On the Formulation of Non-local Multiaxial High-Cycle Fatigue Models based on the Theory of Critical Distances

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ABSTRACT. *This paper investigates the formulation of non-local multiaxial high-cycle fatigue models based on the Theory of Critical Distances. In the present framework, the critical distance is identified in a manner consistent with the multiaxial fatigue model under consideration – which may imply critical distances not equal to the widely used value given by half of the El Haddad's intrinsic crack length. The methodology is applied to well-known fatigue models, namely the ones proposed by Susmel & Lazzarin (Modified Wöhler Curve Method), Crossland and Dang Van, and compared with available experimental data obtained from small and/or sharp notches under proportional loading. The Modified Wöhler Curve Method showed the best agreement with the experimental data.*

INTRODUCTION

The Theory of Critical Distances (TCD) [1,2] has become a useful engineering approach to the problem of estimating the fatigue strength in the presence of high stress gradients. Among its important applications, we can mention the fatigue analysis of sharp and small notches [3-5] as well as mechanical contacts [6].

Fatigue models based on the TCD state that failure by high-cycle fatigue will not occur before a specified number of loading cycles if

$$\frac{1}{V}\int_V F(\Delta)\,\mathrm{d}V \le 0\,, \tag{1}$$

where Δ denotes the elastic stress history at a material point, $F(\cdot)$ is a function written in terms of equivalent stress measures and V is a material volume. The shape of the material volume is chosen *a priori* and its size is assumed to be a material parameter. Roughly speaking, Eq. 1 expresses that the driven forces which damage the material should be a non-local measure evaluated in a finite material volume, instead of being computed at a dimensionless point (hot-spot method).

The simplest approximation for Eq. 1, called The Point Method, may be written as

$$F(\Delta(L_c)) \le 0\,, \tag{2}$$

where $\Delta(L_c)$ is the elastic stress history at a certain distance L_c from the hot-spot. For simplicity and widespread use in applications, we shall only deal with The Point Method throughout the paper.

The crucial aspect of the TCD regards the identification of the material parameter L_c. D. Taylor [1] argued that since Eq. (2) holds for any elastic stress distribution (i.e. from uniform distributions to ones with high stress gradients), then it is possible to find this parameter by matching the estimated threshold stress intensity factor range with the one observed experimentally. Within this setting and writing the fatigue model as a function of the maximum principal stress, one can shown that the critical distance is given by

$$L_c = 0.5L, \quad L := \frac{1}{\pi}\left(\frac{\Delta K_{I,th}}{\Delta\sigma_{-1}}\right)^2. \tag{3}$$

The parameter L, called El Haddad's intrinsic crack length, depends on the threshold stress intensity factor range, $\Delta K_{I,th}$, and the fully reversed axial fatigue strength range, $\Delta\sigma_{-1}$, at a high number of loading cycles (conventionally, greater than 10^6 cycles).

Although the TCD has been successful in predicting the fatigue strength of sharply notched specimens subjected to uniaxial loading [3], its extension to complex loadings is not yet consolidated [7]. In this paper, we investigate the formulation of non-local multiaxial high-cycle fatigue models based on The Point Method. It is proposed that the critical distance should not always be given by half of El Haddad's intrinsic crack length. Instead, its value should depend on the multiaxial fatigue model under consideration.

NON-LOCAL MULTIAXIAL FATIGUE MODELS BASED ON THE THEORY OF CRITICAL DISTANCES

Non-local formulation of the Modified Wöhler Curve Method

The application of The Modified Wöhler Curve Method [8] into Eq. 2 yields

$$F_{SL}(\Delta(L_c)) = \tau_a + \alpha\frac{\sigma_{n,max}}{\tau_a} - \beta \le 0, \tag{4}$$

where τ_a is the shear stress amplitude on the material plane experiencing the maximum shear stress amplitude, $\sigma_{n,max}$ is the maximum normal stress in this plane, and α, β and L_c are material parameters. The shear stress amplitude on a material plane is defined as the radius of the minimum circumference enclosing the path described by the shear stress vectors.

