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Damage Evolution and Compliance Variation for Variable Multiaxial Stress States

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ABSTRACT: The aim of this paper is to propose damage accumulation and fracture condition with the associated compliance variation description. The damage distribution within the representative element is specified on each physical plane by a scalar or vector value. The elastic compliance variation due to damage can be neglected when multiaxial fatigue problems are considered. However, this variation can be determined by introducing interfacial strain components due to damage and averaging over all plane orientations within the element.
A simple model is presented by assuming tension and shear fatigue modes on each plane specified in terms of tractions components and the respective failure function. To provide uniform treatment of crack initiation and propagation from stress concentrations, the non-local stress and energy condition was proposed and applied to predict both critical load value for crack initiation and also crack orientation. This simple model was next extended to predict damage accumulation and fatigue crack initiation under multiaxial cyclic loading. It is assumed that the failure function can be used in predicting damage on each physical plane by following the evolution rule associated with this function. The strength and compliance variation depend on damage accumulated on each physical plane.

Introduction

The problem of simulation of progressive damage and fracture of structural elements is of fundamental importance in assessing the operation life and reliability of elements. For multiaxial variable loading, both crack initiation and propagation are affected by the loading history and non-proportionality inducing varying traction components on each physical plane. Whereas for high-cycle fatigue problems the effect of compliance variation due to damage accumulation can be neglected, for low cycle fatigue problems, the effect of both plasticity and damage should be accounted for in the analysis of local stress and strain distribution.

The complete formulation of problem should contain constitutive equations with elastic compliance moduli dependent on damage state, the rule of damage evolution, and the condition of damage localisation along critical planes with subsequent macrocrack initiation and propagation. There are numerous damage models employing scalar, vector or tensor damage variables, cf. monographs [1,2]. However, for modelling the fatigue damage accumulation, it can be assumed that the strength parameters are affected by damage but the compliance moduli are constant. Hence, the linear elastic analysis can be used in predicting crack initiation and the linear fracture mechanics can be applied in predicting crack initiation and the linear fracture mechanics can be applied in predicting crack propagation. Such simplified formulation can essentially facilitate the solution of problems of service life prediction of machine or structural elements.

Figure 1 presents the calculation scheme of a structural element using damage or fracture mechanics concepts.

