# Multiaxial fracture model for weak fiber\matrix interface composite within the three bending fatigue test.

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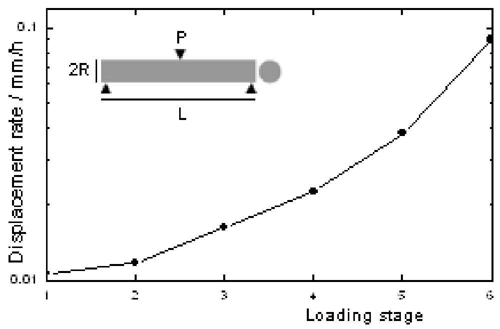
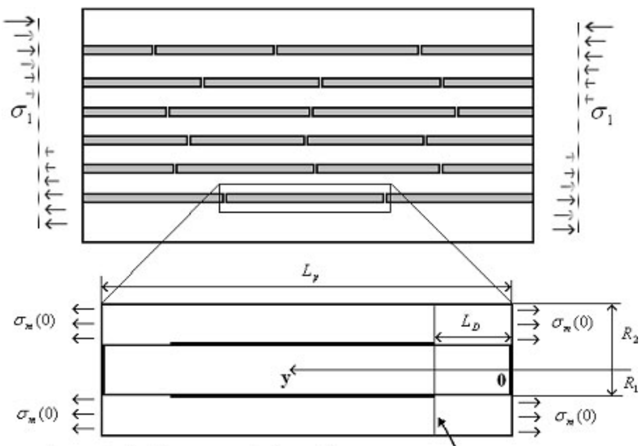
***ABSTRACT.*** *The aim of the present paper is to interpret theoretically three bending fatigue test results for weak fiber\matrix interface composite. Composite materials such as oxide/oxide composites with brittle fiber and brittle matrix are perspective modern materials, which can be used in high temperature regimes. Low steady-state creep rates and high crack propagation durability are two sides of their manufacture problem. Weak fiber/matrix interface stimulates high energy dissipation near the crack tip due the pull-out mechanism but very weak interface provides an interesting and dramatic effect of its rapid degradation during the 5-10 cycle loadings. So, we would like to propose a model of cyclic interface degradation and to analyze optimal conditions for fracture energy dissipation in brittle/brittle composites due the fibers pull-out.*

## INTRODUCTION

Multiaxial fracture model was built up to theoretically evaluate the behaviour of composites with the weak (island-like) fiber/matrix interface. Three bending test leads to the micromechanical model of interface degradation is described and experimental data of rapid increasing of steady-state rate of the displacement of an oxide/nickel composite specimen is analysed.




Changes in the interface strength during a cycle loading and, correspondingly, changes in the creep resistance of a composite are described due the abrasive deterioration process in the fibre/matrix interface. Fatigue strength estimation at a specified number of loading cycles is obtained.



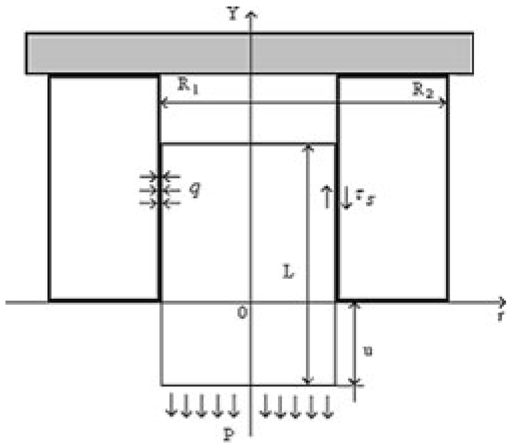

$L_D$ - interface crack length

Volume of energy dissipation due the pull-out mechanizm for different stresses is obtained. Area of effective maximum energy dissipation depending on the weakness of interface is estimated. In this case pull-out energy due the multiaxial fracture in brittle-brittle composites can provide such dissipation effect as crack tip plastic zone in metals.

## THE MODEL

### Abrasive deterioration model.

Let us consider a cylindrical piece of the fiber extended with tension P from a cylindrical site of a matrix, fixed from above. Fiber and a matrix forces by Coulomb's law of friction due the normal tensions in fiber/matrix boundary. The primary normal tension appears due the manufacture process and is considered set. During the pull-out process normal tensions are redistributing and the fiber/matrix contact surface is deteriorating simultaneously, that leads to change of boundary conditions and further friction weakening.




