A study of the stability of sharp notches in the orthotropic heterogeneous media

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ABSTRACT. *The bi-material notch composed of two orthotropic parts is considered. The radial and tangential stresses and strain energy density are expressed using the Stroh-Eshelby-Lekhnitskii formalism for the plane elasticity. The stress singular exponents and corresponding eigenvectors are the solution of the eigenvalue problem leading from the prescribed notch boundary and compatibility conditions. In generally, there is more than one solution of this eigenvalue problem and consequently the generalized stress intensity factors. The potential direction of the crack initiation is determined from the maximum mean value of the tangential stresses and local minimum of the mean value of the generalized strain energy density factor in both materials. Following the assumption of the same mechanism of the rupture in the case of the crack and the notch, an expression can be obtained for the critical values of the generalized stress intensity factor.*

INTRODUCTION

In practical engineering structures or in parts of electronic devices joints of different materials occur (e.g. layered composite materials, constructions with protective surface layers, thermal barriers). They enable achievement of properties which could not be attained by means of homogeneous materials. In the case of composite materials, parts of the joints often exhibit orthotropic material properties. The stress field in closed vicinity of such material joints has singular character and complicated form. In comparison to a crack in homogeneous media, in the case of bi-material joints, the stress singularity exponent is different from 1/2 and can generally be complex. The stress is mostly characterized by more singular terms and at the same time each singular term covers combination of both normal and shear modes of loading.
Such stress concentrators preclude any application of the fracture mechanics approaches originally developed for a crack in isotropic homogeneous materials, so the assessment of such singular stress concentrators becomes topical [1-3].

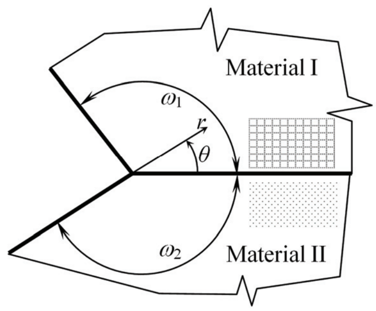

Figure 1. Bi-material orthotropic notch with corresponding polar coordinate system

The orthotropic material properties seriously complicate procedures for assessment of bi-material notch stability. Most such discontinuities can be mathematically modelled as bi-material notches composed of two orthotropic materials. The edge of a protective coating, a free edge stress singularity or other shapes of notches can be modelled for varying angles ω_1 and ω_2, see figure 1. In the contribution the orthotropic bi-material notch is analysed from the perspective of generalized linear elastic fracture mechanics, i.e. the validity of small-scale yielding conditions is assumed. It is further assumed ideal adhesion at the bi-material interface and the notch radius $R \rightarrow 0$ (the sharp bi-material notch tip).

The contribution aims to suggest a procedure for the determination of the crack initiation direction from an orthotropic bi-material notch. Further the critical loading conditions are estimated. The approach is based on knowledge of the stress distribution in place of the concentration. Combined analytical and numerical approaches are employed for the stress field determination. A criterion of the maximum tangential stress and the criterion of the minimum strain energy density factor are modified and adapted to particularities of the nature of the stress concentrator.

STRESS DISTRIBUTION

The necessary step for the crack initiation assessment is detailed knowledge of the stress distribution. Within plane elasticity of anisotropic media the Lekhnitskii-Eshelby-Stroh (LES) formalism based on [4] can be used. Complex potentials satisfying the equilibrium and the compatibility conditions as well as the linear stress-strain dependence and given boundary conditions are the basis for the determination of stress and deformation fields. In the case of general plane anisotropic elasticity all the components of the stress and deformation tensors have to be considered. In the case of orthotropic materials symmetry in the stiffness and compliance matrices occur. Thus the stress and strain tensor is significantly reduced. According to the LES theory for an orthotropic material, the relations for displacements and stresses can be written as follows

$$u_i = 2\,\mathrm{Re}\left\{A_{ij} f_j(z_j)\right\}, \sigma_{2i} = 2\,\mathrm{Re}\left\{L_{ij} f_j'(z_j)\right\}, \sigma_{1i} = -2\,\mathrm{Re}\left\{L_{ij}\mu_j f_j'(z_j)\right\} \quad (i,j=1,2) \quad (1)$$

where Re denotes the real part of the complex expression and $z_j = x + \mu_j y$. Complex numbers $\mu_j = \mu_j' + \mathrm{i}\mu_j''$ are the eigenvalues of the material. For matrices A_{ij} and L_{ij} holds

