A Critical Plane Theory For Multiaxial Fatigue of Elastomers

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ABSTRACT. *In this paper, a critical plane theory dedicated to fatigue crack initiation in elastomers is derived. The approach is based on experimental observations which state that the initiation of a macroscopic fatigue crack results from the growth of small flaws initially present in a bulk material. Our derivation is based on the definition of an idealized Representary Volume Element that consists in the distribution of small cracks in the bulk material. In order to solve the problem without considering singularities, these cracks are assimilated to material surfaces defined in the undeformed reference configuration. With the help of the Configurational Mechanics theory, the virtual energy release rate associated with all possible changes of position of all possible material planes is derived to mimic the growth of small cracks. The critical plane, i.e. the plane in which the macroscopic crack will develop, is then defined as the material plane that maximizes this quantity. A careful solving of the corresponding optimization problem establishes that the critical plane orientation is the eigenvector of the configurational (Eshelby) stress tensor associated with its smallest eigenvalues. The approach is illustrated on the simple problem of simultaneous tension and torsion for which we explicitly calculate fatigue crack orientation.*

INTRODUCTION

In the last few years, number of papers has proposed predictors for fatigue crack nucleation in elastomers subjected to multiaxial loading conditions. By nature, a *crack nucleation approach* claims to be able to capture the onset of a macroscopic crack in a yet idealized defect-free material; practically it must be able to determine the most probable locus for the growth of a fatigue crack but it can also be able to predict its direction. In this latter case, the approach is referred to as a *critical plane theory*. Most of the critical plane theories consist in combining stress components that exert on a given geometrical plane, and to determine the plane on which this combination is maximized. For metallic materials, number of theories has been derived and most of them highlighted the coupled influence of shear and normal stresses; the case of polymers has received highly less attention.

Our study is devoted to rubber materials. From an experimental point of view, it is recognized that the nucleation of a macroscopic fatigue crack is the consequence of the growth of microscopic flaws initially present in the material [1,2], and the macroscopic crack grows in a plane defined by the multiaxiality of loading conditions [3]. Even if some authors recently considered the explicit influence of flaws on mechanical fields [4,5], such approaches cannot be managed in structural mechanics. Thus, crack nucleation approaches must focus on defect free derivations. Recent phenomenological investigations have stated that the relevant mechanical quantities to predict fatigue damage in such materials are the true stress tensor but also energy based tensors [6,7,8]. For the latter approach, the authors consider the energy necessary for the growth of microscopic flaws at the macroscopic scale, *i.e.* with Continuum Mechanics quantities. Concerning the orientation of fatigue cracks, a few critical plane theories have established that macroscopic cracks develop in the plane perpendicular to the largest nominal strain under proportional multiaxial loading conditions [9,10,11].

Our objective is to overcome the apparent incoherence between studying flaws and considering continuous fields. The key concept consists in adopting a realistic but implicit model of the distribution of microstructural inhomogeneities. In this way, at a given material point we consider all possible material planes and we derive the change in energy involved by the change of position of these planes in the undeformed configuration. Then, we determine the material plane that maximizes this energy. It leads to the critical plane for the growth of microscopic flaws and thus for the nucleation of a macroscopic fatigue crack. The theory is derived in the general case of homogeneous, incompressible hyperelastic materials under large strain.

DERIVATION OF THE THEORY

The Model

The present theory is based on a simple model of an idealized microstructure which evolution under multiaxial fatigue loading conditions is investigated. The definition of this model is schematized in Figure 1.

We consider a body in the sense of the Continuum Mechanics as shown in Fig. 1(a). At this scale, the continuous fields can be defined once a Representary Volume Element (RVE) (and then a material point *P*) has been defined. Classically, the RVE of soft materials includes the bulk material, hard inclusions and cavities (Fig. 2(b)). Under repeated loading conditions the nucleation of a macroscopic fatigue crack is the result of the propagation of microscopic defects that can be idealized as cracks: we define an idealized RVE that consists in an isotropic distribution of small cracks as shown in Fig. 1(c). Finally, from an engineering point of view, one expects to investigate the fatigue properties without considering explicit defects, thus we greatly simplify the study of the single crack by replacing it by its corresponding material plane, *i.e.* the oriented plane defined in the reference configuration and that contains the crack, as shown in Fig. 1(d).