Coulomb's law of friction. $\tau_S = \mu \cdot q$

Depth of deterioration of a matrix in a surface of contact corresponds to model of abrasive deterioration: $u_W = l_i \int_0^v \tau_S dv$

Distribution of normal pressure in a direction of an axis 0y goes from:

$$\frac{d\sigma_m}{dy} = \frac{2\gamma}{R_1}\tau_S, \quad \frac{d\sigma_f}{dy} = -\frac{2}{R_1}\tau_S$$

For any cross-section we have classical problem:

$$\sigma_{rr} = \frac{B}{r^2} + C, \quad \sigma_{\theta\theta} = -\frac{B}{r^2} + C, \quad \varepsilon_{\theta\theta} = \frac{1}{E_m}\left(\sigma_{\theta\theta} - \upsilon_m(\sigma_{rr} + \sigma_m)\right)$$

$$u_r = r\varepsilon_{\theta\theta} = -\frac{1+\upsilon_m}{E_m r}B + \left(\frac{1-\upsilon_m}{E_m}C - \frac{\upsilon_m}{E_m}\sigma_m\right)r$$

$$\sigma_{rr} = \sigma_{\theta\theta} = A, \quad u_r = \left(\frac{1-\upsilon_f}{E_f}A - \frac{\upsilon_f}{E_f}\sigma_f\right)r$$

with boundary conditions depending on depth of deterioration:

$$u_{r+} - u_{r-} = u_0 - u_w$$

$$\sigma_{rr+} = \sigma_{rr-}$$

Therefore we obtain normal tension dependence of normal stresses and depth of deterioration:

$$q(y,u) = q(u_W, \sigma_f, \sigma_m) = \frac{E_m u_0 / R_1}{\alpha(1-\upsilon_f)+1+2\gamma+\upsilon_m} - \frac{E_m u_W / R_1}{\alpha(1-\upsilon_f)+1+2\gamma+\upsilon_m} + \frac{\upsilon_m \sigma_m}{\alpha(1-\upsilon_f)+1+2\gamma+\upsilon_m} - \frac{\alpha\upsilon_f\sigma_f}{\alpha(1-\upsilon_f)+1+2\gamma+\upsilon_m} \quad (1)$$

Using following

$$u_W = l_i \mu \int_0^u q(y,t)dt\,, \sigma_m(y) = \int_0^y \frac{2\gamma}{R_1}\tau_S dy = \frac{2\mu\gamma}{R_1}\int_0^y q(t,u)dt, \quad \sigma_f(y) = \frac{P}{\pi R_1^2} - \frac{\sigma_m(y)}{\gamma}$$

finally we obtain the defining equation

$$q(y,u) = q_0 - A_a \int_0^u q(y,t)dt - A_2 \int_0^{L-u} q(t,u)dt + A_3 \int_0^y q(t,u)dt\,,\ u \in [0;L], y \in [0;L-u]; q(y,u) \ge 0$$

We can solve this equation numerically or can involve simplifications such as:

Linearization for tension *q:* $\quad q(y,u) = K(u)y + M(u)$

Thus we have:

$$\begin{cases} K(u) = -A_1 \int_0^u K(t)dt + B_3 \\ M(u) = q_0 - A_1 \int_0^u M(t)dt - A_2 K(u)\dfrac{(L-u)^2}{2} - A_2 M(u)(L-u) \end{cases}$$

Solution of this system can be obtained analytically:

$$K(u) = B_3 e^{-A_1 u}$$

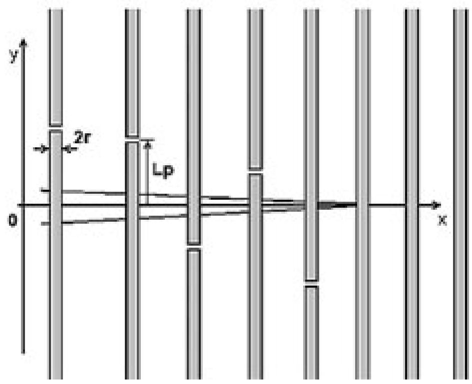
$$M(u) = C\left(1 + A_2 L - A_2 u\right)^{\frac{A_3 - A_2}{A_2}} + \frac{A_2 B_3}{2}\left(1 + A_2 L - A_2 u\right)^{\frac{A_3 - A_2}{A_2}} \int_0^u \frac{(L-t)(A_1 L + 2 - A_1 t)}{\left(1 + A_2 L - A_2 t\right)^{\frac{A_3}{A_2}}} e^{-A_1 t}\, dt$$

$$C = \frac{2q_0 - A_2 B_3 L^2}{2(1 + A_2 L)^{\frac{A_3}{A_2}}}$$

The second simplification is averaging of normal tensions *q* by *y,* after that we have

$$q_1(u) = q_0 - A\int_0^v q_1(u)du\,, \quad \text{therefore } q_1(u) = q_0 e^{-Au}$$