$$\mathbf{A} = \begin{bmatrix} s_{11}\mu_1^2 + s_{12} & s_{11}\mu_2^2 + s_{12} \\ s_{12}\mu_1 + s_{22}/\mu_1 & s_{12}\mu_2 + s_{22}/\mu_2 \end{bmatrix}, \qquad \mathbf{L} = \begin{bmatrix} -\mu_1 & -\mu_2 \\ 1 & 1 \end{bmatrix}, \quad (2)$$

where s_{ij} are the elastic compliances. Introducing the stress function

$$\phi = [\varphi_i], \quad \varphi_i = 2\,\mathrm{Re}\left\{L_{ij} f_j(z_j)\right\} \quad (i,j=1,2) \quad (3)$$

the radial and tangential stresses can be expressed as

$$\sigma_{rr} = \mathbf{n}\cdot\mathbf{t}_r, \quad \sigma_{r\theta} = \mathbf{m}\cdot\mathbf{t}_r = \mathbf{n}\cdot\mathbf{t}_\theta, \quad \sigma_{\theta\theta} = \mathbf{m}\cdot\mathbf{t}_\theta, \quad (4)$$

where n_i and m_i are the normal and tangential vectors to the selected curve or boundary. If the comma denotes differentiation, for vectors $(t_r)_i$, $(t_\theta)_i$, m_i and n_i one can write

$$\mathbf{t}_r = -\frac{1}{r}\phi_{,\theta}, \mathbf{t}_\theta = \phi_{,r}, \mathbf{n}^T = [\cos\theta, \sin\theta], \mathbf{m}^T = [-\sin\theta, \cos\theta]. \quad (5)$$

In the case of the studied notch, the potential $f_j(z_j)$ has the following form

$$\mathbf{f} = H\left\langle z_*^\delta \right\rangle \mathbf{v}, \quad (6)$$

where H is the generalized stress intensity factor, $v_i = v_i' + \mathrm{i}v_i''$ is an eigenvector corresponding to the eigenvalue $\delta = \delta' + \mathrm{i}\delta''$ representing the exponent of the stress singularity at the notch tip. Eigenvector v_i and eigenvalue δ are the solution of the eigenvalue problem leading from the prescribed notch boundary and compatibility conditions. The expression $\left\langle z_*^\delta \right\rangle$ denotes a diagonal matrix

$$\left\langle z_*^\delta \right\rangle = \mathrm{diag}\left[z_1^\delta, z_2^\delta\right]. \quad (7)$$

In most practical cases, as well as in the cases studied in the paper, there are two singular terms corresponding to two stress singularity exponents δ_k, $k = 1,2$. Note that for the final determination of the stress field in the bi-material notch vicinity the generalized stress intensity factors have to be estimated by means of numerical approaches. Their values result from an analytical-numerical solution for a certain construction with given material properties, geometry,boundary conditions, see also [5].

CRACK INITIATION DIRECTION

Mean value of the tangential stress

The stress field around a bi-material notch inherently covers combined normal and shear modes of loading. For mixed mode fields a crack may grow along the interface or at a certain angle θ_0 with the interface into material I or II. In the present paper where the two orthotropic materials are assumed as perfectly bonded, only crack propagation into materials I or II is supposed. Erdogan and Sih [6] proposed and Smith *et al*. [7] modified the MTS theory in a study on the slant crack under mixed mode I/II loading, see also [1]. This criterion states that the crack is initiated in the direction θ_0 where the circumferential stress $\sigma_{\theta\theta}$ at some distance from the crack tip has its maximum and reaches a critical tensile value. The local maximum of the tangential stress $\sigma_{\theta\theta}$ in the case of a bi-material orthotropic notch depends on the radial distance r from the notch tip. In order to suppress the influence of the distance r, the mean value of the tangential stress is evaluated over a certain distance d

$$\overline{\sigma}_{\theta\theta}(\theta)=\frac{1}{d}\int_0^d \sigma_{\theta\theta}(r,\theta)\mathrm{d}r=\frac{1}{d}\sum_{k=1}^{2}\left(m_1\varphi_1(d,\theta,\delta_k)+m_2\varphi_2(d,\theta,\delta_k)\right). \tag{8}$$

