (1980). The latter analysis was initiated by Zorin et al. (1988) and developed in (Movchan et al. 1991; Movchan, 1992). The analysis is based on the concept of Pólya-Szegö (1951) matrix, which characterises the far-field effects of the defect. An experimental program has been developed for comparing real fracture patterns in traditional and advanced ceramic materials to analytical simulation of crack trajectory. Surface cracks have been induced in various ceramic materials using Vicker indenter and subsequently observed using SEM. The observations regard porous ceramic materials, namely, porcelain stoneware and glaze, and Zirconia/Alumina composites. In all these materials fracture trajectories are far from linearity. Porcelain stoneware and glaze contain a dilute near-spherical or elliptical porosity, whereas the composites material consists of a Zirconia matrix which contains 20% of near-spherical Alumina particles. As a conclusion, it can be pointed out that the analytical solution shows impressive agreement with experiments.

2. ANALYTICAL MODEL

2.1 Problem formulation

A semi-infinite, plane crack is considered, curved in the portion extending from the tip till a point where it becomes rectilinear, as indicated in Fig.1. With reference to a coordinate system having the origin at that point and the axis x_1 tangent to the rectilinear crack surfaces, the curved portion of the crack has length l. If the curved portion of the crack is sufficiently regular and close to rectilinearity, it can be treated as *a perturbed straight crack*. In this case, the crack geometry can be specified by introducing a smooth function h of x_1, which, when multiplied by a perturbation parameter, say, α, specifies the x_2-coordinate of the curved portion of the crack. Therefore, the semi-infinite crack is described by the set $M_\alpha=\{(x_1, x_2) : x_1 < l, x_2 = \alpha\, h(x_1)\}$, with $0 < \alpha << 1$.

A defect is considered in the form of a cavity or an inclusion, in a position which is to some extent arbitrary, in the sense that it can be placed in an arbitrary point $\boldsymbol{x}^0$, but the ratio between the diameter of the defect and the minimal distance from crack trajectory has to be small enough to allow the use of a perturbed solution, that will be introduced in the following. In particular, we can use stretched coordinates $\varepsilon^{-1}\boldsymbol{x}$ and define the defect by the set $\Omega_\varepsilon=\{(x_1, x_2):\varepsilon^{-1}(x_1-x_1^0, x_2-x_2^0)\in\Omega\subseteq\mathbf{R}^2\}$, where $\varepsilon=\frac{1}{2}\mathrm{diam}\Omega_\varepsilon/\mathrm{dist}(\Omega_\varepsilon, M_\alpha)<<1$.

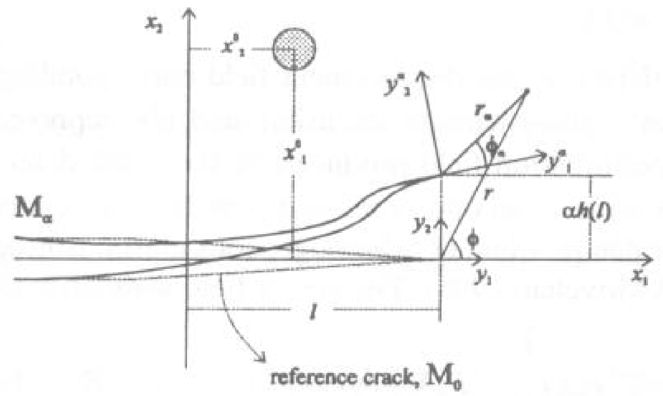


Fig.1 - Crack geometry and reference systems

The above-described crack problem regards plane strain deformation of linear elastic, isotropic materials, characterised by the Lamé constants λ and μ, for the matrix material, and λ_0 and μ_0, for the inclusion. A displacement vector $\boldsymbol{u}$, solution of the above problem, satisfies the system of equilibrium equations