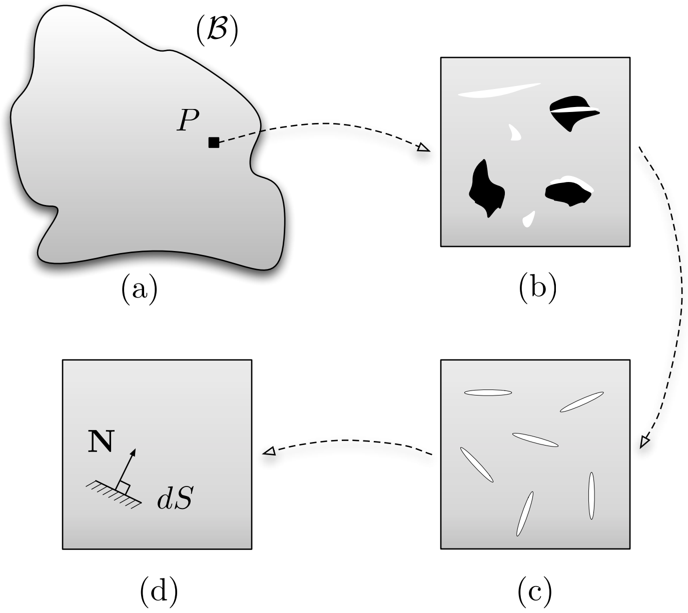

Figure 1. (a) Continuous body. (b) Real RVE. (c) Idealized RVE. (d) Model.

Energy Release Rate of a Material Plane

In Fracture Mechanics where cracks are explicitly considered, the key quantity is the energy release rate, *i.e.* the energy dissipated during crack growth per unit of newly created crack surface area. Although no crack is considered in the present study, it turns out that the concept of energy release rate has to be extended to our model. Note that along the same line of thinking Mars came up with the "Cracking Energy Density" which is an heuristic thermodynamical quantity that attempts to estimate the amount of strain energy that is available for the growth of an arbitrarily oriented small crack from a Continuum Mechanics point of view [7].

Inspired by Eshelby [12], and Kienzler and Herrmann [13], we define the energy release rate of a material plane as the change in energy caused by a material displacement of the considered plane. In this way, we consider the RVE in the material point P of Fig. 1(d) and we study its deformation.

Consider this RVE in the reference configuration (C_R) and its deformed state in the current configuration (C) as shown in Figure 2(a) and (b), respectively. The material plane in the reference configuration is an infinitesimal planar oriented surface $dS\mathbf{N}$ and it is transformed into $ds\mathbf{n}$ through the deformation gradient $\mathbf{F}$. In order to estimate the energy release rate of this material plane, we consider a replica of the RVE in which the material plane is displaced of $\delta\mathbf{U}$ in the reference configuration as shown in Fig. 2(c). Following the classical vocabulary of Configurational Mechanics, this displacement takes place in the Material space, *i.e.* the abstract set of particles that constitute the body [14]. As the mechanical quantities are considered uniform and equal to their macroscopic counterparts, the material plane in the replica is also deformed into $ds\mathbf{n}$

(Fig. 2(d)). The displacement of the plane $ds\mathbf{n}$ between the original RVE and its replica is denoted $\delta\mathbf{u}$ and is equal to $\mathbf{F}\delta\mathbf{U}$.