Inserting Eq. 3, 5 and 6 into Eq. 8 one gets for material $m=\mathrm{I},\mathrm{II}$

$$\overline{\sigma}_{\theta\theta}(\theta)=H_1F_{\theta\theta 1m}(\theta)+H_2F_{\theta\theta 2m}(\theta), \tag{9}$$

where

$$\begin{aligned}
F_{\theta\theta km}(\theta)=-2\sin\theta\Big\{&-d^{\delta_k'-1}R_1^{\delta_k'}\mathrm{e}^{-\delta_k''\Psi_1}\big[(\mu_1'\nu_1'-\mu_1''\nu_1'')\cos(\delta_k''\ln d+\delta_k''\ln R_1+\delta_k'\Psi_1)\\
&-(\mu_1'\nu_1''+\mu_1''\nu_1')\sin(\delta_k''\ln d+\delta_k''\ln R_1+\delta_k'\Psi_1)\big]\\
&-d^{\delta_k'-1}R_2^{\delta_k'}\mathrm{e}^{-\delta_k''\Psi_2}\big[(\mu_2'\nu_2'-\mu_2''\nu_2'')\cos(\delta_k''\ln d+\delta_k''\ln R_2+\delta_k'\Psi_2)\\
&-(\mu_2'\nu_2''+\mu_2''\nu_2')\sin(\delta_k''\ln d+\delta_k''\ln R_2+\delta_k'\Psi_2)\big]\Big\}\\
+2\cos\theta\Big\{&d^{\delta_k'-1}R_1^{\delta_k'}\mathrm{e}^{-\delta_k''\Psi_1}\big[\nu_1'\cos(\delta_k''\ln d+\delta_k''\ln R_1+\delta_k'\Psi_1)\\
&-\nu_1''\sin(\delta_k''\ln d+\delta_k''\ln R_1+\delta_k'\Psi_1)\big]\\
&-d^{\delta_k'-1}R_2^{\delta_k'}\mathrm{e}^{-\delta_k''\Psi_2}\big[\nu_2'\cos(\delta_k''\ln d+\delta_k''\ln R_2+\delta_k'\Psi_2)\\
&-\nu_2''\sin(\delta_k''\ln d+\delta_k''\ln R_2+\delta_k'\Psi_2)\big]\Big\},
\end{aligned} \tag{10}$$

$$R_i^2 = (\cos\theta + \mu_i' \sin\theta)^2 + (\mu_i'' \sin\theta)^2, \tag{11}$$

$$\Psi_i = \begin{cases} 0 & \text{for } \theta = 0 \\ \operatorname{arccot}((\cos\theta + \mu_i' \sin\theta) / \mu_i'' \sin\theta) & \text{for } \theta \in (0, \pi) \\ \operatorname{arccot}((\cos\theta + \mu_i' \sin\theta) / \mu_i'' \sin\theta) - \pi & \text{for } \theta \in (-\pi, 0) \\ -\pi & \text{for } \theta = -\pi \end{cases} \tag{12}$$

The distance d has to be chosen in dependence on the mechanism of a rupture, e.g. as a dimension of a plastic zone or as a size of material grain. The distance d can also be chosen by means of the theory of critical distances, see [2]. The mean value of the tangential stress is determined in dependence on the polar angle θ. The potential direction of crack initiation is determined from the maximum of the mean value of tangential stress in both materials

$$\left(\frac{\partial \bar{\sigma}_{m\theta\theta}}{\partial \theta} \right)_{\theta_0} = 0, \quad \left(\frac{\partial^2 \bar{\sigma}_{m\theta\theta}}{\partial \theta^2} \right)_{\theta_0} < 0. \tag{13}$$

The crack initiation angle θ_0 is independent of the absolute values of GSIFs and depends only on their ratio $\Gamma_{21} = H_2 / H_1$. Generally, the value of the angle θ_0 depends on the averaging distance d. The distance d has to be chosen depending on the corresponding rupture mechanism and the microstructure of the material.

Strain energy density factor

Similarly, the crack initiation direction can be derived via the generalization of the mean value of the strain energy density factor defined in [8]

$$\bar{\Sigma}_m(\theta) = \frac{1}{d} \int_0^d \Sigma_m(r, \theta) \mathrm{d}r, \tag{14}$$

where

$$\Sigma_m(r, \theta) = \frac{1}{4\pi} \left(r^{2\delta_1' - 1} H_1^2 U_{1m} + r^{2\delta_2' - 1} H_2^2 U_{2m} + 2 r^{\delta_1' + \delta_2' - 1} H_1 H_2 U_{12m} \right), \tag{15}$$

$$\begin{aligned} U_{1m} &= s_{11m} F_{111m}^2 + s_{22m} F_{221m}^2 + 2 s_{12m} F_{111m} F_{221m} + s_{66m} F_{121m}^2, \\ U_{2m} &= s_{11m} F_{112m}^2 + s_{22m} F_{222m}^2 + 2 s_{12m} F_{112m} F_{222m} + s_{66m} F_{122m}^2, \\ U_{12m} &= s_{11m} F_{111m} F_{112m} + s_{22m} F_{221m} F_{222m} + s_{12m} (F_{112m} F_{221m} + F_{111m} F_{222m}), \end{aligned} \tag{16}$$