Identification of material parameters

We shall start with the identification of the material parameter L_c. Let's consider a fatigue specimen with a long crack subjected to a fully reversed Mode I stress intensity factor given by

$$K_I(t) = 0.5\Delta K_I \sin\Omega t, \tag{5}$$

where ΔK_I is the stress intensity factor range. The elastic stress history at a distance L_c from the crack tip (assuming a plane stress state) is given by

$$\sigma_x(t) = \sigma_y(t) = \frac{1}{2}\frac{\Delta K_I}{\sqrt{2\pi L_c}}\sin\Omega\mathrm{t}. \tag{6}$$

This is a proportional stress history so that the evaluation of the stress measures in Eq. 4 are easily computed as

$$\tau_a = \frac{1}{4}\frac{\Delta K_I}{\sqrt{2\pi L_c}}, \qquad \rho = 1. \tag{7}$$

The substitution of these expressions into Eq. 4 provides that the estimated fatigue threshold range is

$$\Delta K_{I,th}^{SL} = 4(\beta - \alpha)\sqrt{2\pi L_c}\ . \tag{8}$$

Finally, to identify L_c, we impose the restriction

$$\Delta K_{I,th}^{SL} = \Delta K_{I,th}, \tag{9}$$

which implies that

$$L_c = \frac{1}{32\pi}\left(\frac{\Delta K_{I,th}}{\beta - \alpha}\right)^2. \tag{10}$$

The identification of the material parameters α and β can be accomplished with two independent fatigue strengths obtained from plain specimens. Notice that when the fully reversed axial and torsional fatigue tests are considered, then we have

$\beta - \alpha = 0.5\sigma_{-1}$ and $\beta = \tau_{-1}$. Therefore, in this particular case, the critical distance expressed by Eq. 3 is recovered. However, this may not be the case if different fatigue tests are selected for material identification.

Non-local formulation of Crossland and Dang Van fatigue models

In this section we present the critical distances obtained when the material identification procedure outlined above is applied to the multiaxial fatigue models proposed by Crossland [9] and Dang Van [10,11]. For a complete description of the identification procedure the reader should consult reference [12].

A non-local format for Crossland's fatigue model is written as

$$F_{CL}(\Delta(L_c)) = \tau_a + \alpha\sigma_{h,max} - \beta \leq 0, \tag{11}$$

where $\tau_a := \left(1/\sqrt{2}\right)\mathrm{rmb}\left(\Delta^{\mathrm{dev}}\right)$ is the shear stress amplitude defined as the radius of the minimum ball enclosing the deviatoric stresses (multiplied by $1/\sqrt{2}$), $\sigma_{h,max}$ is the maximum value of the hydrostatic stresses and α, β and L_c are material parameters. The critical distance which provides perfect agreement between this model and the experimental fatigue threshold range is given by

$$L_c = \frac{1}{2\pi}\left(\frac{\sqrt{3}+2\alpha}{6\beta}\right)^2 \Delta K_{I,th}^2. \tag{12}$$

If α and β are obtained from axial and torsional fatigue tests under fully reversed loading, then this expression becomes

$$L_c = \frac{1}{2\pi}\left(\frac{6r-\sqrt{3}}{3r}\right)^2 \left(\frac{\Delta K_{I,th}}{\Delta\sigma_{-1}}\right)^2, \tag{13}$$

where $r := \tau_{-1}/\sigma_{-1}$.