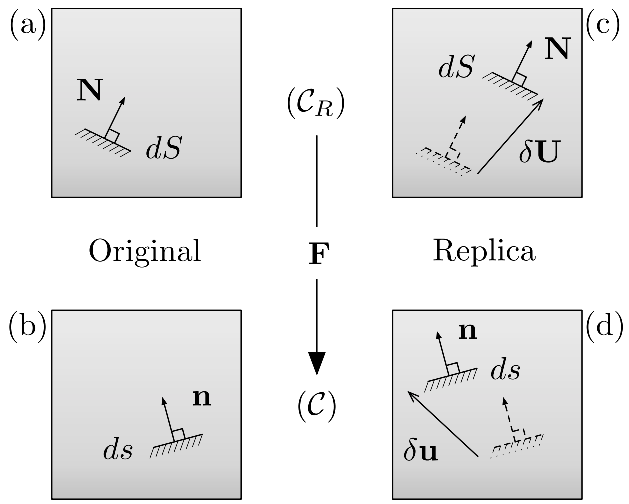

Figure 2. Deformation of a RVE and its replica.

We now estimate the energy δD that could be dissipated in the material displacement $\delta\mathbf{U}$ of the material plane $dS\mathbf{N}$. Similarly as in Fracture Mechanics, δD is equal to the opposite of the change in potential energy between the RVE and its replica

$$\delta\mathcal{D} = -\left(\delta W_{\text{int}} - \delta W_{\text{ext}}\right) \tag{1}$$

where δW_{int} is the change in strain energy in the RVE and δW_{ext} is the change in external forces that exert on the material plane. They are calculated as follows:

- The stress is uniform in the RVE, δW_{ext} is the scalar product of the Cauchy stress vector $\boldsymbol{\sigma}\mathbf{n}$ acting on the surface by its displacement vector

$$\delta W_{\text{ext}} = \delta\mathbf{u}\cdot\boldsymbol{\sigma}\mathbf{n}ds = \delta\mathbf{U}\cdot\mathbf{F}^T\mathbf{P}\mathbf{N}dS \tag{2}$$

  in which $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor.
- The change in strain energy in the RVE, δW_{int}, is associated with the topological change of the material plane $dS\mathbf{N}$. This change in strain energy is the product of the strain energy density $W(\mathbf{F})$ (per unit of undeformed volume) by the volume deformed throughout the material, *i.e.* the volume swept by the plane throughout the motion (Fig. 2(c)):

$$\delta W_{\text{int}} = W(\mathbf{F})\,\delta\mathbf{U}\cdot dS\mathbf{N} \tag{3}$$

Finally, using Eqs (2-3), the dissipation due to the material translation $\delta\mathbf{U}$ of the material plane $dS\mathbf{N}$ is given by

$$\delta\mathcal{D} = -\delta\mathbf{U}\cdot\left(W(\mathbf{F})\mathbf{I} - \mathbf{F}^T\mathbf{P}\right)dS\mathbf{N} = -\delta\mathbf{U}\cdot\mathbf{\Sigma}(P)dS\mathbf{N} \tag{4}$$

where $\mathbf{I}$ is the identity tensor, and $\mathbf{\Sigma}=W\mathbf{I}\text{-}\mathbf{F}^T\mathbf{P}$ is the configurational (or Eshelby, or Material) stress tensor at point P [15]. Introducing the unit vector $\mathbf{\Theta}$ that defines the direction of the material displacement and inspired by the definition of the energy release rate of existing cracks, we define the *energy released during the material displacement in the direction* $\mathbf{\Theta}$ *of a material surface with normal* $\mathbf{N}$*, per units of material surface and displacement* as

$$G^*\left(\mathbf{N},\mathbf{\Theta}\right) = \lim_{dS\to 0}\lim_{\|\delta\mathbf{U}\|\to 0}\frac{\delta\mathcal{D}}{\|\delta\mathbf{U}\|\,dS} = -\mathbf{\Theta}\cdot\mathbf{\Sigma}\mathbf{N} \tag{5}$$

Determination of the Critical Material Plane

In amorphous materials, defects are oriented in every possible direction in space. Thus, all material planes have the same potentiality to develop a fatigue crack. The only difference between given planes is the mechanical loading conditions they experience. So, $G^*(\mathbf{N},\mathbf{\Theta})$ can be calculated for any directions $\mathbf{N}$ and $\mathbf{\Theta}$ of the space. Thus, it is actually possible to determine the infinitesimal material plane and the associated infinitesimal material motion that maximize the virtual energy release rate in order to deduce the orientation of the material plane that undergoes the most damaging process. This critical material plane is then postulated to be the plane of crack nucleation.

