An Alternative Measure for the Shear Stress Amplitude in Critical Plane Based Multiaxial Fatigue Models

A. P. Dantas, J. A. Araújo, E. N. Mamiya, F.C. Comes & J.L.A. Ferreria

Department of Mechanical Engineering, University of Brasília,Brasília,DF, Brazil.
alex07@unb.br

ABSTRACT. *The goal of this work is to propose and to assess an alternative measure for the shear stress amplitude in critical plane based multiaxial fatigue criteria. Usually such amplitude is characterized by the radius of the minimum circle circumscribing the shear stress history in a material plane. Here, an alternative measure which considers the maximum circumscribed rectangle (MCR) in terms of its Frobenius norm is considered. The computation of the shear stress amplitude by the minimum circle and by the maximum rectangle was conducted for a number of experimental data available in the literature involving proportional and nonproportional stress paths. Then a critical plane criterion was invoked to estimate the fatigue endurance based on such values. It is shown that the multiaxial fatigue estimates were improved for most data evaluated when the shear stress amplitude was computed in terms of the maximum box. Some critical assessment concerning the classical definition of the critical plane was also addressed in this work.*

INTRODUCTION

Design of mechanical components under combined loadings to operate in High Cycle Fatigue (HCF) regime often relies on the predictive capabilities of multiaxial models. In such cases, stress based criteria are appropriate and a number of such models, proposed under different approaches, are available [1]. One approach that has gained increasing interest is the critical plane one. It requires the computation of the shear stress vector history at each possible plane passing through a material point. Usually, the plane experiencing the greatest amplitude of shear stress is defined as the critical plane. The influence of mean normal stresses is also considered. Findley [2], McDiarmid [3], and Susmell and Lazzarin [4] have proposed stress based critical plane criteria. Drawbacks associated with this approach involve the most appropriate manner to define the shear stress amplitude, the cost to search for the critical material plane, and the fact that it is usual to find situations where the amplitude of the shear stress is basically the same for a number of planes, while the mean normal stress acting on these same planes vary a lot. In this paper such drawbacks are addressed by the proposition of an alternative measure for the shear stress amplitude.

SHEAR STRESS AMPLITUDE IN CRITICAL PLANE APROACHES

Basic Concepts

Consider a mechanical element submitted to oscillatory multiaxial loadings. At each time instant a given material point O located on the surface of the loaded element is subjected to internal forces represented by the Cauchy stress tensor $\mathbf{T}(t, O)$. Consider now, a material plane characterized by its unit normal vector $\mathbf{n}$, passing through point O. This is in turn described by its spherical angles ϕ and θ, as shown in Fig. 1(a). Cauchy's theorem establishes that the stress vector $\mathbf{t}$ acting on O depends on the material plane orientation:

$$\mathbf{t}(t) = \mathbf{T}(t)\mathbf{n} . \tag{1}$$

And its normal and shear stress components are given by:

$$\boldsymbol{\sigma}(t) = \left(\mathbf{T}(t)\mathbf{n} \cdot \mathbf{n}\right)\mathbf{n} . \tag{2}$$

$$\boldsymbol{\tau}(t) = \mathbf{T}(t)\mathbf{n} - \boldsymbol{\sigma}(t) . \tag{3}$$

While the computation of the amplitude and mean values of $\boldsymbol{\sigma}(t)$ is a straightforward task, as such vector function varies in magnitude but not in direction, the same can not be said about the calculation of these quantities associated to the shear stress vector path, Ψ (Fig. 1(a)).