$$\boldsymbol{L u} := \mu\nabla^2\boldsymbol{u} + (\lambda + \mu)\,\nabla\nabla\cdot\boldsymbol{u} = 0, \quad \boldsymbol{x} \in \mathbf{R}^2 \setminus (\Omega_\varepsilon \cup \mathrm{M}_\alpha), \tag{1}$$

$$\boldsymbol{L}^0\boldsymbol{u}^0 := \mu_0\nabla^2\boldsymbol{u}^0 + (\lambda_0 + \mu_0)\,\nabla\nabla\cdot\boldsymbol{u}^0 = 0, \quad \boldsymbol{x} \in \mathbf{R}^2 \setminus \Omega_\varepsilon, \tag{2}$$

and the boundary condition on crack faces

$$\boldsymbol{\sigma}^{(n)}(\boldsymbol{u}\,;\boldsymbol{x}) = \boldsymbol{0}, \qquad \boldsymbol{x} \in \mathrm{M}_\alpha^{\pm} \tag{3}$$

and on the inclusion boundary

$$\boldsymbol{\sigma}^{(n)}(\boldsymbol{u}\,;\boldsymbol{x}) = \boldsymbol{\sigma}^{(n,0)}(\boldsymbol{u}^0\,;\boldsymbol{x}), \quad \boldsymbol{u} = \boldsymbol{u}^0, \qquad \boldsymbol{x} \in \partial\,\Omega_\varepsilon. \tag{4}$$

For the case of a cavity, (4) are replaced by the condition

$$\boldsymbol{\sigma}^{(n)}(\boldsymbol{u}\,;\boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \partial\,\Omega_\varepsilon. \tag{5}$$

At infinity, the displacement vector is supposed to have the following asymptotic form

$$\boldsymbol{u}(\boldsymbol{x}) \sim K_{\mathrm{I}}^{\infty}\, r^{1/2}\, \boldsymbol{\Phi}^{\mathrm{I}}(\varphi), \quad \text{as } r \to \infty, \tag{6}$$

where K_{I}^{∞} is given, and the polar components of the vector function $\boldsymbol{\Phi}^{\mathrm{I}} = (\Phi^{\mathrm{I}}_r, \Phi^{\mathrm{I}}_\varphi)^{\mathrm{T}}$ are

$$\Phi^{\mathrm{I}}_r(\varphi) = \frac{(2\kappa - 1)\cos\dfrac{\varphi}{2} - \cos\dfrac{3\varphi}{2}}{4\mu\sqrt{2\pi}}, \qquad \Phi^{\mathrm{I}}_\varphi(\varphi) = \frac{\sin\dfrac{3\varphi}{2} - (2\kappa + 1)\sin\dfrac{\varphi}{2}}{4\mu\sqrt{2\pi}}, \tag{7}$$

where $\kappa = (\lambda + 3\mu)(\lambda + \mu)^{-1}$, is the elastic bulk modulus.

2.2 Crack deflection due to defects

The perturbation introduced by a defect on the near tip crack fields is considered for a rectilinear crack M_0. For this problem, following Movchan et al. (1991) and Movchan and Movchan (1995) the displacement field *around the crack tip* can be represented as

$$u(x) \sim v(x) + \varepsilon^2 w(x), \tag{8}$$

where $v(x) = K_I\, r^{1/2}\, \Phi^I(\varphi)$ is the displacement field corresponding to a rectilinear (i.e. unperturbed) crack in a plane without inclusion, and $\varepsilon^2 w$ represents a correction term associated with the perturbation field produced by the small defect Ω_ε. The reason for which the correction term is second-order in ε can be appreciated by considering the Neumann boundary value problem of a homogeneous elastic isotropic solid containing a defect (Movchan and Movchan,1995). The vector field w satisfies the system of equation

$$Lw(x) = -\sum_{j,k=1}^{3}\left[V_{\frac{\partial}{\partial x}}^{(j)T} v(x) \right]_{x=x^0} P_{jk} V_{\frac{\partial}{\partial x}}^{(j)} \delta(x-x^0), \qquad x \in \mathbf{R}^2 \setminus \mathrm{M}_0. \tag{9}$$

