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Prediction of Crack Initiation Direction for Surface Flaws under Biaxial Loading

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ABSTRACT: The theoretical derivation of the fracture crack initiation angle as a function of induced crack angles of angled surface flaws under biaxial stress field is discussed in this paper. The proposed method is an extension of the Erdogan and Sih approach from uniaxial to biaxial tension-compression stressed models. This method assumes that the crack extends in a radial direction and that the initial fracture crack angle, θ_0, is obtained by maximizing the hoop stress along a circumference of a radius r. A series expansion including higher order terms in the hoop stress expression derived at a distance r from the crack tip with respect to the crack angles, β, is used to determine the initial fracture angles. Mohr's circle is used to determine the final expressions of the stresses applied at an element close enough to the crack tip. The theoretical results of the crack initiation angle as a function of the induced crack angle are compared to uniaxial and biaxial loading cases and show excellent agreement with published results.

Notation

α	biaxiality ratio
β	induced crack angle
μ_f	coefficient of friction
θ_o	initial fracture angle
σ_x	parallel stress
σ_y	normal stress
σ_θ	hoop stress
τ_{xy}	shear stress at an element crack tip

Introduction

A large number of publications dealing with the mixed mode problem was restricted to uniaxial loading conditions; however, in many cases, most structures are subjected to complex loading conditions such as biaxial loading. Erdogan and Sih [1] were among the first investigators who treated the problem of thin sheets with angled central cracks of length 2a under uniaxial uniform tension. In their study, the initial fracture crack direction was determined by assuming that the crack grows in a direction for which the hoop stress, at the crack tip is maximum. A few years later, Williams and Ewing [2] modified their theory by including the non-singular terms in the series expansion of the hoop stress expression for better experimental correlations. Later, Finnie and Saith [3] pointed out an error in the expression derived by Williams and Ewing [2] where the contribution of the normal stress to the crack was not included. It was later accounted for by Swedlow [4]. However, a limited number of publications dealing with mixed mode problems under biaxial stress field is published. Recently, Ling and Woo [5] were among the first investigators who studied angled cracks under biaxial loading condition. Their study was based on the extension and modification of Swedlow's [4] approach from uniaxial to biaxial tension-compression stressed models. In their analysis, the closing and frictional effects were taken into account by introducing a friction coefficient in the series expansion of the stress equations which led to the conclusion that the maximum hoop stress criterion is the most appropriate one for both open and closed crack conditions as compared to other fracture criteria for predicting the initial fracture crack angles.

Although these studies have considered the closing and frictional effects by introducing a friction coefficient that takes account of this mechanism, they never mentioned the possibility of crack propagation delay or even arrest in some cases given the fact that the closing mechanism takes place only for large values of crack angles. Therefore, based on this hypothesis, the present paper predicts the fracture crack angle and focuses on the occurrence and effects of crack closure with respect to the biaxiality ratio and the crack angle.

Proposed Model

The problem of mixed mode fatigue crack initiation has been of growing interest to both designers and researchers to predict the initial fracture crack angle under multiaxial fatigue loading. A good representation of a two-dimensional mixed mode loading case is a straight surface flaw oriented at various crack angles, β, with respect to the maximum tensile stress as shown in Figure 1.