**Pull-out energy dissipation.** By means of the two cylinders shift problem solution received in the first part we can obtain energy dissipating definition due the fiber pull-out mechanism. Let us consider a unidirectional brittle/brittle composite and and classic cross-plane crack, r - is fiber radius, Lp – is average length of broken fiber due the crack approach. Fiber strength is random value and depends on defects randomly distributed along a fiber and presented by Weibull distribution. $\tau$ s - is initial stress along the pull-out fiber.




$$P(y) = 1 - \exp\left[-\frac{1}{L_0}\int_0^y \left(\frac{\sigma(t)}{\sigma_0}\right)^\beta dt\right]$$

Stresses in the fiber we take like in [1]

$$\sigma(y)=\begin{cases}\sigma(0)-\dfrac{2y\tau}{R_1},\ y\in(0;L_D)\\ \sigma(0)-\dfrac{2\tau L_D}{R_1}-\left(\sigma(0)-\dfrac{2\tau L_D}{R_1}-\sigma_\infty\right)(L_C-L_D)^{-1}(y-L_D),\ y\in(L_D;L_C)\end{cases}$$

Therefore $\tilde{\sigma}=\int\limits_{\sigma_\infty}^{\infty}\sigma_C\dfrac{d\tilde{P}}{d\sigma_C}d\sigma_C$ , $L_p=\int\limits_0^{L_C}y\dfrac{dP(y)\big|_{\sigma_C=\tilde{\sigma}}}{dy}dy$

After that we analyze probability of fiber breakage due the pull-out out of the matrix

$$P(\sigma_C)=P(L_P)=1-\exp\left[-\frac{1}{L_0}\int\limits_0^{L_D}\left(\frac{\sigma(t)}{\sigma_0}\right)^{\beta}dt\right]\exp\left[-\frac{1}{L_0}\int\limits_{L_D}^{L_P}\left(\frac{\sigma(t)}{\sigma_0}\right)^{\beta}dt\right]=$$

$$1-\exp\left[\frac{L_D\left(\left(\sigma_C-\frac{2\tau L_D}{R_1}\right)^{\beta+1}-\sigma_C^{\beta+1}\right)}{L_0\left(\frac{2\tau L_D}{R_1}\right)\sigma_0^{\beta}(\beta+1)}\right]\exp\left[\frac{(L_D-L_P)\left(\sigma_C-\frac{2\tau L_D}{R_1}\right)^{\beta}}{L_0\sigma_0^{\beta}(\beta+1)}\right],$$

$$L_{p1}=\int\limits_0^{L_P}y\frac{dP(y)\big|_{\sigma_C=\tilde{\sigma}_2}}{dy}dy$$

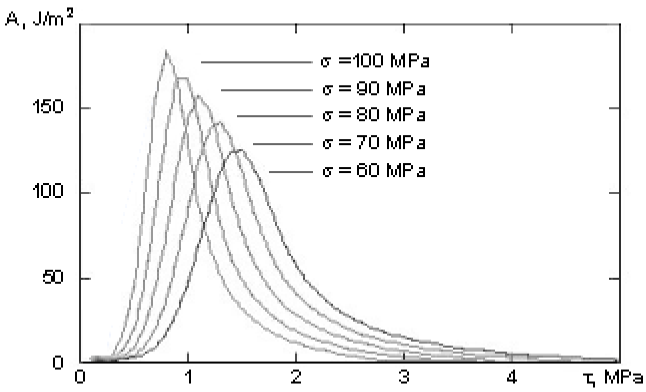
Finally, we can calculate an pull-out energy dissipation range and determine the optimal conditions for its maximum approaching:

$$A(\sigma_1,\tau)=\frac{2V_f\tau}{R_1}\left[\frac{e^{-Ku_\alpha}}{K}\left(u_\alpha-L+\frac{1}{K}\right)+\frac{L}{K}-\frac{1}{K^2}\right]=$$

$$=\frac{2V_f\tau}{R_1}\left[\frac{e^{-K\frac{L_C\sigma_1}{2E}}}{K}\left(\frac{L_C\sigma_1}{2E}-L_{P1}(\sigma_1,\tau)+\frac{1}{K}\right)+\frac{L_{P1}(\sigma_1,\tau)}{K}-\frac{1}{K^2}\right]$$