Alternative Measure to Compute the Shear Stress Amplitude in a Material Plane

A classic methodology to compute the shear stress amplitude, τ_a, in a material plane was proposed first by *Dang Van et al.* [5] and later by *Papadopoulos* [6]. In this method τ_a is provided by the radius of minimum circle circumscribing Ψ (Fig. 2(a)). Space only precludes us to yield a more detailed explanation on the process to determine the Minimum Circumscribed Circle (MCC). Here only a brief overview will be provided. Basically the methodology consists of two steps. First, it is necessary to find the center of the MCC with respect to O, which corresponds to the mean shear stress, $\boldsymbol{\tau}_m$. This is basically a *min max* type o problem:

$$\boldsymbol{\tau}_m = \arg\left\{ \min_{\boldsymbol{\tau}^*} \max_{t} \left\| \boldsymbol{\tau}(t_i) - \boldsymbol{\tau}^* \right\| \right\}, \tag{4}$$

where $\boldsymbol{\tau}(t_i)$ is a vector element of the time discretized shear stress path Ψ, $\boldsymbol{\tau}^*$ is any shear vector on Δ with origin in O and $\|\bullet\|$ denotes the vector norm. Next, the circumference with minimum radius is determined as:

$$\tau_a = R := \max_{t_i} \|\boldsymbol{\tau}(t_i) - \boldsymbol{\tau}_m\|. \tag{5}$$

Althoug the MCC is unique and hence numerically constitutes a well posed problem there are some drawbacks associate to its use to calculate τ_a. One of them is that the *min max* problem requires more elaborate algorithms to compute the center of the circumference, but perhaps more important is the fact that a proportional and a non-proportional shear stress path in a material plane may be circumscribed by the same circunferece (Fig. 1(b)), i.e., the amount of damage provoked by these different stress paths would be equivalent. Experimental work on multiaxial fatigue have shown that depending on material and and type of solicitation, non-proportional load histories may be significantly more damaging than proportional ones [1].

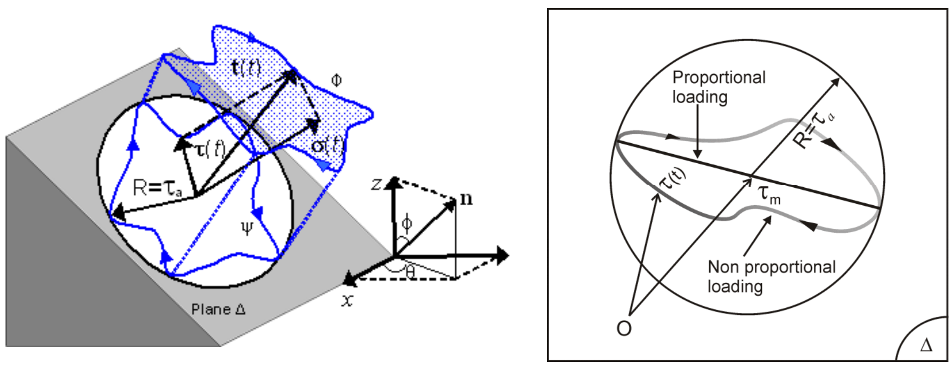


Figure 1. (a) Stress vector $\mathbf{t}(t)$, its normal, $\boldsymbol{\sigma}(t)$, and shear, $\boldsymbol{\tau}(t)$, components in a material plane Δ and Minimum Circumscribing Circle (MCC) to the shear stress vector path Ψ projected on Δ. (b) Drawbacks associated to the MCC method: a proportional and a non-proportional shear stress path with the same MCC.

In this setting, a simple and alternative method to compute τ_a is now proposed. We claim here that the shear stress amplitude, which correctly characterizes the solicitation to fatigue under multiaxial loadings, is given by the Maximum Circumscribed Rectangle (MCR) to the shear stress vector path in a material plane Δ in terms of its Frobenius norm (Fig. 2(a)):

$$\tau_a = \max_{\varphi} \sqrt{a_1^2(\varphi) + a_2^2(\varphi)}, \tag{6}$$

where

$$a_i = \frac{1}{2}\left[\max_t \tau_i(\varphi,t) - \min_t \tau_i(\varphi,t)\right],\ i = 1,2, \tag{7}$$

correspond to half of the sides of a rectangle with orientation φ circumscribing the shear stress path Ψ (Fig. 2(b)).

