A damage mechanics approach to multiaxial fatigue

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ABSTRACT. *The fatigue assessment of structural components under a complex multiaxial stress history is one of the most difficult engineering challenge. As is well-known, several materials present a conventional fatigue strength at a given number of cycles under constant amplitude fatigue loading, while a conventional fatigue strength at a given number of cycles cannot be defined in the case of arbitrary varying and/or multiaxial stress histories. Several approaches have been developed to study this problem: empirical models, critical plane approaches, average stress criterion and stress invariant approaches. The damage phenomena can also be assessed by using an endurance function. In the present paper, a model for fatigue damage evaluation in the case of an arbitrary multiaxial loading history is proposed by using a damage function which allows us to evaluate the final failure of the material. By introducing an evolution equation for the material damage* D*, the final collapse of the material is assumed to occur when the damage is complete, that is to say, when* D *reaches the unity. The parameters of the model are determined through a Genetic Algorithm (GA) once a complex stress history and its effects on the material fatigue life are known. The proposed model presents the advantage to avoid any evaluation of a critical plane and any cycle counting algorithm to quantify the fatigue life, because it simply considers the loading process step by step and its effect in terms of damage.*

INTRODUCTION

Fatigue failure is one of the most analysed collapse mechanism in modern engineering applications since variable load histories are often involved in practical problems. From the pioneering researches conducted by Wöhler in the XIX century, enormous advances have been made in fatigue assessment of solids and structures [1, 2]. Nevertheless, the problem of fatigue is not completely solved because of the complex mechanical phenomena which are involved (fatigue damage is influenced by the environmental conditions, size of structural components, stress values, stress gradient, etc.), and remains still open, especially when the case of arbitrary loading history is considered. Even in high-cycle fatigue regime, where the material behaviour is macroscopically elastic, fatigue failure occurs after a certain number of loading cycles due to the nucleation and growth of micro-cracks, voids or micro-plasticized zones up to the formation of a dominant crack, subsequent stable fracture propagation, and complete failure when the macro-crack reaches a critical size.

The problem of fatigue assessment in the case of a complex multiaxial stress history is one of the most difficult since damage accumulation depends on all the components of the stress tensor and on their variation during the whole fatigue phenomenon. Several approaches have been proposed to examine such a problem: empirical models, critical plane criterion [3-6], average stress criterion [7], energy approach [8], and stress invariant approach [9].

In the present paper, an attempt to model fatigue damage for an arbitrary multiaxial loading history is presented based on a damage function the parameters of which are determined through a genetic algorithm procedure.

THE PROPOSED FORMULATION

Endurance function

As is well-known, several materials present a conventional fatigue strength at a given number of cycles under constant amplitude fatigue loading, i.e. a level of stress below which the expected life is practically unlimited and no damage accumulation occurs. When a cyclic loading is characterised by a maximum stress higher than such a conventional fatigue strength at a given number of cycles, failure occurs after a certain number of cycles due to the progressive development of damage inside the material up to the final collapse. On the other hand, a conventional fatigue strength cannot be defined in the case of arbitrary varying and/or multiaxial stress histories. Nevertheless, damage in such cases develops inside the material due to the complex action of all the stress tensor components, up to final failure of the structure.

As has been proposed by Ottosen et al. [10], the damage phenomena can be supposed to occur when the point *P* representing the stress state - for instance in the principal stress space - is outside a so-called *endurance surface* which can be usefully represented by a mathematical function (Fig. 1). The endurance surface can be assumed to depend on the stress invariants and on the deviatoric stress invariants. A very general endurance surface function can be written as follows:

$$E(\boldsymbol{\sigma},\mathbf{s}_e)=\left[a_1\cdot I_1(\boldsymbol{\sigma})+a_2\cdot I_2^{1/2}(\boldsymbol{\sigma})+a_3\cdot I_3^{1/3}(\boldsymbol{\sigma})+a_4\cdot J_2^{1/2}(\mathbf{s}_e)+a_5\cdot J_3^{1/3}(\mathbf{s}_e)\right]-\sigma_0=0 \qquad (1)$$

with $\mathbf{s}_e=(\mathbf{s}-\mathbf{s}_b)$

where $a_1, a_2, ..., a_5$ are material constants. The dependence of the stress invariants and deviatoric stress invariants (I_1, I_2, I_3, J_2, J_3) on the stress tensor $\boldsymbol{\sigma}$ and on the effective deviatoric stress tensor $\mathbf{s}_e$, respectively, is shown in Eq.(1). Further, note that the stress tensor $\mathbf{s}_b$ plays the role of a back stress tensor, and σ_0 is a material parameter.