and the homogeneous traction boundary conditions on the crack faces. In eqn.(9) δ is the Dirac function and:

$$V_{\frac{\partial}{\partial x}}^{(1)} = \begin{pmatrix} \dfrac{\partial}{\partial x_1} \\ 0 \end{pmatrix}, \qquad V_{\frac{\partial}{\partial x}}^{(2)} = \begin{pmatrix} 0 \\ \dfrac{\partial}{\partial x_2} \end{pmatrix}, \qquad V_{\frac{\partial}{\partial x}}^{(3)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_1} \end{pmatrix}. \tag{10}$$

Function $w(x)$ can be written in components:

$$w^{(j)}(x) = -\sum_{j,k=1}^{3} P_{jk} V_{\frac{\partial}{\partial x}}^{(j)} T(x) + O(|x^{-2}|), \tag{11}$$

where, T is the Somigliana matrix (Fichera, 1972; Parton and Perlin, 1982)

$$T(x) = \frac{\lambda+\mu}{4\pi\mu(\lambda+2\mu)} \begin{pmatrix} -\kappa \ln\sqrt{x_1^2+x_2^2} + \dfrac{x_1^2}{x_1^2+x_2^2} & \dfrac{x_1 x_2}{x_1^2+x_2^2} \\ \dfrac{x_1 x_2}{x_1^2+x_2^2} & -\kappa \ln\sqrt{x_1^2+x_2^2} + \dfrac{x_2^2}{x_1^2+x_2^2} \end{pmatrix}, \tag{12}$$

and P_{ij} is the Pólya-Szegö matrix, which characterises the defect. For instance, when the defect is a circular cavity of radius c, thè Pólya-Szegö matrix becomes

$$P = -\frac{c^2\pi(\lambda+2\mu)}{\mu(\lambda+\mu)} \begin{pmatrix} \lambda^2+3\mu^2+2\lambda\mu & \lambda^2-\mu^2+2\lambda\mu & 0 \\ \lambda^2-\mu^2+2\lambda\mu & \lambda^2+3\mu^2+2\lambda\mu & 0 \\ 0 & 0 & 4\mu^2 \end{pmatrix}. \tag{13}$$

We are now in a position to investigate the main problem, namely, perturbation of the crack trajectory. In the absence of a defect, the crack would propagate rectilinearly under Mode I loading, a condition which trivially satisfies the Sih criterion $K_{II}=0$. The presence of a defect, even small, produces a perturbation in terms of a smooth deflection from linearity of crack trajectory. Let us consider the asymptotic of the displacement field near the tip of perturbed crack M_α. A local system of coordinates y^α can be introduced, which has the origin centred in the tip of the perturbed crack (i.e. in the point of coordinates (l, $\alpha h(l)$) and axis y_1^α directed toward the tangent to the crack trajectory at the crack-tip (Fig.1). The asymptotic of the displacement vector u, relative to the perturbed crack, can be represented as follows

$$u(y^\alpha) \sim \sum_{j=1}^{2} r_\alpha^{1/2}\, K_j(\alpha)\, \Phi^{(j)}(\varphi_\alpha), \qquad K_j(\alpha) \sim K_j(l) + \alpha K_j'(l), \tag{14}$$

where quantities evaluated in l are referred to the unperturbed problem, so that $K_{II}(l)=0$. A Taylor series expansion near $\alpha = 0$ yields

$$u(y,\alpha) \sim v(y) + \alpha \Bigg[-\frac{1+\kappa}{4\mu} h(l)\, K_I(l)\, r^{-1/2}\, \Psi^{II}(\varphi) + K_I'(l)\, r^{1/2}\, \Phi^{I}(\varphi) + \left(K_{II}'(l) - \frac{1}{2} K_I(l)\, h'(l) \right) r^{1/2} \Phi^{II}(\varphi) \Bigg], \tag{15}$$