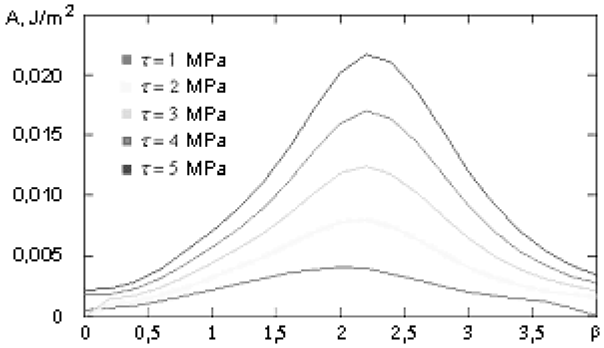
An example calculation for concrete data

| Fiber | | Matrix | | Specimen | |
|---|---|---|---|---|---|
| r | $0.5\cdot10^{-4}\,m$ | $E_m$ | $10^8\,Pa$ | $L_C$ | $10^{-1}\,m$ |
| $\sigma_0$ | $5\cdot10^8\,Pa$ | | | $\tau$ | $5\cdot10^6\,Pa$ |
| $L_0$ | $10^{-2}\,m$ | | | $V_f$ | $1/3$ |
| $\beta$ | 3,6 | | | $V_m$ | $2/3$ |
| $E_f$ | $10^9\,Pa$ | | | | |




Area of effective maximum energy dissipation depending on the weakness of interface is estimated. But the certain shift in interface properties can completely reduce this effect.

Average length of broken fiber and, correspondingly, pull-out lenth in unidirectional composite with cross-plane crack is estimated. The range of optimal Weibull parameter β is established as 2<β<2,5.

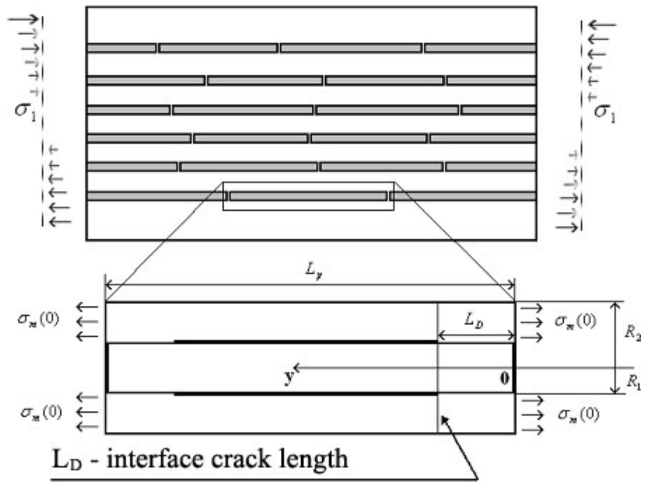



So, we can compare maximum value of pull-out energy dissipation and energy of ceramic matrix destruction.

| Ceramic matrix destruction energy | | Fiber pull-out energy | |
|---|---|---|---|
| SiC | $10\,J/m^2$ | $\sigma_1 = 50\,MPa, \tau = 2\,MPa$ | $100\,J/m^2$ |
| CAS | $25\,J/m^2$ | $\sigma_1 = 100\,MPa, \tau = 1\,MPa$ | $150\,J/m^2$ |

**Interface degradation.**

Let $L_D$ is a distance of fibre-matrix debonding area with shear stress defined by the Coulomb's friction law. Leading the (1) we can get normal stresses on the fiber surface.