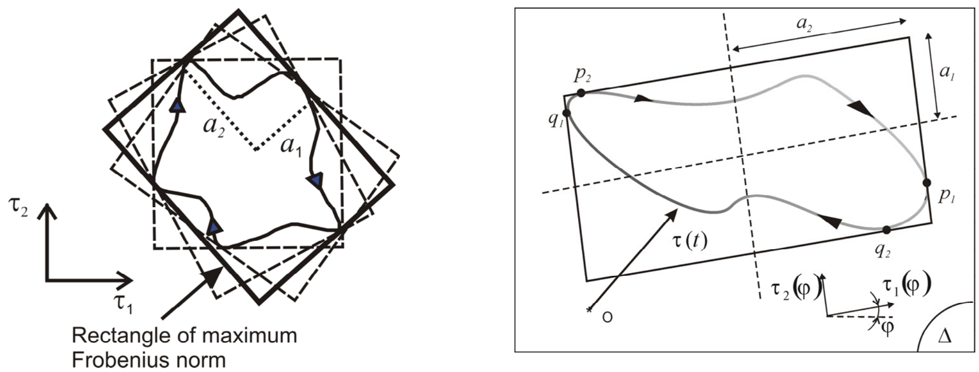
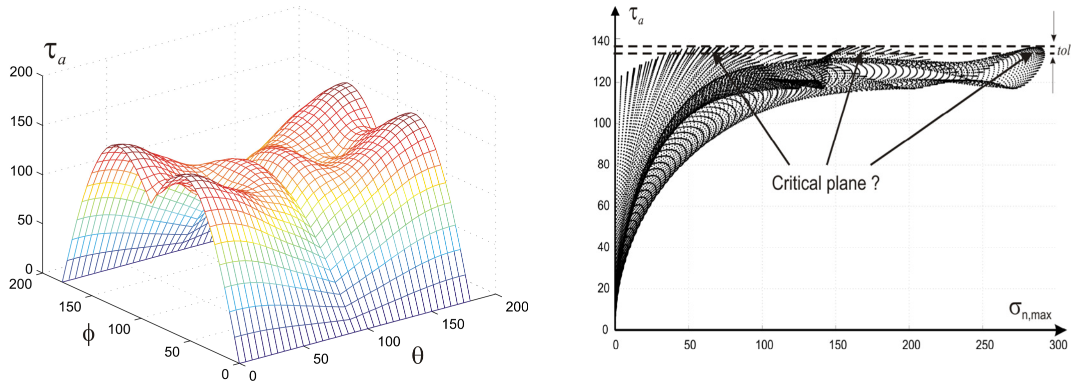


Figure 2. (a) Illustration of the Maximum Circumscribed Rectangle to the shear stress vector path. (b) Half sides a_i, i = 1,2 of a rectangle with orientation φ circumscribing the shear stress path Ψ in a material plane.

Critical Plane in Stress Based Multiaxial Fatigue Models

Usually the critical plane in stress based multiaxial fatigue criteria has been classically defined as the material plane where the shear stress amplitude reaches its maximum value. However, such definition constitutes an ill posed problem in numerical terms as there are cyclic stress states where there are more than one material plane with maximum and identical values of τ_a. As an example Fig. 3(a) depicts a graph of τ_a for each material plane (ϕ,θ) in a multiaxial state of stress. It can be clearly observed that there are four material planes with rigourously the same τ_a value. To further illustrate such characteristic, one plots τ_a *vs* $\sigma_{n,max}$ for each material plane $(\textit{plane increments}\ \Delta\phi = \Delta\theta = 1°)$ in such stress state (see Fig. 3(b)). This graph shows there are a number of material planes experiencing nearly the same level of τ_a, but with maximum normal stress to the plane varying from 50 MPa to 270MPa. In this setting, it seems sensible to consider that the fatigue solicitation will be more severe in a plane where (i) τ_a is close to its highest value (but not necessarily the maximum) and (ii) $\sigma_{n,\max}$ is more significant. In mathematical terms it is claimed here that the candidate plane is among the ones where τ_a reaches 99% of its maximum value, being the critical one the plane with the largest $\sigma_{n,\max}$.



(a) (b)

Figure 3. (a) Shear stress amplitude in a number of material planes (ϕ,θ) for a multiaxial state of stress, (b) set of τ_a x $\sigma_{n,\max}$ points, each point representing such stresses in a material plane.