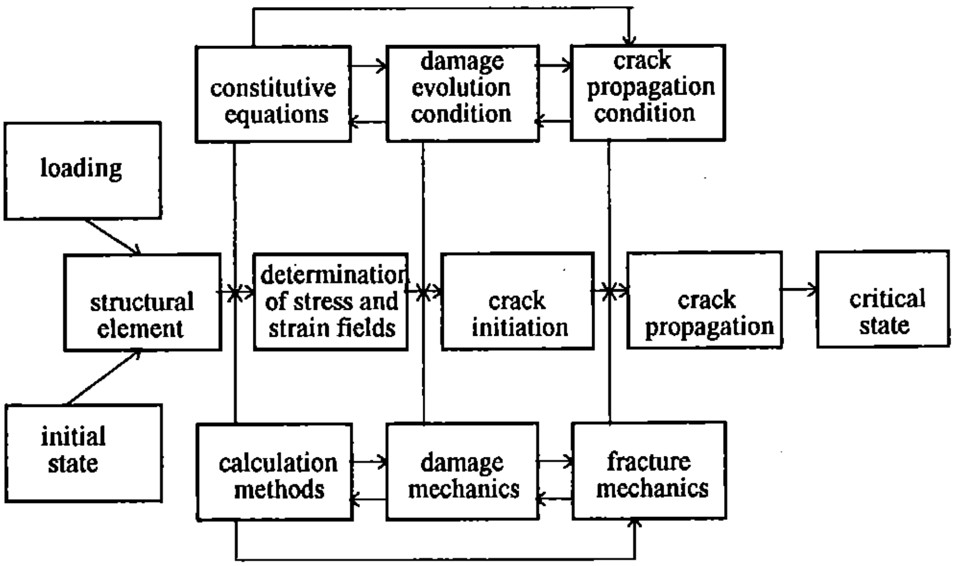

Fig. 1. Calculation scheme of failure of an element

It is seen that depending on the type of problem and material different approaches can be applied. However, in each case the crack initiation and propagation condition should be formulated in terms of local stress variation and the direction of propagation should be predicted. The effect of accumulated damage on crack initiation and propagation should be accounted for, so the theoretical model could be applied for both proportional and non-proportional loading histories. In most damage mechanics descriptions the damage is quantified by a state variable represented by scalar or tensor variable referred to a representative volume element. However, the other approach is possible where the damage is specified on each physical plane in terms of traction of strain components related to this plane. In this case the directional damage distribution is obtained and the conditions of crack initiation are related to the critical plane with the highest damage density. The present work is concerned with the local and non-local formulation of crack initiation and propagation conditions with the associated damage distribution, starting from the concept of a surface distribution of damage and its effect on material properties. The present work complements and extends the previous papers on this topic [2-10].

Non-local failure criteria

Non local elasticity, plasticity, or damage formulations have been recently proposed in literature. The yield or damage condition at a material element depends on the stress distribution in the neighbourhood of this element, cf. [11,12]. Alternatively, stress or strain gradients are assumed to affect the yield or damage condition, cf. [13].

In crack propagation problems, the non-local criteria have been formulated by averaging the normal stresses on the physical plane near the crack edge over a specified area, cf. [14,15], or by analysing stress and strain states at a specified distance from the crack tip, cf. [16,17]. The non-local approach is related to the cohesive crack models initiated by Barenblatt [18] and Dugdale [19] and next extended by numerous researchers, cf. [20-23]. In the process zone ahead of the crack tip the normal tractions are related to the displacement discontinuity and the softening response terminates by crack opening. This model provides a new dimensions, the length of the process zone which could be compared with the averaging length in the non-local model.

In the present work two non-local criteria of brittle failure are considered. First, it is assumed that the failure process is specified on the critical plane by averaging the failure function over a specified area. The orientation of the critical plane is also specified by maximising the non-local condition with respect to orientation. Alternatively, the non-local energy release condition is associated with the crack propagation over a specified distance. These conditions were next applied to the analysis of crack initiation from notches or crack propagation in the multiaxial stress field with account for both singular and regular stress states occurring in asymptotic expansion near the crack tip.

Non-local stress failure condition

Consider an arbitrary physical plane Δ and the local coordinate system (ξ_1,ξ_2,ξ_3), Fig. 2. In the global coordinate system (x_1,x_2,x_3), the origin of the local system is specified by the position vector $\mathbf{x}_0(x_{01},x_{02},x_{03})$, and the orientation of the plane Δ is specified by the normal vector $\mathbf{n}(n_1,n_2,n_3)$, where $n_i = \cos(\xi_3,x_i)$.

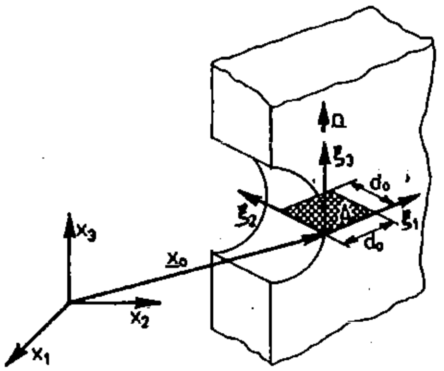

Fig. 2. Physical plane Δ with the associated local coordinate system (ξ_1,ξ_2,ξ_3), and the global reference system (x_1,x_2,x_3).