We now present the following non-local format for Dang Van's fatigue model:

$$F_{DV}(\Delta(L_c)) = \max_t \left\{\sigma_T(\mathbf{S}-\mathbf{A}) + \alpha\sigma_h\right\} - \beta \leq 0, \tag{14}$$

where σ_T denotes Tresca's equivalent stress, $\mathbf{S}$ is the deviatoric stress, $\mathbf{A} := \mathrm{cmb}(\Delta^{\mathrm{dev}})$ is the center of the minimum ball enclosing the deviatoric stresses, σ_h is the hydrostatic stress and α, β and L_c are material parameters. The identification of the critical distance using the fatigue threshold range yields

$$L_c = \frac{1}{8\pi}\left(\frac{3+4\alpha}{6\beta}\right)^2 \Delta K_{I,th}^2. \tag{15}$$

If α and β are obtained from axial and torsional fatigue tests under fully reversed loading, then this expression becomes

$$L_c = \frac{1}{2\pi}\left(\frac{4r-1}{2r}\right)^2 \left(\frac{\Delta K_{I,th}}{\Delta\sigma_{-1}}\right)^2. \tag{16}$$

ASSESSMENT OF THE MODELS

The non-local multiaxial fatigue models were assessed considering fatigue strengths obtained from notched and cracked specimens. The notched specimens were as follows:

1. Five double edge V-notched plates made of mild steel 0.22%C [13,14];
2. Ten plates with circular holes made of SAE 1045 steel [15];
3. Seven circumferentially V-notched bars made of mild steel 0.15%C [14];
4. Six bars with circumferential semi-circular notches made of 2.25Cr-1Mo steel [16].

The fatigue strengths for the data in items 1, 2 and 4 were evaluated at 10 millions cycles, while the number of cycles to failure for the data in item 3 was not reported. Fig. 1 shows the geometry of the notched specimens. Details about the experimental data can be found in reference [12].

The following error indexes measure the difference between theoretical estimations and experimental data:

$$I = \frac{p_{a,th}^{\text{model}} - p_{a,th}}{p_{a,th}}, \quad \text{for notched specimens,} \tag{17}$$

$$I = \frac{\Delta K_{I,th}^{\text{model}} - \Delta K_{I,th}}{\Delta K_{I,th}}, \quad \text{for cracked specimens.} \tag{18}$$

where $p_{a,th}^{\text{model}}$ and $p_{a,th}$ are the theoretical and experimental fatigue strengths relative to the gross area of notched specimens, while $\Delta K_{I,th}^{\text{model}}$ and $\Delta K_{I,th}$ denote the theoretical and experimental threshold stress intensity factor ranges. Notice that $I = 0$ means a perfect agreement between theory and experiment.

In order to identify the material parameters α, β and L_c, a best fitting strategy was considered. For each set of fatigue tests (obtained from plain, notched and cracked specimens made from a given material) the parameters were chosen so as to minimize the sum of the squared error index of each test.

Table 1 presents the basic statistics of the error indexes for each data set. The results show that for notched plates the estimations of the non-local fatigue models compared well with the experimental data. On the other hand, the Modified Wöhler Curve Method provided better estimations than the other models for notched cylindrical bars. This conclusion is illustrated in Fig. 2, where the estimations of the non-local multiaxial fatigue models are compared with the experimental data obtained from V-notched bars.

CONCLUDING REMARKS

A framework for the estimation of multiaxial fatigue strength, based on the Theory of Critical Distances, was investigated in this work. The key feature of the approach is that the critical distance is identified in a manner consistent with the multiaxial fatigue model under consideration. Among the multiaxial fatigue models considered in this work, the Modified Wöhler Curve Method showed the best estimations. Aiming at a further development of the proposed approach, one should investigate situations involving non-proportional multiaxial loading and/or notches with a more complex geometry.