We define:

$$u_W = l_i \mu \int_0^u q(y,t)dt,$$

$$\sigma_f(y) = \sigma_f(L_D) + \frac{2\mu}{R_1}\int_{L_D}^{y} q(t,u)dt,$$

$$\sigma_m(y) = \sigma_1(\gamma+1) - \gamma\sigma_f$$

$$q(y,u) = q(u_W, \sigma_f, \sigma_m) = \frac{E_m u_0 / R_1}{\alpha(1-\upsilon_f)+1+2\gamma+\upsilon_m} - \frac{E_m u_W / R_1}{\alpha(1-\upsilon_f)+1+2\gamma+\upsilon_m} +$$

$$+\frac{\upsilon_m \sigma_m}{\alpha(1-\upsilon_f)+1+2\gamma+\upsilon_m} - \frac{\alpha \upsilon_f \sigma_f}{\alpha(1-\upsilon_f)+1+2\gamma+\upsilon_m},$$

and fiber-matrix displacement will be:

$$u(y) = u_m - u_f = \int_{L_D}^{y} \varepsilon_m dy - \int_{L_D}^{y} \varepsilon_f dy = \frac{1}{E_m}\int_{L_D}^{y} \sigma_m dy - \frac{1}{E_f}\int_{L_D}^{y} \sigma_f dy =$$

$$= \left(\frac{\sigma_1(\gamma+1)}{E_m} + \left(\frac{1}{E_f} + \frac{\gamma}{E_m}\right)\sigma_f(L_D)\right)(L_D - y) - \left(\frac{1}{E_f} + \frac{\gamma}{E_m}\right)\int_{L_D}^{y} \sigma_f dy$$

So, we can get a foolowing system:

$$\begin{cases} q(y,u) = q_0(y) + T - A_1 \int_0^u q(y,t)dt - A_2 \int_{L_D}^y q(t,u)dt \\ u(y) = A_3(L_D - y) - A_4 \int_{L_D}^y \int_0^y q(t,u)dt \end{cases}$$ , where

$$T = \frac{\upsilon_m \sigma_1(\gamma+1) - (\alpha\upsilon_f + \gamma\upsilon_m)\sigma_f(L_D)}{\alpha(1-\upsilon_f) + 1 + 2\gamma + \upsilon_m},$$

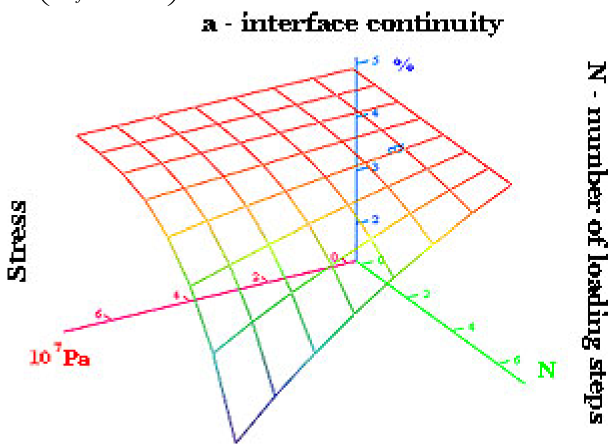
$$A_1 = \frac{\mu E_m l_i / R_1}{\alpha(1-\upsilon_f) + 1 + 2\gamma + \upsilon_m}, \quad A_2 = \frac{2(\gamma\upsilon_m + \alpha\upsilon_f)\,{}^{\mu}\!/\!{}_{R_1}}{\alpha(1-\upsilon_f) + 1 + 2\gamma + \upsilon_m},$$

$$A_3 = \frac{\sigma_1(\gamma+1)}{E_m} + \left(\frac{1}{E_f} + \frac{\gamma}{E_m}\right)\sigma_f(L_D), \quad A_4 = \frac{2\mu}{R_1}\left(\frac{1}{E_f} + \frac{\gamma}{E_m}\right)$$

If we numerically solve this system, we can obtain the level of interface degradation:

$$\frac{a}{a_0} = \frac{\int_0^{L_p} q(t,u(t))dt}{q_0 L_p},$$

where $a$ - is interface continuity, included in steady-state creep rate conditions in the form [7].




$$\frac{\sigma}{\sigma_m} = K_f a^{\frac{m}{n}} \left(\frac{\dot{\varepsilon}}{\eta_m}\right)^{\frac{1}{n}} + K_m \left(\frac{\dot{\varepsilon}}{\eta_m}\right)^{\frac{1}{m}}$$

where

$$K_f = \left(\frac{2}{3}\right)^{\frac{1}{m}} \left(\frac{m}{2m+1}\right) \left[\left(\frac{2\sqrt{3}}{\pi}\right)^{\frac{1}{2}} - 1\right]^{-\frac{1}{m}} \left[\left(\frac{\sigma_0^{(f)}}{\sigma_m}\right)^{\beta} \left(\frac{L_0}{2R_1}\right)\right]^{\frac{m+1}{n}} V_f, \quad K_m = V_m$$

**DAMAGE FUNCTION**

Let us consider a damage function: $\omega(x, y, y_0)$, depends on the of fiber's damage level.