where the components of Φ^{II} and Ψ^{II} are given by

$$\Phi^{II}{}_r(\varphi) = \frac{(1-2\kappa)\sin\frac{\varphi}{2} + 3\sin\frac{3\varphi}{2}}{4\mu\sqrt{2\pi}}, \qquad \Phi^{II}{}_\varphi(\varphi) = \frac{3\cos\frac{3\varphi}{2} - (2\kappa+1)\cos\frac{\varphi}{2}}{4\mu\sqrt{2\pi}}, \tag{16}$$

$$\Psi^{II}{}_r(\varphi) = \frac{(1+2\kappa)\sin\frac{3\varphi}{2} - \sin\frac{\varphi}{2}}{\sqrt{8\pi}(1+\kappa)}, \qquad \Psi^{II}{}_\varphi(\varphi) = \frac{(2\kappa-1)\cos\frac{3\varphi}{2} - \cos\frac{\varphi}{2}}{\sqrt{8\pi}(1+\kappa)}. \tag{17}$$

The stress components associated to (15) exhibit a strong unphysical singularity, a fact related to the presence of a boundary layer near the crack tip. It is a well-known expedient (Bueckner, 1970) to introduce a weight function:

$$\zeta^{II} = r^{-1/2}\, \Psi^{II}(\varphi), \tag{18}$$

and to integrate in a ring $\Xi_R = \{ y : 1/R \le \| y \| \le R\}$

$$\int_{\Xi_R \setminus M_0} \zeta^{II}(y) \cdot L w^*(y) - w^*(y) \cdot L\zeta^{II}(y) = \sum_{j,k=1}^{3} V^{(j)}_{\frac{\partial}{\partial y}} v(y^0)\, P_{jk}\, V^{(k)}_{\frac{\partial}{\partial y}} \varsigma^{II}(y^0) \tag{19}$$

Where the field:

$$w^*(y) = u'(y) + \frac{1+\kappa}{4\mu} h(l)\, K_I(l)\, r^{-1/2}\, \Psi^{II}(\varphi), \tag{20}$$

satisfies the same Lamé system and the traction boundary conditions on M_0 as $\boldsymbol{u}'$, which is the term of (15) that multiplies α, but does not have a singularity at the crack tip $(l,0)$. Unperturbed crack field (8) has to match with (15), thus $\alpha = \varepsilon^2$. Moreover, using the Betti formula to the l.h.s. of (20), taking the limit $R \to \infty$ and integrating, gives:

$$K_{II}'(l) = K_I(l)\, \{1/2\, h'(l) - \sum_{j,k=1}^{3} V^{(j)}_{\frac{\partial}{\partial y}} v(y^0)\, P_{jk}\, V^{(k)}_{\frac{\partial}{\partial y}} \varsigma^{II}(y^0)\}. \tag{21}$$

The Sih criterion of crack propagation, $K_{II} = 0$, yields

$$h'(l) = 2 \sum_{j,k=1}^{3} V^{(j)}_{\frac{\partial}{\partial y}} v(y^0)\, P_{jk}\, V^{(k)}_{\frac{\partial}{\partial y}} \varsigma^{II}(y^0), \tag{22}$$

which may be easily integrated, obtaining, finally

$$h(l) = \frac{4\mu}{y_2^0(1+\kappa)}\{cos\theta\, l(\theta)Pl(\theta) - l(0)Pl(0)\}, \tag{23}$$

where

$$cos\theta = \frac{-l + y_1^0}{\sqrt{(y_2^0)^2 + (-l + y_1^0)^2}}, \tag{24}$$

and

$$l(\varphi) = \begin{pmatrix} \frac{1}{4\mu\sqrt{2\pi}} cos\frac{\varphi}{2}\left[\kappa - 1 - 2sin\frac{\varphi}{2} sin\frac{3\varphi}{2}\right] \\ \frac{1}{4\mu\sqrt{2\pi}} cos\frac{\varphi}{2}\left[\kappa - 1 + 2sin\frac{\varphi}{2} sin\frac{3\varphi}{2}\right] \\ \frac{1}{4\mu\sqrt{\pi}} sin\varphi cos\frac{3\varphi}{2} \end{pmatrix}. \tag{25}$$

Equation (23) gives the crack trajectory h, as a function of the crack tip position l and of the coordinates of the defect centre, $\mathbf{y}^0$. The defect is characterised by the Pólya-Szegö matrix and more than one defect can be considered with a straightforward generalisation of the above computations.