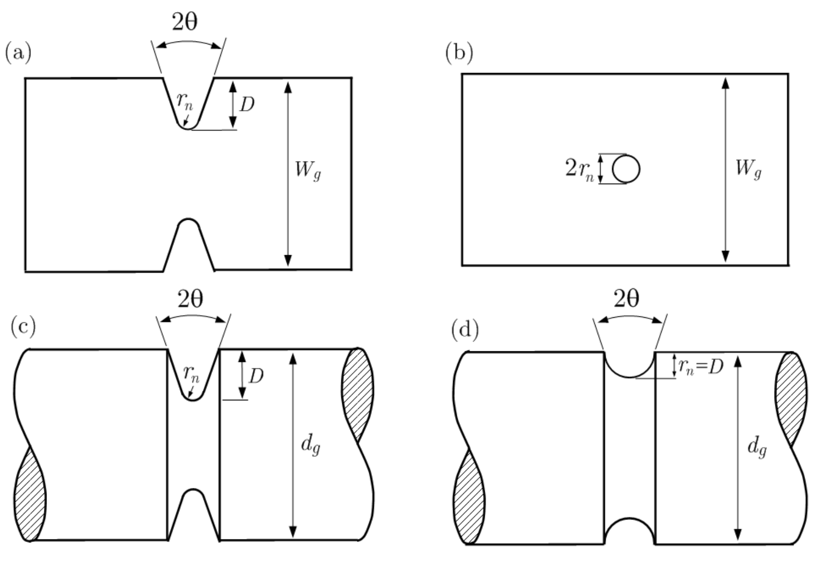

Figure 1. Specimens geometry. (a) V-notched plate, (b) plate with circular hole, (c) circumferentially V-notched bar and (d) circumferentially semi-circular notched bar.

Table 1. Statistics of the error indexes for each data set.

	Mean (%)	Standard deviation (%)	Maximum deviation (%)	Sum of squared indexes
Plates with V-notches				
Crossland	0.8	4.1	5.3	0.012
Dang Van	-0.2	3.9	6.5	0.011
Susmel & Lazzarin	-1.4	6.9	11.1	0.034
Plates with central hole				
Crossland	-1.4	9.9	24.0	0.14
Dang-Van	-2.3	10.4	22.5	0.16
Susmel & Lazzarin	-1.6	12.3	25.8	0.22
Bars with V-notches				
Crossland	15.1	14.7	31.8	0.401
Dang Van	6.7	9.7	15.7	0.125
Susmel & Lazzarin	0.2	5.1	10.0	0.024
Bars with semi-circular Notches				
Crossland	5.1	15.2	34.4	0.206
Dang-Van	2.4	13.4	27.9	0.148
Susmel & Lazzarin	-0.8	7.5	16.5	0.046

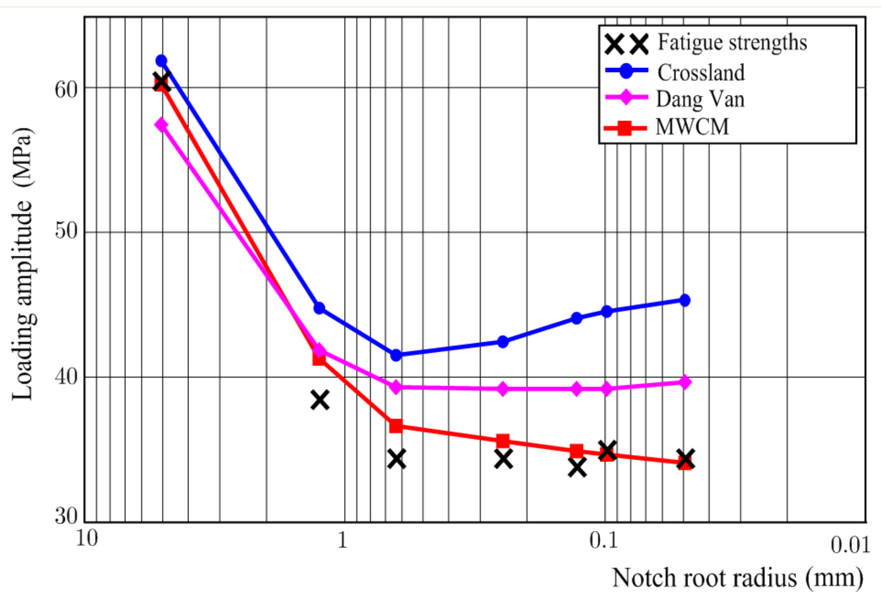

Figure 2. Comparison between estimations of non-local multiaxial fatigue models and experimental data obtained from V-notched bars.

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