A New Formulation of the C-S Multiaxial Fatigue Criterion in the Frequency Domain

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ABSTRACT. *In the present paper, a new computationally-efficient frequency domain formulation of the critical plane-based Carpinteri-Spagnoli (C-S) criterion is proposed to evaluate the fatigue lives of smooth metallic structures subjected to multiaxial random loading. The critical plane orientation is here proposed to depend on the Power Spectral Density (PSD) matrix of the stress tensor. Then, the PSD function of an equivalent normal stress is defined by considering a linear combination of the PSD functions of the normal stress and the projected shear stress along the direction of maximum variance, with such stresses acting on the critical plane. The equivalent PSD function obtained allows us to apply the Tovo-Benasciutti method in order to determine the fatigue life of the structure being examined. The frequency domain formulation of the C-S criterion is applied to some relevant random fatigue tests related to smooth specimens under non-proportional bending and torsion random loading.*

INTRODUCTION

Engineering structures prone to fatigue failure are often exposed to cyclic loading which are characterized by randomly varying amplitudes. The assessment of structural integrity, fatigue strength and reliability under random loading is a critical issue in the design of such structures. However, despite the numerous research papers in the field, a correct quantification of the relationship between fatigue damage and load fluctuation features is still lacking. The problem is even more complex in the case of multiaxial loading.

In general, when dealing with random loading having specific statistical characteristics, a large effort has been spent in many research works to correlate fatigue damage with power spectral density characteristics of stress components. This approach in the frequency domain of loads represents an alternative approach (computationally appealing) to the classical one based on some cycle-counting methods in the time domain.

In the present paper, an efficient frequency domain formulation of the Carpinteri-Spagnoli (C-S) criterion is proposed to evaluate the fatigue lives of smooth metallic structures subjected to multiaxial random loading.

The critical plane orientation, originally correlated to weighted mean directions of the principal stresses [1-4], is here assumed to be dependent on the Power Spectral Density (PSD) matrix of the stress vector [5]. Then, the criterion presented in Refs [6, 7] for random loading is modified to evaluate the PSD function of an equivalent normal stress [8]. Accordingly, the shear stress acting on the critical plane is projected along the direction that maximises the variance of such a stress (note that the projected shear stress obtained is time-varying in modulus, but its direction does not change with time), and the PSD function of the equivalent stress is defined by a linear combination of the PSD functions of the normal stress and the projected shear stress, both acting on the critical plane. The equivalent PSD function obtained allows us to apply the Tovo-Benasciutti method [9] in order to determine the fatigue life of the structural component being examined.

The frequency domain formulation of the C-S criterion is applied to some relevant random fatigue experimental results available in the literature [10], related to smooth specimens under non-proportional bending and torsion random loading.

CRITICAL PLANE ORIENTATION

Let us consider a point in the structural component exposed to a general time-varying stress state. The stress tensor, defined with respect to the fixed frame PXYZ in Fig. 1, is described by the time-varying vector $\mathbf{s}(t) = \{s_1, s_2, s_3, s_4, s_5, s_6\}^T = \{\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}\}^T$.

Now assume that the random features of the stress tensor can be described by a six-dimensional stationary Gaussian stochastic process with zero mean values. Further let us assume that we know the matrix of PSD functions or, alternatively, that approximate discrete spectra are obtained from the Fast Fourier Transform (FFT) of stress time histories. The PSD matrix is expressed as follows:

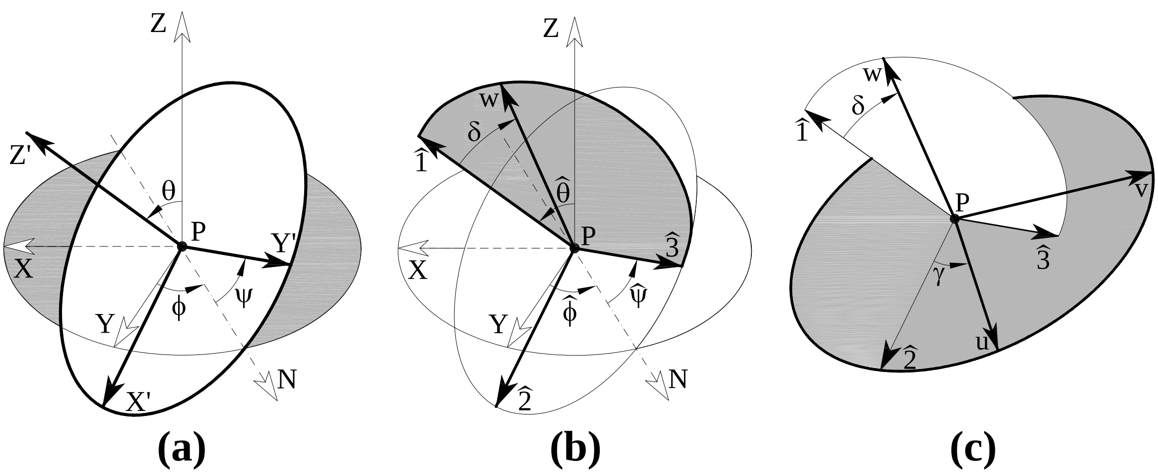

Figure 1. (a) Fixed frame PXYZ with rotated frame $\mathrm{PX'Y'Z'}$; (b) averaged principal stress frame $\mathrm{P}\hat{1}\hat{2}\hat{3}$; (c) frame Puvw attached to the critical plane.

$$\mathbf{S_{xyz}}(\omega) = \begin{bmatrix} S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} & S_{1,5} & S_{1,6} \\ S_{2,1} & S_{2,2} & S_{2,3} & S_{2,4} & S_{2,5} & S_{2,6} \\ S_{3,1} & S_{3,2} & S_{3,3} & S_{3,4} & S_{3,5} & S_{3,6} \\ S_{4,1} & S_{4,2} & S_{4,3} & S_{4,4} & S_{4,5} & S_{4,6} \\ S_{5,1} & S_{5,2} & S_{5,3} & S_{5,4} & S_{5,5} & S_{5,6} \\ S_{6,1} & S_{6,2} & S_{6,3} & S_{6,4} & S_{6,5} & S_{6,6} \end{bmatrix} \tag{1}$$

where the coefficients of the matrix are given by

$$S_{i,j}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_{i,j}(\tau)e^{-i\omega\tau}d\tau \qquad i,j = 1,\ldots 6 \tag{2}$$

and the auto/cross-correlation function $R_{i,j}$ is given by

$$R_{i,j}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{0}^{T} s_i(t)\,s_j(t+\tau)dt \tag{3}$$

with T = observation period.

If one considers a rotated coordinate system $\mathrm{P\,X'Y'Z'}$, e.g. defined by the three rotation Euler angles ϕ,θ,ψ (representing three sequential counterclockwise rotations about the *Z*-, *N*- and Z'-axis, respectively, where *N* is the so-called line of nodes, Fig. 1), the PSD matrix can be easily computed from the following relationship since the stress components in the rotated frame are linear combinations of the stress components in the original frame:

$$\mathbf{S_{x'y'z'}}(\omega) = \mathbf{C}\,\mathbf{S_{xyz}}(\omega)\,\mathbf{C}^T \tag{4}$$

where $\mathbf{C} = \mathbf{C}(\phi,\theta,\psi)$ is the rotation matrix, and the coefficients of the $\mathbf{S_{x'y'z'}}$ matrix are indicated as $S'_{i,j}(\omega)$, with $\mathrm{i,j} = 1,\ldots 6$, in the following.

According to the concept adopted in the original version of the C-S criterion [4] (see also Ref. [11] for a review of the criterion), the critical plane is linked to averaged principal stress directions. In the case of random loading, such directions are here proposed to be computed as follows. For given values of the angles ϕ,θ, the PSD function $S_{z',z'} = S'_{3,3}$ of the normal stress related to the general direction Z' can be computed through Eq. 4. Then, the number of up-crossing of level zero (zero-mean stresses are considered) is determined according to the Rice's formula [12]:

$$\nu_0^+ = \frac{1}{2\pi}\sqrt{\frac{\lambda_2}{\lambda_0}} \tag{5}$$

where λ_0 and λ_2 are the spectral moments of order 0 (variance) and 2, respectively, of the PSD function:

$$\lambda_0 = \int_{-\infty}^{+\infty}\phi'_{3,3}(\omega)d\omega, \qquad \lambda_2 = \int_{-\infty}^{+\infty}\omega^2\phi'_{3,3}(\omega)d\omega \tag{6}$$

The number of loading cycles over the observation period is hence:

$$N_1 = \nu_0^+ T \tag{7}$$

and, according to Davenport [13], the expected value of the extreme of the Gaussian process $\sigma_{z'}$ (over the observation period T) is given by:

$$E\left[\max_{0\le t\le T} \sigma_{z'}(t)\right] = \sqrt{\lambda_0}\, F(N_1) \tag{8}$$

with $F(N_1) = \sqrt{2\ln(N_1)} + \dfrac{0.5572}{\sqrt{2\ln(N_1)}}$.