However, when searching for the maximum of G^*, it should be made sure that all restrictions on the admissible vectors $\mathbf{N}$ and $\mathbf{\Theta}$ are taken into consideration: they are unit vectors, they must satisfy $-\mathbf{\Theta}\cdot\mathbf{\Sigma}\mathbf{N}\geq 0$ (the process is dissipative) and, due to symmetry their scalar product must be positive (they are in the same semi-space). Thus, the critical plane is defined by the normal vector $\mathbf{N}_\mathrm{D}$ and the corresponding material displacement direction $\mathbf{\Theta}_\mathrm{D}$ such that the virtual energy release rate is maximum:

$$\mathbf{\Theta}_\mathrm{D}\cdot\mathbf{\Sigma}\mathbf{N}_\mathrm{D} = \min\left\{\mathbf{\Theta}\cdot\mathbf{\Sigma}\mathbf{N}\leq 0 \text{ with } \|\mathbf{N}\|=1, \|\mathbf{\Theta}\|=1, -\mathbf{N}\cdot\mathbf{\Theta}\leq 0\right\} \tag{6}$$

This optimization problem has been carefully solved by Aït-Bachir [16]. In the case of isotropic hyperelastic materials, the set of solutions is

$$\mathcal{S}_1 = \left\{(\mathbf{\Theta}_\mathrm{D},\mathbf{N}_\mathrm{D}) = (\mathbf{v},\mathbf{v}) : \mathbf{v}\in\ker\left(\mathbf{\Sigma}-\Sigma_{\min}\mathbf{I}\right)\right\} \tag{7}$$

with

$$\Sigma_{\min} = \min\left\{\lambda\in\mathrm{sp}\left(\mathbf{\Sigma}\right) \ : \ \lambda < 0\right\} \tag{8}$$

where "ker" and "sp" stand for the kernel of the linear transformation associated with a tensor, and its spectrum, *i.e.* the set of its eigenvalues. From Eqs (7-8), it is predicted that the macroscopic fatigue crack will initiate in the plane perpendicular to the eigenvector of $\mathbf{\Sigma}$ associated with its smallest eigenvalues Σ_{min}, and that the normal material traction drives the crack nucleation process.

Another capability of the predictor that has yet been left out so far is that it also predicts the location where the first macroscopic crack initiates. Indeed, the location of the first macroscopic crack is simply understood as the location of the maximum of $G^*(\mathbf{N}_D,\mathbf{\Theta}_D)$ over the body.

APPLICATION: SIMULTANEOUS UNIAXIAL TENSION AND TORSION

In order to illustrate our theory, we consider the problem of fatigue under simultaneous uniaxial tension and torsion. Such loading conditions are the most widely considered to experimentally investigate multiaxial fatigue for rubber materials (see for example [7]). Here, the calculation steps are not detailed due to a lack of writing space; only important results are given.

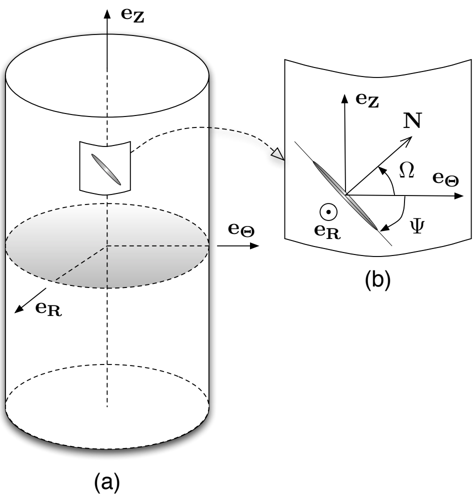

Figure 3. Simultaneous tension and torsion of a cylinder.