The traction vector $\Sigma = \sigma\,\mathbf{n}$ and resultant shear stress τ_n on the plane Δ is expressed in the local system as follows

$$\Sigma_i\left(\tau_{n1},\tau_{n2},\sigma_n\right) = N_{ij}\sigma_{jk}n_k\,,$$
$$\tau_n = \left[\tau_{n1}^{\ 2} + \tau_{n2}^{\ 2}\right]^{1/2}, \qquad (1)$$

where $\underline{\sigma}$ is the stress tensor and the matrix $N_{ij} = \cos(\xi_i, x_j)$ specifies the transformation to the local system. The shear tractions τ_{n1} and τ_{n2} are oriented along the axes ξ_1 and ξ_2.

Assume the crack initiation and propagation process to depend on the traction components on the physical plane. The local failure condition is assumed in the form

$$R_{f\sigma} = \max_{(\mathbf{n},\mathbf{x}_0)} R_\sigma\left(\sigma_n / \sigma_c, \tau_n / \tau_c\right) = 1, \qquad (2)$$

that is expressed in the terms of traction components. Here σ_c, τ_c denote the rupture stresses in tension and shear. The critical plane is determined by maximising function R_σ with respect to the position $\mathbf{x}_0$ and plane orientation $\mathbf{n}$. Depending the material, the local failure function R_σ may be expressed in the different form. For instance, Seweryn and Mróz [6] assumed this function as follows

$$R_\sigma\left(\frac{\sigma_n}{\sigma_c},\frac{\tau_n}{\tau_c}\right)=\begin{cases}\left[\left(\frac{\sigma_n}{\sigma_c}\right)^2+\left(\frac{\tau_n}{\tau_c}\right)^2\right]^{0.5}, & \sigma_n\geq 0,\\ \frac{1}{\tau_c}\left(\left|\tau_n\right|+\sigma_n\tan\varphi\right), & \sigma_n<0.\end{cases} \tag{3}$$

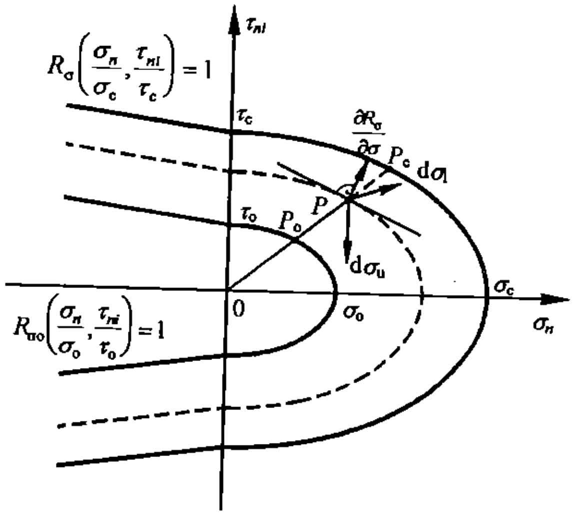

Fig. 3. The failure stress function: elliptic condition for normal tensile stress combined with the Coulomb condition for compressive normal stress

that is elliptic condition for $\sigma_n > 0$ and Coulomb condition for $\sigma_n \leq 0$.

For singular or quasi-singular stress regimes in the front of cracks, sharp notches, or interfaces, the stress function $R_\sigma(\sigma_n/\sigma_c, \tau_n/\tau_c)$ is averaged over the area $d_0 \times d_0$, Fig. 1. The non-local stress failure condition has the form, cf. [6]

$$R_{f\sigma}=\max_{(\mathbf{n},\mathbf{x}_0)}\overline{R}_\sigma\left(\sigma_n/\sigma_c,\tau_n/\tau_c\right)=$$
$$=\max_{(\mathbf{n},\mathbf{x}_0)}\left[\frac{1}{d_0^{\,2}}\int_0^{d_0}\int_0^{d_0}R_\sigma\,d\xi_1 d\xi_2\right]=1, \tag{4}$$

where $\overline{R}_\sigma$ is now a non-local failure stress function. The non-locality parameter d_0 can be specified for a brittle material by identifying the Griffith-Irwin condition for a plane crack under remote tension, thus obtaining the relation, cf. [3]

$$d_0=\frac{2}{\pi}\left(\frac{K_{Ic}}{\sigma_c}\right)^2. \tag{5}$$

However, d_0 can be specified from experiments for a combined mode fracture.