The Modified Wohler Curve Method (MWCM)

To evaluate the impact of the shear stress amplitude computation by the MCR and MCC approaches on the estimation of the multiaxial fatigue strength it is necessary to invoke an appropriate model. Susmel and Lazzarin [4] observed that the multiaxial high-cycle fatigue behaviour of metallic materials could successfully be predicted by using a simple τ_a vs. $\frac{\sigma_{n,\max}}{\tau_a}$ relationship, where such a fatigue damage parameter had to be calibrated using material fatigue properties generated under simple loading conditions. In more detail, the MWCM can be formalised as follows [4]:

$$\tau_a\left(\phi^c,\theta^c\right)+\kappa\frac{\sigma_{n,\max}}{\tau_a}\left(\phi^c,\theta^c\right)\leq\lambda. \qquad (8)$$

In the above equation, $\tau_a\left(\phi^c,\theta^c\right)$ is the shear stress amplitude in the critical plane $\left(\phi^c,\theta^c\right)$ (as defined in the prior section), $\sigma_{n,\max}$ is the maximum stress perpendicular to this plane and, finally, κ and λ are material constants that can be obtained from two fatigue strengths generated under different loading conditions. For instance, if the uniaxial, f_{-1}, and the torsional, t_{-1}, fully-reversed plain fatigue strengths (at 2 x 10^6 cycles) are used to calibrate Eq. (8), constants κ and λ turn out be [3]:

$$\kappa=t_{-1}-\frac{f_{-1}}{2},\ \lambda=t_{-1}. \qquad (9)$$

In order to evaluate the accuracy of the proposed methodology, the following error index was adopted:

$$i(\%) = \frac{\tau_a\left(\phi^c,\theta^c\right) + \kappa\frac{\sigma_{n,\max}}{\tau_a}\left(\phi^c,\theta^c\right) - \lambda}{\lambda}\cdot 100\,. \tag{10}$$

A negative value of the above error index indicates that fatigue failure should not occur up to at 2 x 10^6 cycles. It is interesting to observe also that, from a engineering point of view, a negative value of the index is an indication of the fact that the component dimensions could be reduced down to the limiting condition given by $i(\%) = 0$.

EXPERIMENTAL DATA FROM LITERATURE AND RESULTS

Biaxial fatigue data from six different steel alloys were collected from the literature [8-13] to assess whether the new definition for the shear stress amplitude can improve the performance of the MWCM model in estimating fatigue strength under a high number of cycles. These data are reported in Table 1 and correspond to experiments under in phase and out-of-phase sinusoidal combined loadings on hard metals $1.3 \leq f_{-1}/t_{-1} \leq \sqrt{3}$, as defined by Papadopoulos [6]. The following nomenclature was adopted in this table: the subscript a stands for the amplitude of stresses. As usual, σ and τ are normal and shear stresses while β_{xy} is the phase difference and λ_{xy} is the frequency ratio between σ and τ. The stress values reported in each table correspond to the maximum combination of stresses that the specimen can stand without failing (up to a limit of 2x10^6 cycles). Fatigue strength under fully reversed bending f_{-1} and torsion t_{-1} to this limit life are also provided within Table 1 for each material.

The calculation of the shear stress amplitude was conducted by the MCC and by the MCR methods and then the error index could be evaluated. Figure 4 shows a bar diagram of $i(\%)$ for each test condition. It is worth of notice that for all tests under proportional loading (tests 1, 5 and 7) $i(\%)$ was rigorously the same independently of the method used to calculate $\tau_a\left(\phi^c,\theta^c\right)$. For all asynchronous and for a number of synchronous non-proportional data computation of $i(\%)$ by the MCR method provided significantly better multiaxial fatigue estimates than the classical MCC. For instance, application of the MWCM to the GGG60 Steel tested under different frequencies of loading provided $i(\%) = 28.4$ and $i(\%) = 26$ (tests 12 and 13, respectivelly) when $\tau_a\left(\phi^c,\theta^c\right)$ was computed by the MCC while the use of the MCR to these same data improved the estimates provided by the MWCM to $i(\%) = -0.5$ and $i(\%) = -0.9$. The search for the critical plane in all cases was conducted considering plane increments $\Delta\phi = \Delta\theta = 1°$. In the the algorithm to determine the MCC it was established a coefficient of expansion $\chi = 0.05$ and a convergence factor $tol = 1\times 10^{-6}$ [5]. In the

MCR method, the rectangle was rotated in steps $\Delta\varphi = 1°$. Under such conditions the MCR consumed only 13% of the computational time spent by the MCC.