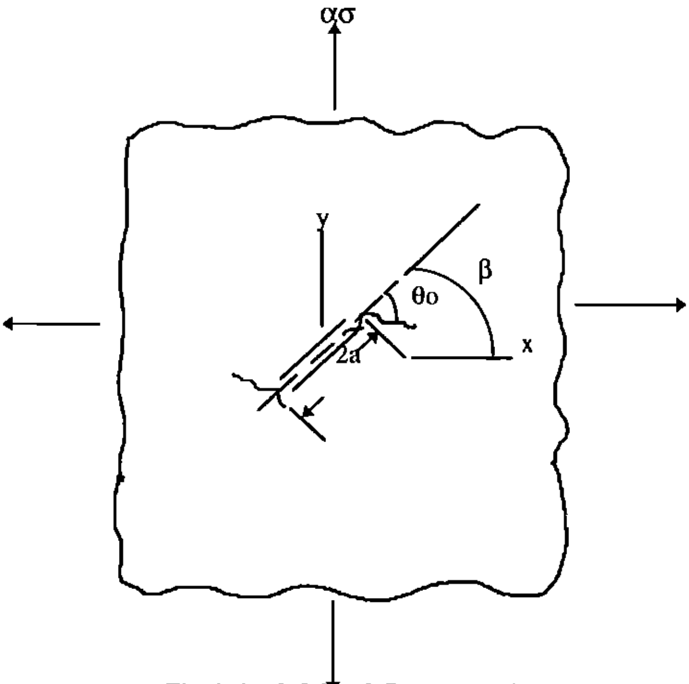

Fig. 1 Angled Crack Representation

This figure also shows an inclined induced surface flaw where β, θ_o, and α designate, respectively, the angle of the surface flaw relative to the maximum tensile stress, the initial fracture crack angle relative to the original crack direction and the biaxiality ratio defined by ($\alpha = \sigma_2/\sigma_1$ where σ_1 is the maximum tensile stress).

Fracture criteria applied to this problem have assumed that the initial fracture crack angle, θ_o, can, in general, be determined by the stress field near the crack tip. The proposed model is, therefore, based on the expression of the hoop stress near the crack-tip for a biaxial stress field. The stress equation, in the form of a series expansion as presented in

[2], takes the following form:

$$\sigma_\theta = \sqrt{\frac{a}{2r}} \cos\frac{\theta}{2} \left(\sigma_{y'} \cos^2\frac{\theta}{2} - \frac{3\tau_{x'y'}}{2} \sin\theta\right) + \sigma_{x'} \sin^2\theta + \sqrt{\frac{2r}{a}} . F(\sigma_{y'}, \tau_{x'y'}, \theta) + \ldots \tag{1}$$

where $\sigma_{x'}$, $\sigma_{y'}$ and $\tau_{x'y'}$ represent, respectively, the parallel stress, the normal stress and the shear stress applied at an element ahead of the crack tip, as shown in Figure 1. It is worth noting that the first term in Equation (1) gives the singularity as r approaches zero, where r and a express the variation of both $\sigma_{x'}$ and $\tau_{x'y'}$ due to the presence of the crack. On the other hand, the second term representing the stress parallel to the crack is simply superimposed on the stress distribution; hence, it is unaffected by r and a. For a localized region near the crack tip, only the first term is considered since it dominates the solution.
Using the principal of the maximum hoop stress criterion postulated by Erdogan and Sih [1], the initial fracture crack angle, θ_o, can be obtained if the following conditions are satisfied:

$$\sigma_\theta|_{\theta_0} > 0, \qquad \left[\frac{\partial\sigma_\theta}{\partial\theta}\right]_{\theta_0} = 0, \qquad \left[\frac{\partial^2\sigma_\theta}{\partial\theta^2}\right]_{\theta_0} < 0 \tag{2}$$

by setting $\lambda = (2r/a)^{1/2}$ in Equation (1) and using the conditions of Equation (2), the direction of the initial fracture crack growth, θ_o, can be obtained by maximizing Equation (1) and setting $\frac{\partial \sigma_\theta}{\partial \theta} = 0$ resulting in:

$$\sigma_{y'} - \left[\frac{1-3\cos\theta}{\sin\theta}\right] \tau_{x'y'} - \frac{16\lambda}{3}\left[\frac{\sin\frac{\theta}{2}}{\tan\theta}\right]\sigma_{x'} = 0 \tag{3}$$

Equation (3) is the general expression for the maximization of the hoop stress, σ_θ, which can be solved for the initial fracture crack angle, θ_o, for a given stress field. The expressions for $\sigma_{x'}$, $\sigma_{y'}$ and $\tau_{x'y'}$ depend on the nature of the loading conditions (uniaxial or biaxial).