$$
\begin{aligned}
F_{11km} = \sqrt{8\pi}\Big\{ & R_{1m}^{\delta_1'-1} e^{-\delta_k''\Psi_{1m}} \Big[\Big((\mu_{1m}'^2 - \mu_{1m}''^2)(v_{1m}'\delta_k' - v_{1m}''\delta_k'') - 2\mu_{1m}'\mu_{1m}''(v_{1m}'\delta_k'' + v_{1m}''\delta_k')\Big)\cos(\Theta_{1km}) \\
& + \Big((\mu_{1m}'^2 - \mu_{1m}''^2)(-v_{1m}'\delta_k'' - v_{1m}''\delta_k') - 2\mu_{1m}'\mu_{1m}''(v_{1m}'\delta_k' + v_{1m}''\delta_k'')\Big)\sin(\Theta_{1km})\Big] \\
& + R_{2m}^{\delta_k'-1} e^{-\delta_k''\Psi_{2m}} \Big[\Big((\mu_{2m}'^2 - \mu_{2m}''^2)(v_{2m}'\delta_k' - v_{2m}''\delta_k'') - 2\mu_{2m}'\mu_{2m}''(v_{2m}'\delta_k'' + v_{2m}''\delta_k')\Big)\cos(\Theta_{2km}) \\
& + \Big((\mu_{2m}'^2 - \mu_{2m}''^2)(-v_{2m}'\delta_k'' - v_{2m}''\delta_k') - 2\mu_{2m}'\mu_{2m}''(v_{2m}'\delta_k' + v_{2m}''\delta_k'')\Big)\sin(\Theta_{2km})\Big]\Big\}, \\
F_{22km} = \sqrt{8\pi}\Big\{ & R_{1m}^{\delta_k'-1} e^{-\delta_k''\Psi_{1m}} \big[(v_{1m}'\delta_k' - v_{1m}''\delta_k'')\cos(\Theta_{1km}) + (-v_{1m}'\delta_k'' - v_{1m}''\delta_k')\sin(\Theta_{1km})\big] \qquad (17) \\
& + R_{2m}^{\delta_k'-1} e^{-\delta_k''\Psi_{2m}} \big[(v_{2m}'\delta_k' - v_{2m}''\delta_k'')\cos(\Theta_{2km}) + (-v_{2m}'\delta_k'' - v_{2m}''\delta_k')\sin(\Theta_{2km})\big]\Big\}, \\
F_{12m} = -\sqrt{8\pi}\Big\{ & R_{1m}^{\delta_k'-1} e^{-\delta_k''\Psi_{1m}} \Big[\Big(\mu_{1m}'(v_{1m}'\delta_k' - v_{1m}''\delta_k'') - \mu_{1m}''(v_{1m}'\delta_k'' + v_{1m}''\delta_k')\Big)\cos(\Theta_{1km}) \\
& + \Big(\mu_{1m}'(-v_{1m}'\delta_k'' - v_{1m}''\delta_k') - \mu_{1m}''(v_{1m}'\delta_k' + v_{1m}''\delta_k'')\Big)\sin(\Theta_{1km})\Big] \\
& + R_{2m}^{\delta_k'-1} e^{-\delta_k''\Psi_{2m}} \Big[\Big(\mu_{2m}'(v_{2m}'\delta_k' - v_{2m}''\delta_k'') - \mu_{2m}''(v_{2m}'\delta_k'' + v_{2m}''\delta_k')\Big)\cos(\Theta_{2km}) \\
& + \Big(\mu_{2m}'(-v_{2m}'\delta_k'' - v_{2m}''\delta_k') - \mu_{2m}''(v_{2m}'\delta_k' + v_{2m}''\delta_k'')\Big)\sin(\Theta_{2km})\Big]\Big\},
\end{aligned}
$$

$$
\Theta_{ikm} = \delta_k'' \ln r + \delta_k'' \ln R_{im} + (\delta_k' - 1)\Psi_{1m}. \qquad (18)
$$

To find the minimum of $\overline{\Sigma}_m(\theta)$ and consequently the crack initiation direction in materials I or II, following conditions have to be determined

$$
\left(\frac{\partial \overline{\Sigma}_m}{\partial \theta}\right)_{\theta_0} = 0, \quad \left(\frac{\partial^2 \overline{\Sigma}_m}{\partial \theta^2}\right)_{\theta_0} > 0. \qquad (19)
$$