From the above assumption to consider the effective deviatoric stress tensor $\mathbf{s}_e$, the endurance surface can change in a fashion similar to that for kinematic or isotropic hardening in the context of plasticity. The present model, simply based on the damage evaluation, has the advantage that it does not require any determination of a critical plane and any cycle counting algorithm to quantify the fatigue life, because it simply considers the progressive loading damaging effects developed step by step on the material.

Damage evolution

Damage increment can be assumed to occur when the stress state (represented by $\boldsymbol{\sigma},\mathbf{s}_e$) is outside the endurance function (when $E(\boldsymbol{\sigma},\mathbf{s}_e)>0$), while no damage is assumed to

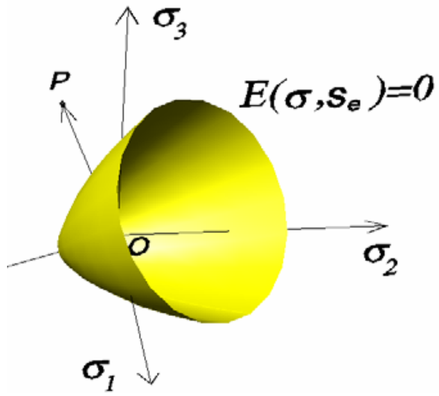

Fig. 1. A schematic view of the endurance function in the principal stress space

occur when $E(\boldsymbol{\sigma}, \mathbf{s}_e) \leq 0$. Such a damage phenomenon occurs if both the following conditions hold:

$$E(\boldsymbol{\sigma}, \mathbf{s}_e) > 0 \ , \quad dE(\boldsymbol{\sigma}, \mathbf{s}_e) > 0 \tag{2}$$

i.e. when the stress state is outside the endurance function and such a representative point moves far away from such a surface.

The amount of damage D at a given point of the structure must be evaluated by considering a so-called 'deterioration process' of the mechanical characteristics of the material.

Since damage is a nondecreasing function during the load history, i.e. $D \geq 0$, the condition $dD \geq 0$ is verified at every load step during the whole fatigue process. The final collapse of the material occurs when damage is complete, that is to say, when D reaches the unity ($D = 1$).

In order to describe the progressive damage phenomenon, an evolution equation is needed to measure the damage increment. It can be postulated that the amount of damage rate, dD, depends on the increment dE (between two subsequent time history stress states, $i-1$ and i, Fig. 2) of the endurance function values: $dE = E(P_i) - E(P_{i-1})$.

The damage increment can be assumed in the following form:

$$dD = A \cdot E^B \cdot dE \tag{3}$$

where A and B are material constants. Two different cases can arise as far as the damage increment calculation is concerned:

(a) The two subsequent stress values $i-1$ and i lead to $E(\boldsymbol{\sigma}_i) > 0$ and $E(\boldsymbol{\sigma}_{i-1}) < 0$. In such a case, the above difference can be defined as follows:

$$dE = E(P_i) - E(P_{i-1}) = E(P_i) - 0 = E(P_i) > 0 \tag{4}$$

(b) The two subsequent stress values $i-1$ and i lead to $E(\boldsymbol{\sigma}_i) > 0$ and $E(\boldsymbol{\sigma}_{i-1}) > 0$. In such a case, the above difference can be defined as follows:

$$dE = E(P_i) - E(P_{i-1}) > 0 \tag{5}$$

Endurance function evolution

As is stated above, the endurance function $E(\boldsymbol{\sigma}, \mathbf{s}_e)$ depends on the stress tensor $\boldsymbol{\sigma}$ and