The proposed model, based on the approaches of Erdogan and Sih [1] and Finnie and Saith [3] extends both methods from uniaxial to biaxial loading conditions. In this model, Mohr's circle is used to determine the stresses applied at an element ahead of the crack tip of angled cracks. The normal, tangential, and shear stresses are given by:

$$\begin{aligned} \sigma_{y'} &= \sigma(\sin^2\beta + \alpha\cos^2\beta) \\ \sigma_{x'} &= \sigma(\cos^2\beta + \alpha\sin^2\beta) \\ \tau_{x'y'} &= (1-\alpha)\sigma\sin\beta\cos\beta \end{aligned} \tag{4}$$

Note that $\sigma_{y'}$ and $\tau_{x'y'}$, are equivalent to Mode I and Mode II stress intensity factors, K_I and K_{II}, respectively. To better understand the presence of the stress acting ahead of the crack tip, Mohr's circle was constructed as shown in Figure 2, which shows the stresses along the x-y and the x'-y' planes for various biaxiality ratios.

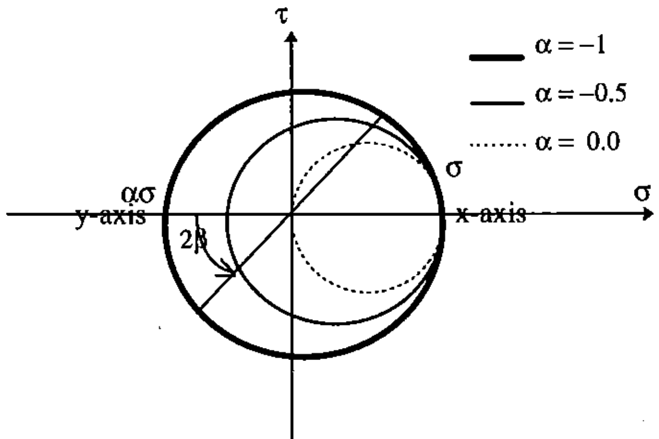

Fig. 2 Mohr's Circle Configurations for Various Biaxiality Ratios

Note that the stresses applied at the plate, σ and $\alpha\sigma$, as shown in Figure 1, are principal stresses. Therefore, it is necessary to rotate the crack element by an angle, β, in order to determine the stresses applied at the element ahead of the crack tip. The angle, β, is the orientation angle of the induced surface flaw with respect to the maximum tensile stress. By substituting the values of $\sigma_{x'}$, $\sigma_{y'}$, $\tau_{x'y'}$ from Equations (4) into Equation (1), we have

$$\begin{aligned} &(\alpha+\tan^2\beta)-(1-\alpha)\left[\frac{1-3\cos\theta}{\sin\theta}\right]\tan\beta \\ &-\frac{16\lambda}{3}\cdot\left[\frac{\sin\frac{\theta}{2}}{\tan\theta}\right](1+\alpha\tan^2\beta)=0 \end{aligned} \tag{5}$$

Equation (5) is the proposed model for relating the crack initiation angle, θ, to the induced crack angle, β, and the biaxiality ratio, α, in a biaxial stress field. This equation is solved using Newton-Raphson method and the obtained roots are denoted by θ_o.

To check whether the derived equation is appropriate, it is reduced to the uniaxial case when $\alpha = 0$, taking the following form:

$$\tan^2\beta - \left[\frac{1-3\cos\theta}{\sin\theta}\right]\tan\beta - \frac{16\lambda}{3}\left[\frac{\sin\frac{\theta}{2}}{\tan\theta}\right] = 0 \tag{6}$$

which is similar to the equation derived by Williams and Ewing [2] for uniaxial loading.

Theoretical Results and Discussion

The solution of Equation (5) is presented in Figure 3 showing the initial fracture angle, θ_o, as a function of the induced crack angle, β, for values of biaxiality ratios, α, ranging between -1.0 and 0.

Note that, for small values of crack angles, β, at a given biaxiality ratio, α, a wide deviation is observed giving high values of θ_o. This deviation is due to the effects of $\sigma_{y'}$ which causes the crack to close. This mechanism takes place whenever the stress value of $\sigma_{y'}$ becomes negative depending on the values of the crack angle, β, as well as the biaxiality ratio, α. It is interesting to note that the closing effect ($\sigma_{y'} < 0$) gets delayed as the biaxiality ratio changes from -1 (compression-tension biaxial loading case) to 0 (uniaxial loading case) since the stress values become negative for $\beta < 45°$. This phenomenon is illustrated by Mohr's circle where the maximum shear stress is getting smaller and smaller as α changes from -1 to 0 as shown in Figure 2. A relationship, describing this delay mechanism, between the biaxiality ratio and the crack angle for which the normal stress, $\sigma_{y'}$, becomes negative is given by:

$$\frac{1+\alpha}{1-\alpha} + \cos 2\beta \le 0, \quad \text{for} \quad \alpha \le 0 \tag{7}$$