By varying the angles ϕ, θ ($0 \le \phi \le 2\pi,\ 0 \le \theta \le \pi$) with the aim of searching the direction Z' experiencing in a statistical sense the maximum normal stress, the maximum of Eq. 8 is determined, and such a direction is regarded as the averaged principal direction $\hat{\mathbf{1}}$ (hence defined by the angles $\hat{\phi}$ and $\hat{\theta}$).

The angle ψ, representing a rotation about the axis $\hat{\mathbf{1}}$, is made to vary with the aim of determining the direction (on the plane with normal $\hat{\mathbf{1}}$) where the corresponding shear stress component (which can also be regarded as the resolved shear stress along a direction) attains the maximum variance. This procedure is performed through Eq. 4 (where $\mathbf{C}(\hat{\phi}, \hat{\theta}, \psi)$ is a function of the angle ψ only), by maximazing the variance of the process $\tau_{y'z'}$, namely by maximizing $\int_{-\infty}^{+\infty} S'\phi_{6,6}(\omega) d\omega$ where $S'_{6,6} = S_{y'z',y'z'}$. The obtained direction Y' is regarded as the averaged principal direction $\hat{\mathbf{3}}$ (hence defined by the angles $\hat{\phi}$, $\hat{\theta}$ and $\hat{\psi}$) describing the plane $\hat{\mathbf{1}}\hat{\mathbf{3}}$ of averaged maximum shear. Therefore, the direction X' is regarded as the averaged principal direction $\hat{\mathbf{2}}$.

The normal to the critical plane is defined by the off-angle δ (clockwise rotation), function of the ratio between fully reversed shear and normal stress fatigue limits [4], about the axis $\hat{\mathbf{2}}$:

$$\delta = \frac{3\pi}{8}\left[1 - \left(\frac{\tau_{af,-1}}{\sigma_{af,-1}}\right)^2\right] \tag{9}$$

EQUIVALENT PSD FUNCTION

The frame Puvw, attached to the critical plane (**u** and **v** belong to the critical plane, and **w** is the normal to the critical plane), is obtained by performing five successive rotations of the angles $\hat{\phi}$, $\hat{\theta}$ and $\hat{\psi}$ (see the Section entitled Critical plane orientation), δ (see Eq. 9) and γ (where γ represents a counterclockwise rotation about **w**-axis, so that the **v**-axis defines the direction of maximum variance, i.e. the direction where the variance of the process τ_{vw}, i.e. $\int_{-\infty}^{+\infty} S_{vw,vw}(\omega)d\omega$, is maximum). The PSD matrix $\mathbf{S_{uvw}}(\omega)$ related to the coordinate system Puvw can be worked out through Eq. 4 ($\mathbf{S}_{uvw} = \mathbf{C}\mathbf{S}_{xyz}\mathbf{C}^T$), where now the rotation matrix $\mathbf{C}$ is a function of five angles:

$$\mathbf{C} = \begin{bmatrix} c_\gamma^2 & s_\gamma^2 & 0 & 0 & 0 & 2c_\gamma s_\gamma \\ s_\gamma^2 & c_\gamma^2 & 0 & 0 & 0 & -2c_\gamma s_\gamma \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_\gamma & s_\gamma & 0 \\ 0 & 0 & 0 & -s_\gamma & c_\gamma & 0 \\ -c_\gamma s_\gamma & c_\gamma s_\gamma & 0 & 0 & 0 & c_\gamma^2 - s_\gamma^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_\delta^2 & s_\delta^2 & -2c_\delta s_\delta & 0 & 0 \\ 0 & s_\delta^2 & c_\delta^2 & 2c_\delta s_\delta & 0 & 0 \\ 0 & c_\delta s_\delta & -c_\delta s_\delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_\delta & s_\delta \\ 0 & 0 & 0 & 0 & -s_\delta & c_\delta \end{bmatrix} \hat{\mathbf{C}} \quad (10)$$

where $\hat{\mathbf{C}} = \mathbf{C}(\hat{\phi}, \hat{\theta}, \hat{\psi})$ is the rotation matrix related to the 'principal' Euler angles ($\hat{\phi}, \hat{\theta}, \hat{\psi}$). Further: $c_\gamma = \cos\gamma$, $s_\gamma = \sin\gamma$, $c_\delta = \cos\delta$ and $s_\delta = \sin\delta$.