Consider, for instant a plane problem of crack initiation from notch, Fig. 4. Using the local coordinate system (r,ϑ) with the origin located at $\mathbf{x}_0(x_0,y_0)$, the non-local failure condition is expressed as follows

$$R_{f\sigma}=\max_{(\vartheta,\mathbf{x}_0)}\overline{R}_\sigma\left(\frac{\sigma_{\vartheta\vartheta}}{\sigma_c},\frac{\tau_{r\vartheta}}{\tau_c}\right)=\max_{(\vartheta,\mathbf{x}_0)}\left[\frac{1}{d_0}\int_0^{d_0}R_\sigma\left(\frac{\sigma_{\vartheta\vartheta}}{\sigma_c},\frac{\tau_{r\vartheta}}{\tau_c}\right)\mathrm{d}r\right]=1. \tag{6}$$

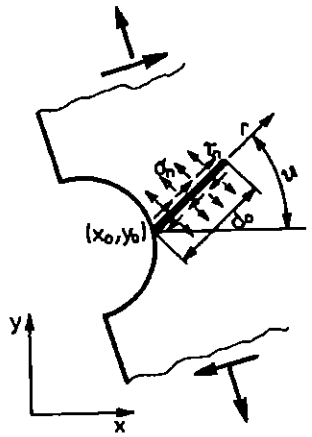

Fig. 4. The non-local damage zone emanating from the notch root.

The rupture stresses σ_c and τ_c are assumed to depend a temperature T_0 and the damage state $\omega_{n\sigma}$ on the physical plane Δ, thus

$$\begin{aligned}\sigma_c&=\sigma_c\left(T_0,\omega_{n\sigma}\right)=\sigma_{c0}\left(T_0\right)\left(1-\omega_{n\sigma}\right)^p,\\ \tau_c&=\tau_c\left(T_0,\omega_{n\sigma}\right)=\tau_{c0}\left(T_0\right)\left(1-\omega_{n\sigma}\right)^p,\end{aligned} \tag{7}$$

where σ_{c0} and τ_{c0} are the rupture stress values not affected by the accumulated on the physical plane, and p is the material parameter.

The damage state variable $\omega_{n\sigma}$ on the physical plane is thus explicitly related to the rupture stress

$$\omega_{n\sigma}=1-\left(\frac{\sigma_c}{\sigma_{c0}}\right)^{1/p}=1-\left(\frac{\tau_c}{\tau_{c0}}\right)^{1/p}, \tag{8}$$

where it is assumed that the effect of damage on the reduction of σ_c and τ_c is the same.

The accumulated damage rule should be formulated to provide the complete model formulation. In general, the total damage accumulation will be dependent on plastic or viscous strain and chemical factors (e.g. corrosion), so we can write

$$\Omega_{n\sigma}=\omega_{n\sigma}+\Phi_\sigma\left(\omega_{np},\omega_{nv},\omega_{nh}\right), \tag{9}$$

where ω_{np}, ω_{nv}, and ω_{nh} are the accumulated damage portions due to plastic and viscous strain, and chemical processes. The function Φ_σ provides the total damage measure due to irreversible processes. Now, the expression (7) is replaced by the formula

$$\sigma_c=\sigma_{c0}\left(T_0\right)\left(1-\Omega_{n\sigma}\right)^p,\qquad \tau_c=\tau_{c0}\left(T_0\right)\left(1-\Omega_{n\sigma}\right)^p. \tag{10}$$

The experimental results for notched specimens under combined tension and shear provided good agreement with theoretical predictions of limit load values and crack orientations obtained from non-local stress brittle failure criterion, cf. Seweryn et al. [10].