$\omega(x, y - y_0) = q(x)(y - y_0)^2 + p(x)(y - y_0)^4$

$\omega(x, 0) = 0$, where x – axis along the specimen within the three bending test, y – cross-section coordinate, $y_0$ - is neutral axis displacement.

Steady state creep rate regime depends on damage function:

$\dot{\varepsilon} = \eta\left(\frac{\sigma_x}{\sigma_n(\omega)}\right)^n$

$\sigma_n(\omega) = (1-\omega)\sigma_n + \omega\sigma_M$.

Within the flat crossections hypothesis we obtain longitudinal stresses:

$\sigma = \sigma_n(\omega)\left(\frac{k(x)(y-y_0)}{\eta}\right)^{\frac{1}{n}}$, where κ is velocity of the neutral axis curvature changing.

Therefore equilibrium equations for cross-section loading and the bending moment M(x) are:

$$\int_{y_0}^{c} b\sigma_n \left(\frac{k(x)}{\eta}\right)^{\frac{1}{n}} (y-y_0)^{\frac{1}{n}} dy + \int_{-c}^{y_0} b\big(\big((1-\omega(y)\big)\sigma_n + \omega(y)\sigma_M\big)\left(\frac{k(x)}{\eta}\right)^{\frac{1}{n}} (y-y_0)^{\frac{1}{n}} dy = 0,$$

$$\int_{y_0}^{c} b\sigma_n \left(\frac{k(x)}{\eta}\right)^{\frac{1}{n}} (y-y_0)^{\frac{n+1}{n}} dy + \int_{-c}^{y_0} b\big(\big((1-\omega(y)\big)\sigma_n + \omega(y)\sigma_M\big)\left(\frac{k(x)}{\eta}\right)^{\frac{1}{n}} (y-y_0)^{\frac{n+1}{n}} dy = M(x),$$

$\left(\frac{c\kappa(x)}{\eta}\right)^{\frac{1}{n}} = \frac{M(x)}{M_n}$, where $\eta = c\kappa_n$

For the determination of $\omega$ we need to solve the system:

$$\int_{a}^{0} \omega(\zeta)\zeta^{\frac{1}{n}} d\zeta = A,$$

$$\int_{a}^{0} \omega(\varsigma)\zeta^{\frac{n+1}{n}} d\zeta = B,$$

where $\zeta = y - y_0$ and $a = -c - y_0$,

$$A = \frac{n}{n+1}\frac{\sigma_n}{(\sigma_n - \sigma_M)}\left((a+2c)^{\frac{n+1}{n}} - a^{\frac{n+1}{n}}\right),$$

$$B = \frac{n}{2n+1}\frac{\sigma_n}{(\sigma_n - \sigma_M)}\left((a+2c)^{\frac{2n+1}{n}} - a^{\frac{2n+1}{n}}\right) - \frac{M_n c^{\frac{1}{n}}}{b(\sigma_n - \sigma_M)}$$

Following the suggested damage function form $\omega(\zeta) = q\zeta^2 + p\zeta^4$, we can determine the constants $p$, $q$ *out of the system:*

$$q\frac{n}{3n+1}a^{\frac{3n+1}{n}} + p\frac{n}{5n+1}a^{\frac{5n+1}{n}} = -A$$

$$q\frac{n}{4n+1}a^{\frac{4n+1}{n}} + p\frac{n}{6n+1}a^{\frac{6n+1}{n}} = -B$$

Functions $q(x), p(x)$ are reduced to the constants $q, p$ due the the flat crossections hypothesis.

## CONCLUSIONS

Thus, fiber pull-out process at the given stage seems very perspective for manufacturing new generation composite materials. However course of this process strongly depends on properties of a material, in particular fiber coating and distribution of fiber strength properties. For example, as shown, fibers without defects are less perspective than fibers with certain defects distribution.

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