After the above sequence of rotations, we have the PSD functions $S_{w,w}$ and $S_{vw,vw}$ related to the processes σ_w and τ_{vw}, respectively. In order to reduce the multiaxial stress state to an equivalent unixial stress state, an equivalent PSD function is proposed to be determined through the following linear combination:

$$S_{eq} = S_w + \left(\frac{\sigma_{af,-1}}{\tau_{af,-1}} \right) S_{vw,vw} \quad (11)$$

FATIGUE DAMAGE ACCUMULATION

Having performed the above reduction of the multiaxial random stress state to an equivalent unixial one, the frequency approach proposed by Tovo and Benasciutti [9] can be applied to the PSD function S_{eq}. The details are reported in Ref. [9] whereas only the main equations are recalled hereafter.

The spectral moments are:

$$\lambda_m = \int_{-\infty}^{+\infty} \omega^m S_{eq}(\omega)d\omega \qquad m = 0,\ 0.75,\ 1,\ 1.5,\ 2,\ 4 \quad (12)$$

Then the frequency ν_0^+ of up-crossings can be determined by applying Eq. 5, and the frequency of peaks can be obtained from:

$$\nu_P = \frac{1}{2\pi}\sqrt{\frac{\lambda_4}{\lambda_2}} \quad (13)$$

Further, the following spectral band-width parameters can be worked out:

$$\alpha_1 = \frac{\lambda_1}{\sqrt{\lambda_0 \lambda_2}}, \qquad \alpha_{0.75} = \frac{\lambda_{0.75}}{\sqrt{\lambda_0 \lambda_{1.5}}}, \qquad \alpha_2 = \frac{\lambda_2}{\sqrt{\lambda_0 \lambda_4}} \tag{14}$$

According to the so-called narrow band approximation of Wirshing-Light [14], being k and C the parameters of the normal stress S-N curve $S^k N = C$, the unit fatigue damage is given by [9]:

$$D_{NB} = \nu_0^+ C^{-1} \left(\sqrt{2\lambda_0}\right)^k \Gamma\left(1 + \frac{k}{2}\right) \tag{15}$$

where Γ is the gamma function ($\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$). Note that such a relationship is approximate for general processes, but it is exact in the case of narrow band signals.

Then, the unit fatigue damage according to the range-mean counting [15] is approximately given by:

$$D_{RC} = \nu_p C^{-1} \left(\sqrt{2\lambda_0}\,\alpha_2\right)^k \Gamma\left(1 + \frac{k}{2}\right) \tag{16}$$

Finally, according to Ref. [9], a linear combination of the fatigue damage computed according to the above two approximations is proposed to estimate the damage of the rainflow method:

$$D_{RFC} = b D_{NB} + (1 - b) D_{RC} \tag{17}$$

where $b = \left(\alpha_{0.75}^2 - \alpha_2^2\right) / \left(1 - \alpha_2^2\right)$. In a stationary process, the fatigue damage in Eq. 17 is constant and, considering a critical damage equal to unity, the calculated fatigue life is $T_{cal} = 1/D_{RFC}$.

APPLICATION TO EXPERIMENTAL RESULTS

Experimental tests on smooth specimens made of 18G2A steel [10] are now examined. The specimens are submitted to non-proportional random bending ($\sigma = \sigma_x$) and torsion ($\tau = \tau_{xy}$) with zero mean stresses. The random loadings have a dominant frequency of 28.8 Hz for bending and 30 Hz for torsion. The loading histories are independent; further, they are sums of four harmonic components with different amplitudes (following a Gaussian probability distribution) and phases. The power spectral density function is characterized by four frequency peaks. The loading histories have a duration of 820 s and a sampling frequency of 250 Hz.

In such experimental tests, 13 combinations of normal and shear stress loading histories, characterized by different values of the ratio $\lambda_\sigma = \tau_{max}/\sigma_{max}$, are considered.