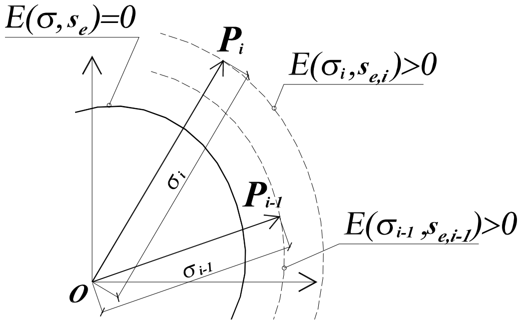

Fig.2. Definition of the endurance function evaluation $E(\boldsymbol{\sigma}_e, \mathbf{s})$ for the stress states $\boldsymbol{\sigma}_i$ and $\boldsymbol{\sigma}_{i-1}$

on the effective deviatoric stress $\mathbf{s}_e$, that is defined through the difference $\mathbf{s}-\mathbf{s}_b$ between the applied deviatoric stress $\mathbf{s}$ and the back stress tensor $\mathbf{s}_b$ ($\mathbf{s}_b$ quantifies the evolution of the endurance function during the loading history). Such a back stress tensor can be assumed to evolve according to the following relationship:

$$d\mathbf{s}_b = C \cdot dE^h \cdot (\mathbf{s}-\mathbf{s}_b) \tag{6}$$

In other words, the back stress evolution occurs in a direction parallel to the deviatoric stress difference $(\mathbf{s}-\mathbf{s}_b)$ and is proportional to the power h (which is assumed to be equal to 1.0 hereafter) of the endurance function increment dE.

MODEL CALIBRATION

The above described model for damage assessment in complex load history situations can be applied to simple uniaxial cyclic constant amplitude stress situations.

Constant amplitude uniaxial fatigue loading: fatigue strength at a given number of cycles

For a periodic stress history (the stress state is always inside the endurance function E), it can be deduced that $dD=0, D=0$ and $\mathbf{s}_b=\mathbf{e}=\mathbf{const}$ since $d\mathbf{s}_b=\mathbf{0}$, i.e. $\mathbf{s}_e=\mathbf{s}-\mathbf{e}$. If the periodic constant amplitude stress is uniaxial, the stress point P moves along one of the principal axes, and its value always ranges into the interval $\sigma_{\min} \le \sigma \le \sigma_{\max}$. In such a case, the endurance function can be simplified as follows:

$$E(\boldsymbol{\sigma}, \mathbf{s}_e) = \left[a_1 \cdot \sigma + \frac{a_4}{\sqrt{3}} \cdot \sqrt{\mathbf{s}_e : \mathbf{s}_e} + a_5 (\det \mathbf{s}_e)^{1/3} \right] - \sigma_0, \quad \text{with } I_1 = \sigma,\ I_2 = 0,\ I_3 = 0,$$
$$J_2 = s_{ii,e}{}^2/2,\ J_3 = s_{11,e} \cdot s_{22,e} \cdot s_{33,e}, \quad \mathbf{s}_e : \mathbf{s}_e = s_{ij,e} \cdot s_{ij,e} \quad \text{and} \quad \mathbf{s}_e = \mathbf{s}-\mathbf{s}_b = \mathbf{s}-\mathbf{e} \tag{7}$$

where the implicit summation notation on repeated indexes has been assumed. The minimum and maximum extreme stress values in a cycle can be written as follows: $\sigma_{\min} = \sigma_m - \sigma_a$ and $\sigma_{\max} = \sigma_m + \sigma_a$ (where σ_m is the mean stress and σ_a the stress amplitude). Equation (7) can be rewritten by considering that, for such a case, $s_{11,b} = 2\sigma_m/3$, $s_{22,b} = -\sigma_m/3$, $s_{33,b} = -\sigma_m/3$. The effective stress tensor and the effective deviatoric stress tensor become:

$$\sigma_{11,e} = \sigma_{11} - s_{11,b} = \sigma - 2\sigma_m / 3, \quad \sigma_{22,e} = \sigma_{22} - s_{22,b} = \sigma_m / 3, \quad \sigma_{33,e} = \sigma_{33} - s_{33,b} = \sigma_m / 3_e, \qquad (8)$$
$$s_{11,e} = 2(\sigma - \sigma_m)/3, \quad s_{22,e} = s_{33,e} = (-\sigma + \sigma_m)/3$$