2.3 Examples of crack trajectories

Crack trajectories for circular voids and elastic inclusions in different relative positions are shown in Fig. 2.

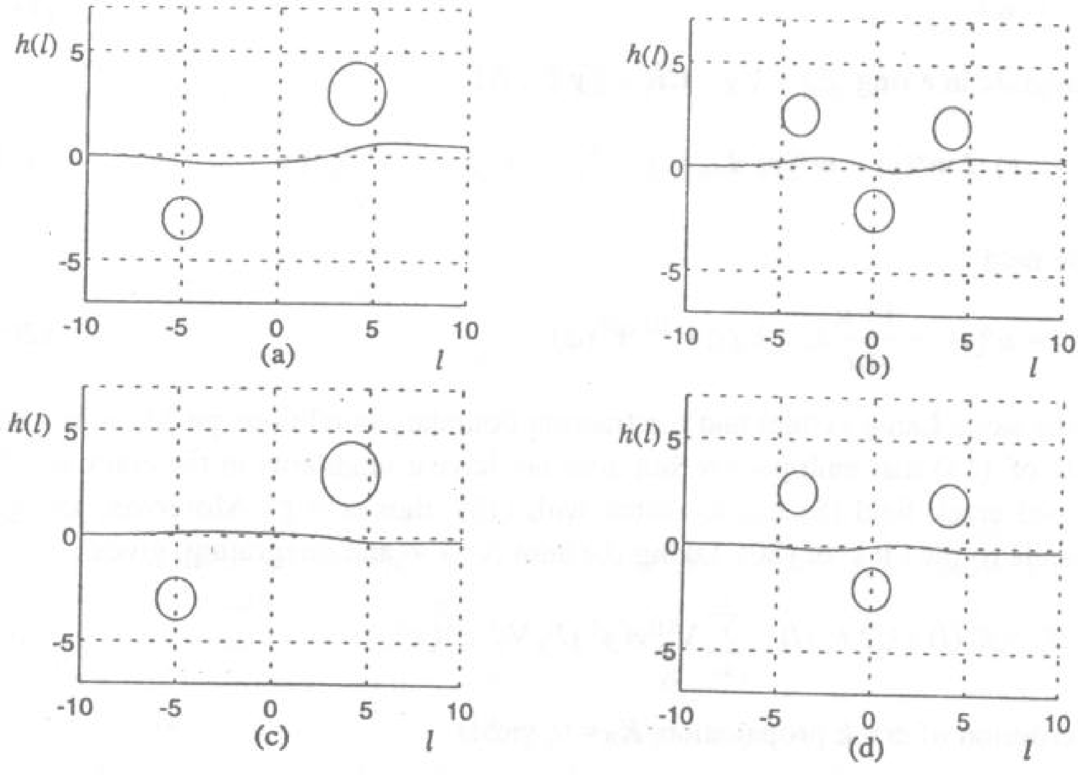
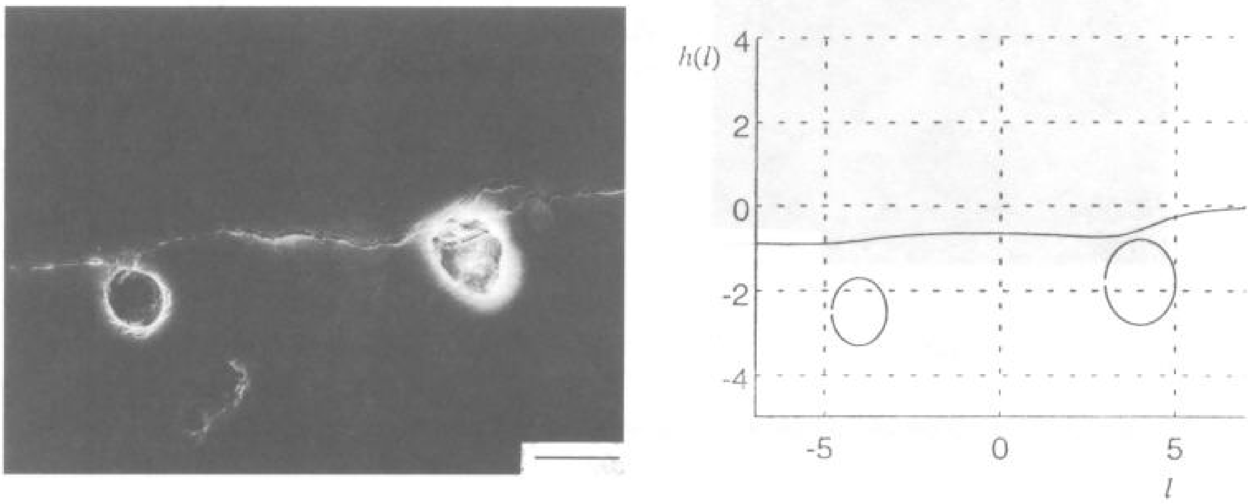