The mechanical properties of the tested steel, reported in Ref. [10] unless otherwise specified, are: Young modulus E = 210 GPa, Poisson ratio ν = 0.3, yield stress f_y = 357 MPa, ultimate tensile strength σ_u = 535 MPa, fully reversed bending fatigue limit $\sigma_{af,-1}$ = 270 MPa, fully reversed torsion fatigue limit $\tau_{af,-1}$ = 170 MPa (see Ref.[16]), inverse slope k = 7.2 (estimated) and coefficient $C = 7.61 \times 10^{23}$ (calculated) of S-N curve for fully reversed bending, reference number $N_0 = 2.375 \times 10^6$ of loading cycles corresponding to fatigue limit $\sigma_{af,-1}$.

The thirteen loading combinations are processed by means of the proposed criterion based on a frequency domain approach (accordingly, the criterion takes into account the loading conditions through the discrete spectra obtained from the FFT of the recorded experimental stress time histories). The comparison, presented in Fig. 2 in terms of experimental against calculated fatigue lives ($T_{\exp} - T_{cal}$), seems to be quite satisfactory, with 89% of the results within the 3x band and with 70% within the 2x band.

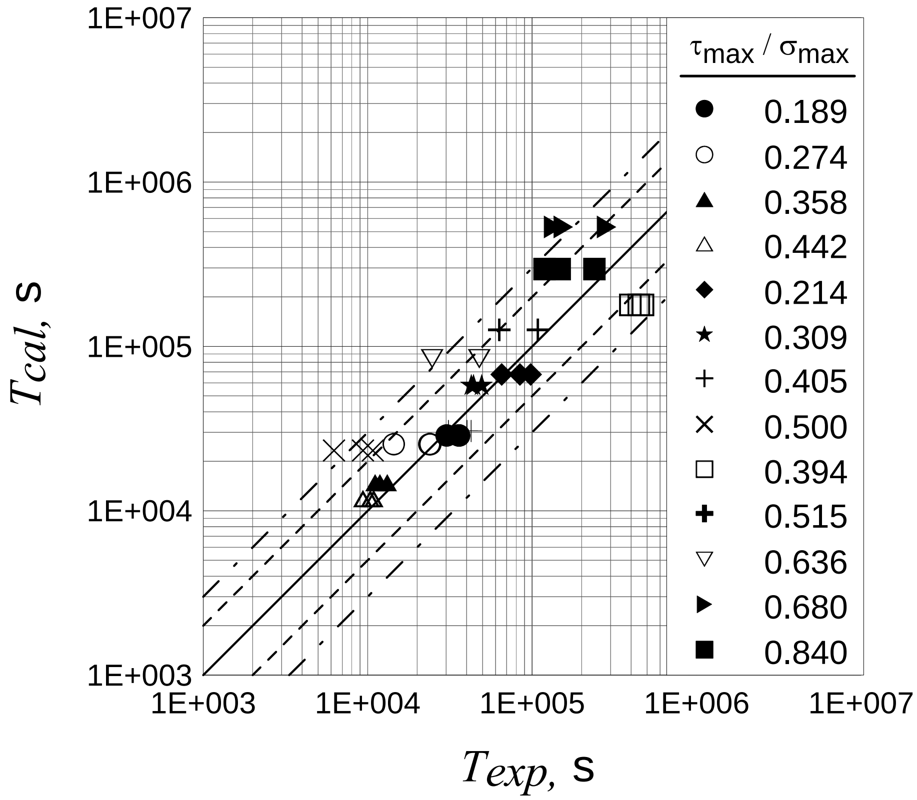

Figure 2. Comparison between calculated and experimental fatigue lives.

CONCLUSIONS

A new computationally-efficient frequency domain formulation of the critical plane-based Carpinteri-Spagnoli (C-S) criterion is proposed in order to evaluate the fatigue life of a smooth metallic structure subjected to multiaxial random loading. The main novel characteristic of such a proposal is the determination of the critical plane orientation through the Power Spectral Density (PSD) matrix of the stress tensor. Then, an equivalent PSD function is suitably defined and processed through the Tovo-Benasciutti method to determine the fatigue life of the structural component being examined. The comparison with some experimental tests is satisfactory, although more complex loading conditions (characterized in particularly by broad-band spectra) need to be processed in order to fully assess the estimation capability of the proposed criterion.

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