The non zero stress invariants and the deviatoric stress invariants assume these values:

$$I_1 = \sigma, \qquad J_2 = (\sigma - \sigma_m)^2 / 3, \qquad J_3 = 2(\sigma - \sigma_m)^3 / 27 \qquad (9)$$

At the two above extreme cases of the stress value, the endurance function is given by:

$$E(\sigma_m - \sigma_a) = a_1(\sigma_m - \sigma_a) + \frac{a_4}{\sqrt{3}}\sqrt{(\sigma_m - \sigma_a - \sigma_m)^2} + (2/27)^{1/3} a_5(\sigma_m - \sigma_a - \sigma_m) - \sigma_0 = 0$$
$$E(\sigma_m + \sigma_a) = a_1(\sigma_m + \sigma_a) + \frac{a_4}{\sqrt{3}}\sqrt{(\sigma_m + \sigma_a - \sigma_m)^2} + (2/27)^{1/3} a_5(\sigma_m + \sigma_a - \sigma_m) - \sigma_0 = 0 \qquad (10)$$

and the mean value of the two above equations is:

$$\frac{E(\sigma_m - \sigma_a) + E(\sigma_m + \sigma_a)}{2} = a_1 \cdot \sigma_m + \frac{a_4}{\sqrt{3}} \cdot \sigma_a - \sigma_0 = 0 \qquad (11)$$

or equivalently:

$$\sigma_a = \frac{\sqrt{3}}{a_4}\sigma_0 + \left(-a_1 \cdot \frac{\sqrt{3}}{a_4}\right)\sigma_m = \sigma_{\lim.Fat} + p \cdot \sigma_m \qquad (12)$$

where $\sigma_{\lim.Fat}$ is the fatigue strength at a given number of cycles for zero mean stress uniaxial cycles, and p is the slope (usually negative) of the straight line in the Haigh diagram representing fatigue strength at a given number of cycles for constant amplitude stress against the mean stress of the loading cycles, which is usually able to satisfactorily fit the experimental data. The parameters σ_0, a_4 can be rewritten as follows:

$$\sigma_0 = \frac{a_4}{\sqrt{3}}\sigma_{\lim.Fat} = \frac{a_1}{p}\sigma_{\lim.Fat}, \qquad \text{and} \qquad a_4 = \frac{a_1\sqrt{3}}{p} \qquad (13)$$

i.e. they can be related to known material constants or to other parameters of the model.

Constant amplitude uniaxial fatigue loading: finite life situation

In the simple case of constant amplitude loading with zero mean value, $\sigma_m = 0$, leading to failure after N loading cycles, some relationships can be found between the damage parameters. In such a particular case, the invariants assume these expressions:

$$I_1 = \sigma, \; I_2 = 0, \; I_3 = 0, \; J_2 = \sigma^2 / 3, \; J_3 = 2\sigma^3 / 27 \qquad (14)$$

and the damage increment becomes:

$$dD = A \cdot \left[a_1 \cdot \sigma + \frac{a_1}{p} \cdot \sigma + a_1 \cdot \sqrt[3]{\frac{2}{27}}\sigma - \frac{a_1}{p} \cdot \sigma_{\lim.Fat} \right]^B \cdot dE = A \cdot \underbrace{[M \cdot \sigma - H]}_{E}^B \cdot dE \qquad (15)$$

where M, H are constants. During one single cycle of loading (Fig. 3), the damage is given by:

$$D_{1cycle} = \int_{1cycle} dD = 1/N = 2 \cdot D_{\alpha-\beta} = 2 \cdot D_{\gamma-\delta} \qquad (16)$$

since failure ($D = 1$) occurs after N loading cycles. Further, $D_{\alpha-\beta} = D_{\gamma-\delta} = 1/2N$.