Fig.2 - Crack trajectories: (a) two circular voids of different radius, (b) three circular voids of equal radius, (c) two circular elastic inclusions (of different radius) stiffer than the matrix, (d) three circular elastic inclusions (of equal radius) stiffer than the matrix.

Fig. 2 (a) and (b) refer to two and three circular cavities and Fig. 2 (c) and (d) to two and three circular inclusions. For the elastic inclusions relative to Fig. 2 (c) and 2 (d), $\mu_0/\mu=1000$ and $(\lambda_0+\mu_0)/(\lambda+\mu)=450$ has been assumed. Note that inclusions stiffer than the matrix repel the crack, whereas softer inclusions or voids attract it.

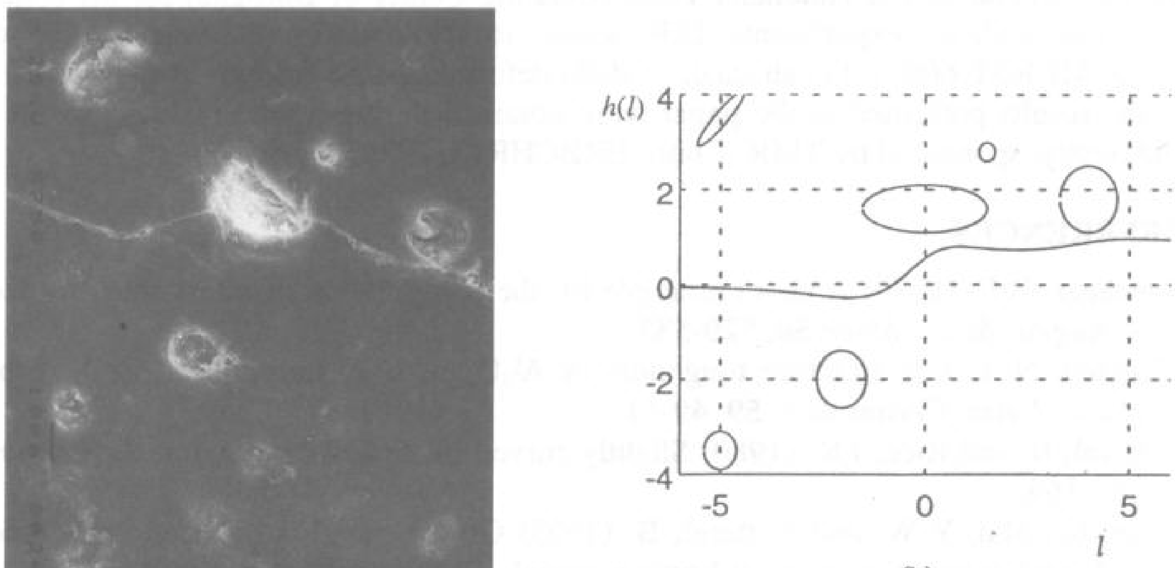
3. EXPERIMENTAL RESULTS

The SEM photographs relative to crack patterns in porcelain stoneware, glaze and Zirconia/Alumina composites are shown in Figs. 3-5 (a). The cracks were induced using Vicker indenter. The same crack trajectories have been simulated using eqn. (23) and represented in Figs. 3-5 (b).