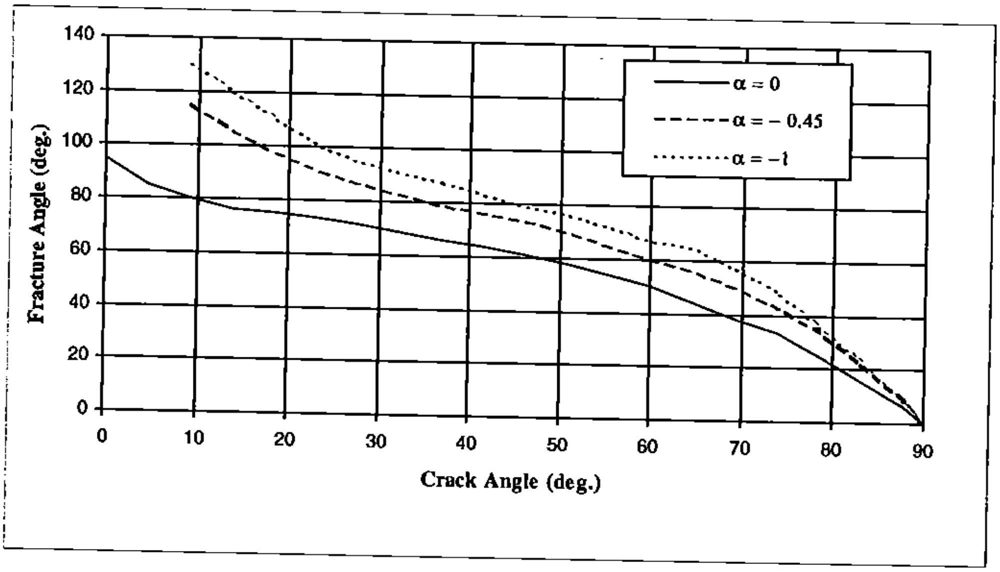

Fig.3 Fracture Angle vs. Crack Angle

This equation describes the conditions for which the normal stress to the crack surface, $\sigma_{y'}$, becomes negative. This condition is valid only for negative biaxiality ratios, α. This delay is best described in Figure 4 which shows the biaxiality ratio, α, as a function of the crack angle, β, which triggers the starting points of the closing mechanism.