STABILITY CRITERION

The stability criterion of the orthotropic bi-material notches defines the loading conditions above that a crack is initiated in the tip of the singular stress concentrator. The suggestion of the stability criterion based on the average stress calculated across a distance d from the wedge tip is presented. The value of the average stress $\overline{\sigma}_{\theta\theta}$ corresponding to the bi-material notch is calculated for the direction of θ_0 and it is compared with the critical stress $\overline{\sigma}_{\theta\theta C}$ corresponding to the crack [9]. Assuming the direction of crack propagation $\theta_0 = 0$ in homogeneous material under mode I of loading, for the critical stress $\overline{\sigma}_{\theta\theta C}$ can be obtained

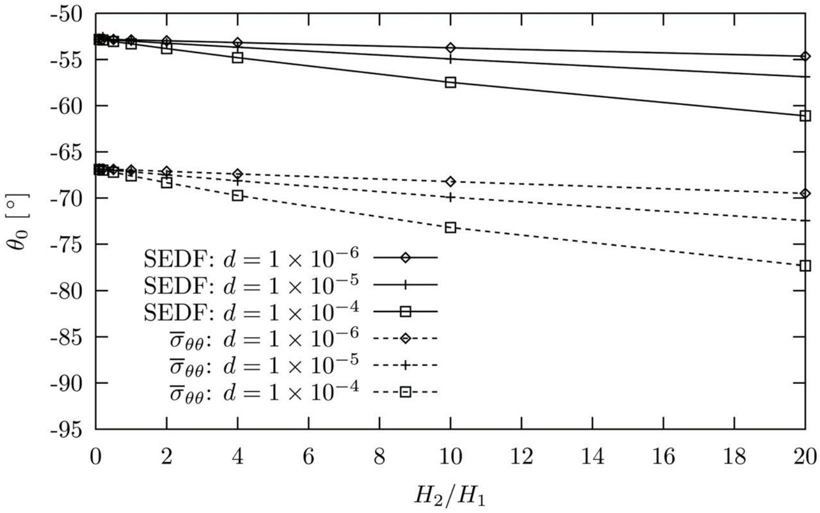

Figure 2. Crack initiation angles for varying loading conditions (expressed by H_2/H_1) and for selected distances d.

$$\overline{\sigma_{\theta\theta\,C}} = \frac{2K_{IC}}{\sqrt{2\pi d}}\,. \qquad (20)$$

Consider the fact that the ratio of the values H_1 and H_2 is constant for a given bi-material configuration and boundary conditions and does not depend on the absolute value of the applied stress σ_{appl} and it holds $\Gamma_{21} = H_2 / H_1 = H_{2C} / H_{1C}$. Inserting the ratio Γ_{21} and the critical value H_{1C} into the Eq. 9 the critical value of the average tangential stress for a bi-material notch is given. Following the assumption of the same mechanism of a rupture in both cases (crack and notch), the critical value of the average tangential stress can be comapred with the Eq. 20 for a crack and it is obtained an expression for H_{1C} value

$$H_{1C} = \frac{2K_{IC}}{\sqrt{2\pi d}\left(F_{\theta\theta 1m}(\theta_0) + \Gamma_{21}F_{\theta\theta 2m}(\theta_0)\right)}\,. \qquad (21)$$

NUMERICAL STUDY OF CRACK INITIATION DIRECTION

Within the paper the parametric study of the crack initiation directions is determined for specific geometry. The rectangular bi-material orthotropic notch characterized by angles $\omega_1 = 90°$ and $\omega_2 = 180°$ often occurs in engineering constructions or electric/electronic devices. It considered the bi-material with the Young's modulus $(E_x)_{\mathrm{I}} = 100$, $(E_y)_{\mathrm{I}} = 50$, $(E_x)_{\mathrm{II}} = 400$, $(E_y)_{\mathrm{II}} = 50$ [MPa]. The corresponding stress singularity exponents have values $\delta_1 = 0.573$ and $\delta_2 = 0.941$. The direction of the applied stress is not specified directly, but it is expressed by varying ratios $\Gamma_{21} = H_2/H_1$. The ratios Γ_{21} are gained from a numerical model by direct or integral methods [1,10,11]. The angles θ_0 are determined on the basis of finding the maximum of the mean value of the tangential stress and the minimum of the strain energy density factor. For the considered material the results of the crack initiation direction are shown in figure 2. The averaging distance d was taken as 1e-4, 1e-5 and 1e-6. It is shown that the direction θ_0 depends on d especially for larger differences between H_1 and H_2.

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