(a) (b)

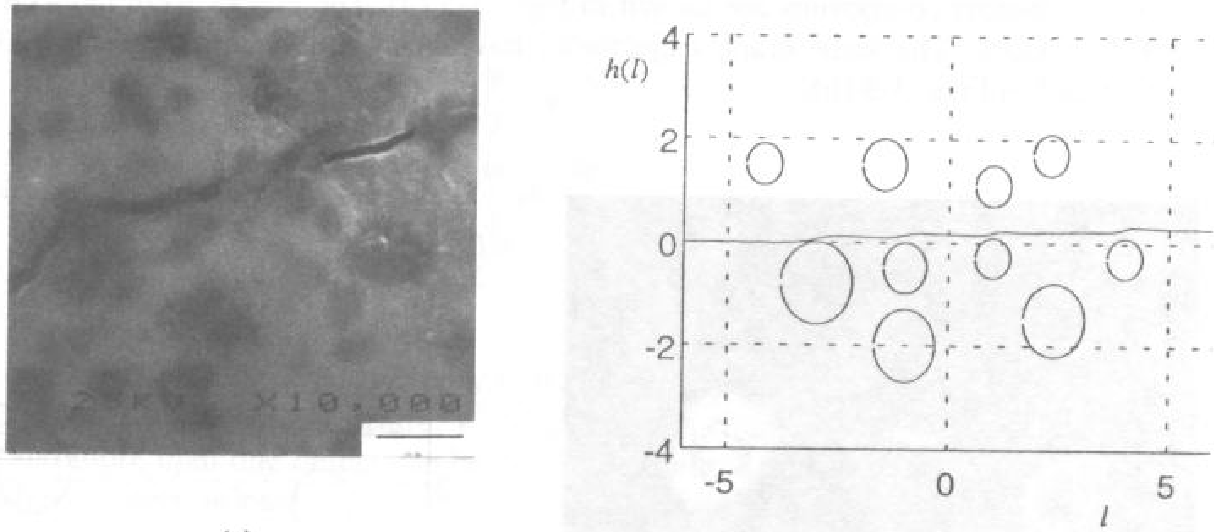
Fig. 3 - (a) SEM photograph of glaze: bar is 10 μm, (b) simulated crack trajectory



(a) (b)

Fig. 4 - (a) SEM photograph of porcelain stoneware: bar is 20 μm, (b) simulated crack trajectory

The defects relative to Figs. 3(a) and 4(a) have been represented through elliptical and circular voids in Figs. 3(b) and 4(b), whereas aggregation of Zirconia inclusions have been represented by large circular stiff inclusions in Fig.5(b). In the case of Fig. 5 the elastic constants of Zirconia and Alumina were known. In particular, $\lambda = 125$ GPa and $\mu = 75$ GPa, have been used for Zirconia matrix and $\lambda_0 = 140$ GPa and $\mu_0 = 180$ GPa for Alumina inclusion.



(a) (b)

Fig. 5 - (a) SEM photograph of ZrO_2/Al_2O_3 composite: bar is 1 μm, (b) simulated crack trajectory

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