Table 1. Experimental data and fatigue strength properties (for different steels tested under combined loadings).

Test #	σ_{xa}	σ_{xya}	λ_{xy}	β_{xy}
Material: Hard Steel; f_{-1}:319.9 MPa; t_{-1}:196.2 MPa; [10]				
1	138.1	167.1	1	0
2	140.4	169.9	1	30
3	145.7	176.3	1	60
4	150.2	181.7	1	90
5	245.3	122.6	1	0
6	249.7	124.8	1	30
7	252.4	126.2	1	60
8	258.0	129.0	1	90
9	299.1	62.8	1	0
10	304.5	63.9	1	90
Material: 34Cr4 Steel ; f_{-1}:415.0 MPa; t_{-1}:259.0 MPa; Rfr [11]				
11	263	132	4	0
Material: GGG60 Steel; f_{-1}:275.0 MPa; t_{-1}:249.0 MPa; Rfr [8]				
12	186	93	0.25	0
13	185	93	4	0
Material: 30NCD16 Steel; f_{-1}:585.0 MPa; t_{-1}:405.0 MPa; Rfr [9]				
14	285	285	0.25	0
15	290	290	4	0
Material: 39NiCrMo3 Steel; f_{-1}:585.0 MPa; t_{-1}:405.0 MPa; Rfr [9]				
16	259.5	150.0	2	0
17	266.0	153.6	3	0
Material: 25CrMo4 Steel; f_{-1}:340.0 MPa; t_{-1}:228.0 MPa; Rfr [7,13]				
18	210	105	0.25	0
19	220	110	2	0
20	242	121	2	90
21	196	98	8	0

CONCLUSIONS

A new method to compute τ_a in a material plane of a multiaxial cyclic stress state was proposed. Such method proved to be very simple to implement as it does not require elaborate algorithms but only simple axes rotation operations. Further, comparisons with available experimental data under a wide range of loading conditions for a number of different steels showed that fatigue estimates provided by the multiaxial MWCM were significantly more accurate when τ_a was computed by the MCR rather than by the MCC. The MCR was also computationally more effecient than the MCC in the

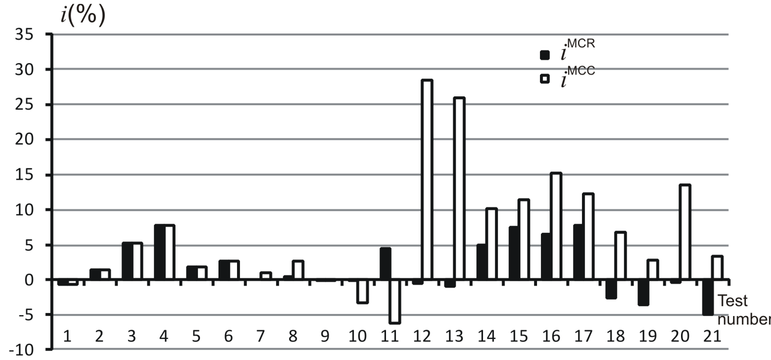


Figure 4. Error index provided by the MWCM for each test condition considering the MCC and MCR approaches.

conditions established in this work. In this setting, the MCR approach to compute τ_a consists in a robust, efficient but simple to use tool to engineers concerned with the design of components in the high cycle multiaxial fatigue regime. It was also shown that the classical definition of the critical as the one where τ_a reaches its maximum value is an ill posed problem. Therefore, using a small tolerance, the critical plane was defined within a range of planes of high τ_a as the one where $\sigma_{n,\max}$ is the largest.

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