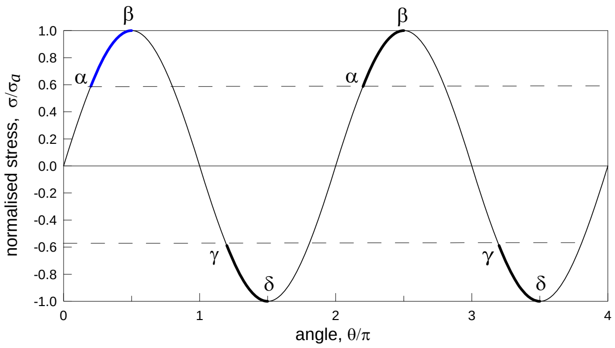

Fig. 3. Constant amplitude cycling loading: zones of damage development

Damage develops only in the intervals $\alpha-\beta$ and $\gamma-\delta$, where the stress state is assumed to cross the endurance function at point α and γ, i.e. $E(\sigma_\alpha=-\sigma_\gamma)=0$. By evaluating the damage developed during the interval $\alpha-\beta$, we get (Fig. 3):

$$D_{\alpha-\beta}=\int_\alpha^\beta A\cdot[M\cdot\sigma-H]^B\cdot dE=\int_\alpha^\beta A\cdot[M\cdot\sigma-H]^B\cdot Md\sigma=A\cdot M\left[\frac{1}{M}\frac{[M\cdot\sigma-H]^{B+1}}{B+1}\right]_{\sigma_\alpha}^{\sigma_\beta} \quad (17)$$

Finally, it can be written the following expression:

$$D_{\alpha-\beta}=\frac{1}{2N}=\frac{A}{B+1}\cdot\left[M\cdot\sigma_\beta-H\right]^{B+1} \quad (18)$$

i.e. the coefficient A results to be related to the coefficient B as follows:

$$A=\frac{B+1}{2N}\cdot\left[M\cdot\sigma_\beta-H\right]^{-(B+1)}=\frac{B+1}{2N}\cdot\left[M\cdot\sigma_a-H\right]^{-(B+1)} \quad (19)$$

where σ_a is the stress amplitude of the constant amplitude stress cycles.

PARAMETER ASSESSMENT BY A GENETIC ALGORITHM

Optimisation setting of design or experimental parameters in several engineering fields can be tackled by algorithms simulating the natural evolutionary process of life, also known as Genetic Algorithms (GAs) [11, 12]. Each GA is based on the Darwinian *survival of the fittest* principle, and iteratively improves the current solution by applying genetic concepts. By repeating the evolution procedure until a given tolerance is attained, the optimal condition can approximately be achieved.

Determination of the model parameters through a Genetic Algorithm

A Genetic Algorithm (GA) is here employed to find out the optimum values of the model parameters $a_1,a_2,....,a_5$ (that define the endurance function E), of the parameters A,B (that define the damage increment) and of the parameters C,h (that define the back stress evolution). By examining a generic multiaxial stress history and by knowing the experimental time t_f at which the failure occurs (i.e. $D=1$) for a given material under such a multiaxial stress history, the estimation error e can be written as follows:

$$e=D(t_f)-1 \quad (20)$$

where $D(t_f)$ is the damage evaluated through the above model by using the experimental time t_f at which the final failure occurs. The parameters $a_1, a_2, ..., a_5$, σ_0, A, B, C, h can be found by minimising such an error [13] through a proper iterative GA procedure.

NUMERICAL TESTS AND DISCUSSION

In the present section, the fatigue failure behaviour of the steel 18G2A under random non-proportional bending and torsion is examined [14]. The mechanical characteristics of such a steel are: conventional fatigue strength $\sigma_{af} = 270 MPa$ at 2.375 10^6 cycles and ulimate stress $\sigma_u = 535 MPa$. Firstly, by considering a simple constant amplitude cyclic uniaxial test, the coefficients a_1, a_4, a_5, A, B, C are determined (whereas the remaining coefficients are assumed for simplicity to be equal to $a_2 = a_3 = 0, h = 1$). The mean values of the coefficients obtained by considering the results related to two points on the Wöhler curve (Fig. 4a) are: $a_1 = 0.03095$, $a_4 = 1.0750$, $a_5 = 0.1945$, $A = 8.7E-14$, $B = 0.1349$, $C = 0.2050$. As can be noted, by introducing the above model parameters in the proposed endurance function, the Wöhler curve pattern is well reproduced by the numerical simulation (see dot symbols in Fig. 4a). By performing the genetic procedure in the cases of multiaxial random loading [14] with $\sigma_{max} = 475 MPa, \tau_{max} = 130 MPa$ and $\sigma_{max} = 420 MPa, \tau_{max} = 210 MPa$ (Fig. 4b), the obtained mean values of the coefficients are: $a_1 = 0.0309, a_4 = 1.0700, a_5 = 0.0301$, $A = 2.05E-14$, $B = 0.0795, C = 0.0371$. The simulations of other different multiaxial stress histories are represented in Fig. 4b making use of the above obtained coefficients.