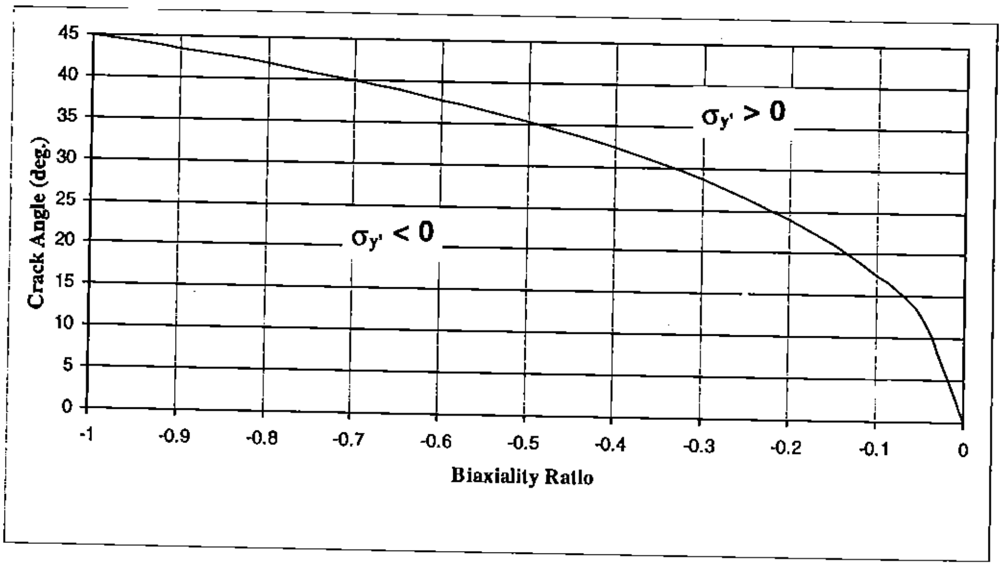

Fig.4 Crack Angle vs. Biaxiality Ratio

For example, for $\beta < 45°$ and $\alpha = -1$, $\sigma_{y'}$ becomes negative causing a closing effect rather than an opening effect on the crack surface. Therefore, there are two cases to be considered: 1) the opening mode as expressed by Equation (5) where $\sigma_{y'}$ is positive, and 2) the closing mode where $\sigma_{y'}$ is negative. The second case results in a Coulomb's friction type occurring at the crack surfaces; therefore, inducing frictional shear stress given by, $\sigma_f = \mu_f \sigma_{y'}$, that must be added to the shear stress expression to take the following form:

$$\tau_{x'y'} = (1-\alpha)\sigma \sin\beta \cos\beta + \mu_f \sigma(\sin^2\beta + \alpha\cos^2\beta) \tag{8}$$

where μ_f is the coefficient of friction. It is worth it to note that the second term in Equation (8) must be included only for $\beta < 45°$, negative values of α, and when it larger than the first term for motion to be impending. Therefore, to account the closing and frictional effect, it is postulated that negative values of $\sigma_{y'}$, similar to K_I, lead to a crack closing, that is, the crack closes on itself creating friction on the surface of the crack. This condition requires the elimination of $\sigma_{y'}$ from Equation (5) and the inclusion of the friction stress present in Equation (8). Similar observations were also made by Swedlow [4] and Ling and Woo [5]. Therefore, substituting Equation (8) and setting $\sigma_{y'} = 0$ for values of $\beta < 45°$, Equation (5) becomes:

$$\left[\frac{1-3\cos\theta}{\sin\theta}\right]\left[(1-\alpha)\tan\beta + \mu_f(\alpha + \tan^2\beta)\right] + \frac{16\lambda}{3}\left[\frac{\sin\frac{\theta}{2}}{\tan\theta}\right](1+\alpha\tan^2\beta) = 0 \tag{9}$$

Therefore, modification of Equation (5) as expressed by (9) shifts the curve downward for better correlations of experimental data; as shown in Figure 5.

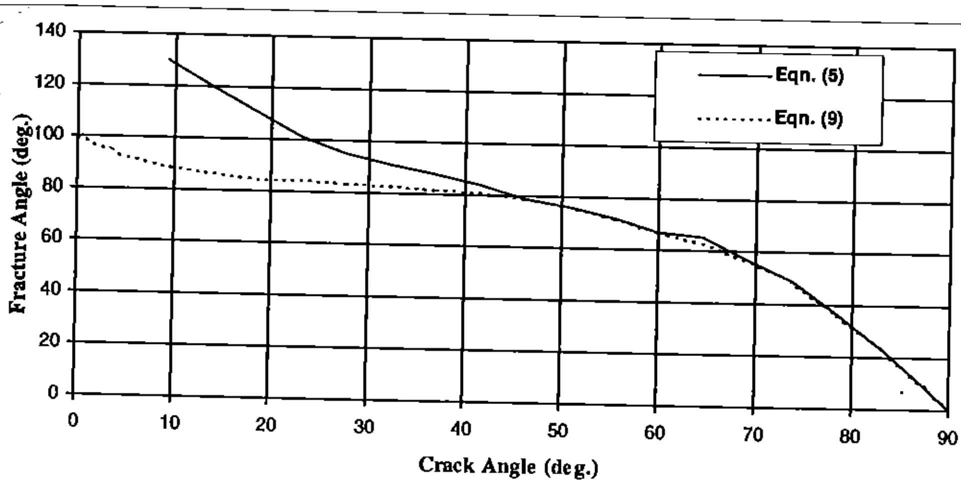

Fig.5 Fracture Angle vs. Crack Angle for Biaxiality Ratio of -1

Note that both curves deviate from one another for $\beta \leq 45°$ and merge together starting from $\beta = 45°$ to $\beta = 90°$ due to $\sigma_{y'}$ effect. The solid curve represents the theoretical results of Equation (5) where $\sigma_{y'}$ is included; whereas, the dashed curve represents the theoretical results of Equation (9) where $\sigma_{y'}$ is disregarded and a friction term is added to the shear stress in order to account for the closing and frictional effect. A comparison of the proposed model as expressed by Equation (9) to Ling and Woo [5] approach, is shown in Figure 6.