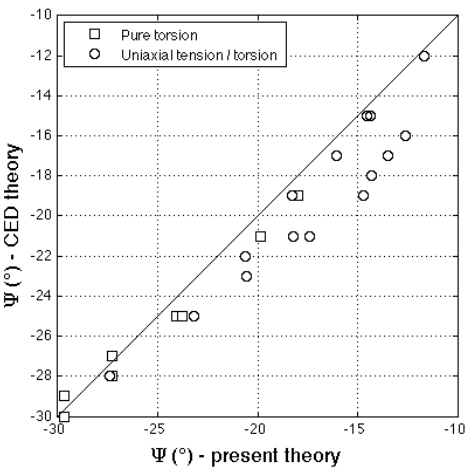

Figure 4. Crack angle: comparison with the CED theory [7].

Consider a hyperelastic cylinder with uniform cross-section; its length is denoted L and its external radius, R_e. (see Figure 3). The aim of the study is the derivation of the orientation of the first macroscopic fatigue crack. Noting (R,Θ,Z) (respectively (r,θ,z)) the coordinates of a material point in the underformed (respectively deformed) configuration, the change in coordinates due to simultaneous torsion and uniaxial extension is [17]

$$r = \frac{R}{\sqrt{\lambda}} \ , \ \theta = \Theta + \lambda\tau Z \ , \ z = \lambda Z \tag{9}$$

where λ is the extension ratio in the length-direction and τ is the twist angle per unit length.

Assuming that the material obeys the incompressible neo-Hookean constitutive equation (with constant *C*) (see for example [18]), it is easy to calculate the deformation gradient **F**, the strain energy *W*, the first Piola-Kirchhoff stress tensor **P** and finally the configurational stress tensor **Σ** (see Eq. (4)). With some algebraic manipulations, we are able to analytically obtain the smallest eigenvalue of **Σ**

$$\Sigma_{\min} = C\left[\frac{3}{\lambda} - 3 - \sqrt{\left(\lambda^2 - \frac{1}{\lambda}\right)^2 + 2\lambda^3\tau^2R_e^2 + \lambda^2\tau^4R_e^4 + 2\tau^2R_e^2}\right] \tag{10}$$

and the corresponding orientation of the critical plane (the angle of the crack plane **Ψ** in Fig. 3)

$$\Psi = \frac{\pi}{2} - \arctan\left(\beta\right) \tag{11}$$

with

$$\beta = \frac{\lambda^2 - \frac{1}{\lambda} + \lambda\tau^2R_e^2 + \sqrt{\left(\lambda^2 - \frac{1}{\lambda}\right)^2 + 2\lambda^3\tau^2R_e^2 + \lambda^2\tau^4R_e^4 + 2\tau^2R_e^2}}{2R_e\tau} \tag{12}$$

Here, we compare the present predictions with those of Mars' CED for pure torsion and simultaneous uniaxial tension/torsion [7]. The results are presented in Figure 4. Results are quiet similar, even if the present predictions lead to smaller values of crack angles. Similar differences have been previously highlighted in [8] and were explained by the empirical nature of the CED. It is interesting to recall that the CED theory necessitates the computation of the predictor for all possible crack orientations and *a posteriori* the determination of the orientation that maximizes it. Here, we prove that a unique equation Eqs (11-12) gives the orientation of the fatigue crack.

Extending this result to finite element computation demonstrates the relevance of the present approach: it highly reduces the necessary number of calculations to determine the locus of the macroscopic fatigue crack and the corresponding critical plane.

CONCLUSION

In this paper, we have proposed a mechanical model that permits the derivation of critical planes for fatigue of elastomers. It is based on an idealized RVE that contains not-interacting flaws modeled by material planes in the reference configuration. By considering the possible material displacements of these planes, we derive the bilinear function that permits to calculate the energy release rates for the growth of microscopic flaws. Then, we derive the constrained optimization problem that leads to the orientation of the macroscopic fatigue crack and to a possible predictor for fatigue life in terms of the configurational (Eshelby) stress tensor.

The rigorous derivation proposed here must be considered as a framework to investigate more complex problems: incremental method to study non-proportional loading conditions, more complex material behaviors such as inelastic or anisotropic constitutive equations,....

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