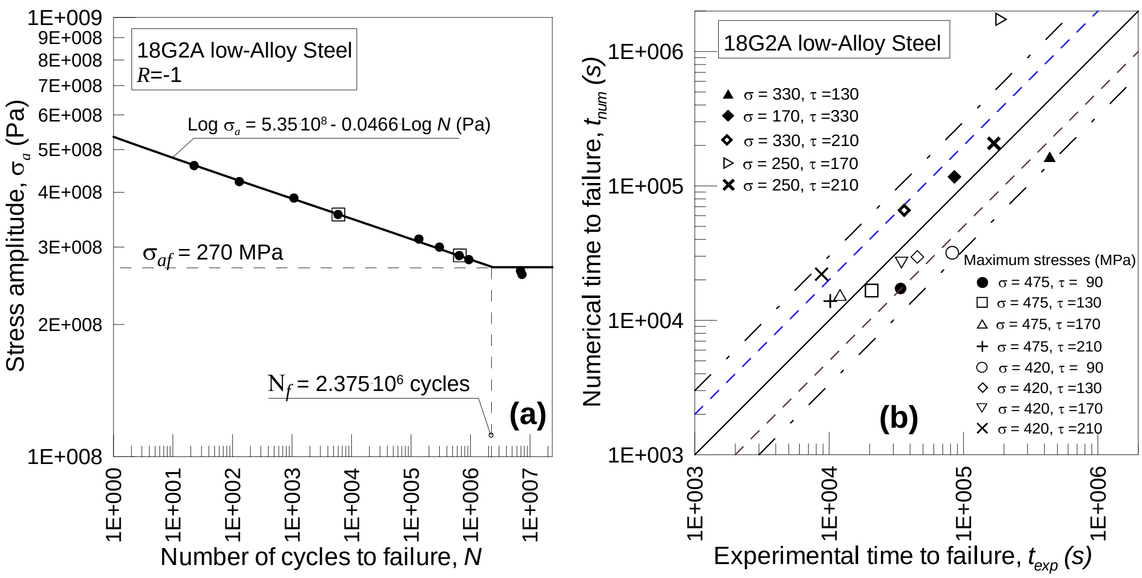

Fig. 4. (a) The Wöhler curve for steel 18G2A: estimation through the present model (dots). The squares indicates the points used for the model calibration. (b) Multiaxial random loading simulation: comparison between experimental [14] and numerical simulations.

Note that the obtained numerical results are within a scatter band with coefficient equal to 2.0 (dashed lines in Fig. 4b) and 3.0 (dash-dot lines in Fig. 4b) with respect to experimental time to failure.

CONCLUSIONS

In the present paper, the problem of fatigue life assessment of structural components under a complex multiaxial stress history has been examined through a fatigue damage evaluation. Such a damage is assessed through the definition of a proper endurance function (which depends on the stress invariants and deviatoric stress invariants), a suitable damage increment expression and the back stress evolution. The parameters of the proposed model are determined through a Genetic Algorithm (GA) once one or more complex stress histories and their effects on the material fatigue life are known.

The above model presents the advantage to not require any evaluation of a critical plane and any cycle counting algorithm to quantify the fatigue life, because it simply considers the loading process step by step and its effect in terms of damage.

Finally, some complex stress histories have been analysed in order to assess the model parameters to be used in the material fatigue strength evaluation under general varying multiaxial stress states. The obtained results show a satisfactorily ability of the model to simulate the fatigue failure behaviour of materials under uniaxial or complex multiaxial stress histories.

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