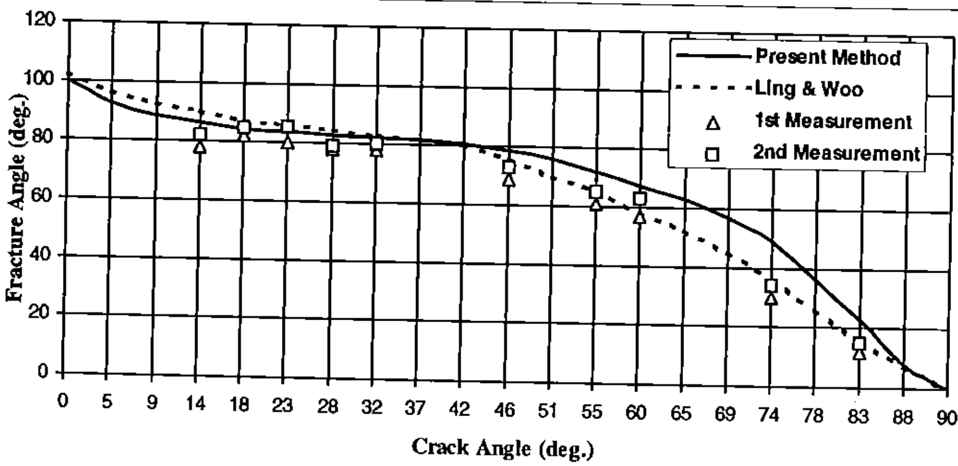

Figure 6 Comparison of the Proposed and Ling & Woo Methods With Experimental Data

It can be seen that both curves are in close agreement and correlates better experimental data especially for $\beta < 45°$ justifying the validity of the proposed model. In addition, results, obtained from Equation (6) for uniaxial loading, were compared to the maximum hoop stress fracture criterion developed by Finnie and Saith [3] and the strain energy density theory developed by Sih [6] as listed in Table 1 and showed good agreement.

It can be seen that there is a slight discrepancy between the maximum hoop stress criterion and the present method as compared to the strain energy density criterion (S-criterion). However, this discrepancy tends to narrow for values of $\beta > 30°$. The difference with the S-criterion is attributed to Poisson's ratio effect which was not accounted for when deriving the hoop stress equation.

The proposed model is also compared with the experimental results conducted by Ling and Woo on PMMA sheets with inclined crack angles as are presented in Figure 6. It can be seen that the present model correlates well with the experimental data.

Table 1 Comparison Between the Proposed Method to Other Fracture Criteria Applied to Uniaxial Loading.

β [deg.]	Present Method	S-Criterion	Percent Difference	Max. Hoop σ-Criterion	Percent Difference
10	-82.88	-76.72	7.4	-77.3	6.7
20	-76.11	-70.0	8.0	-72.7	4.5
30	-70.46	-63.3	10.2	-68.17	3.2
40	-64.56	-56.5	12.5	-63.01	2.4
50	-57.65	-49.3	14.5	-56.74	1.6
60	-48.85	-41.4	15.2	-48.53	0.06
70	-35.51	-31.7	10.7	-37.13	-4.6
80	-20.53	-18.4	9.3	-20.92	-1.8
90	0.0	0.0	0.0	0.0	0.0

Conclusion

A proposed method relating the initial fracture angle, θ_o, to the crack angle, β, was developed. This method compared very well with other fracture criteria as well as experimental results and proved to be as good as other fracture criteria. It was shown that the closing mechanism takes place for crack angles $\beta > 45°$. In addition, it was shown that the closing effect was taken into account by neglecting the applied stress normal to the crack surface which causes surface closure, therefore resisting any crack propagation caused by crack sliding only. It was also demonstrated that the closing effect was delayed by increasing the biaxiality ratio from $\alpha = -1$ which describes the tension-compression biaxial loading case, to $\alpha = 0$ which describes the uniaxial case.

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