Non-local energy condition of brittle failure

The non-local energy criterion is based on the concept that the crack must be generated over a characteristic length d_e. The energy release is then specified not for an infinitesimal crack growth, but for the initiation over the length d_e, cf. [4].

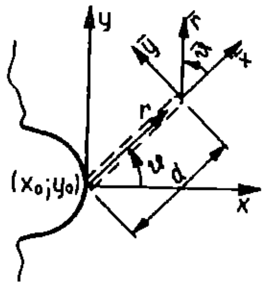

Fig. 5. Crack initiation over the distance d_e from a notch root. Polar coordinate systems (r,ϑ) and $(\bar{r},\bar{\vartheta})$.

Consider the initiation process from the notch root, Fig. 5. The averaged energy release rate due to crack opening mode equals

$$G_\sigma(\vartheta) = -\frac{1}{d_e}\int_0^{d_e} \sigma_{\vartheta\vartheta}(r,\vartheta)\,\bar{u}_\vartheta\left(r = d_e - \bar{r}, \bar{\vartheta} = \pi\right)\mathrm{d}r\,. \quad (11)$$

The averaged energy release rate due to shear mode is expressed similarly

$$G_\tau(\vartheta) = -\frac{1}{d_e}\int_0^{d_e} \tau_{r\vartheta}(r,\vartheta)\,\bar{u}_r\left(r = d_e - \bar{r}, \bar{\vartheta} = \pi\right)\mathrm{d}r\,, \quad (12)$$

where $\bar{u}_r$, $\bar{u}_\vartheta$ are the displacement discontinuities at the crack tip. The crack initiation and propagation occurs when, cf. [4]

$$\Gamma_f = \max_{(\vartheta,x_0)} \Gamma_e\left(G_\sigma, G_\tau\right) = 1, \quad (13)$$

The non-local energy condition for the combined mode fracture can be assumed in the form

$$\Gamma_e\left(\frac{G_\sigma}{G_{\sigma c}}, \frac{G_\tau}{G_{\tau c}}\right) = \frac{G_\sigma}{G_{\sigma c}} + \frac{G_\tau}{G_{\tau c}}, \quad (14)$$

where $G_{\sigma c}$ and $G_{\tau c}$ are the critical values of energy release rate in tension and shear. These values can again be assumed to depend on temperature and the accumulated damage on the physical plane.

The energy release rate for a finite length crack (much larger than d_e) in Mode I equals

$$G_{\sigma c} = \frac{K_{Ic}^{2}}{8\mu}(\kappa + 1), \tag{15}$$

where $\kappa = \begin{cases} (3-\nu)/(1+\nu) & \text{for plane stress state,} \\ 3-4\nu & \text{for plane strain state.} \end{cases}$

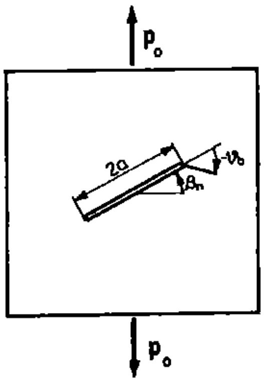

Fig. 6. A plate with the inclined crack

The averaging length can be determined by considering the edge crack of length d_e. The stress intensity factor now equals

$$\overline{K}_I = 1{,}12\ \sigma_{\vartheta\vartheta}(\vartheta = 0)\sqrt{\pi d_e}. \tag{16}$$

and the energy release rate is equal to

$$G_{\sigma c} \approx 0{,}264\frac{\kappa+1}{\mu}\sigma_c^{2} d_e. \tag{17}$$

From (15) and (17) it follows that

$$d_e \approx 0{,}474\left(\frac{K_{Ic}}{\sigma_c}\right)^{2}. \tag{18}$$

The formula (18) is similar to (5) specifying the averaging length in the non-local stress failure condition.

The effect of non-singular terms on crack propagation

Let us include the non-singular terms in the asymptotic stress representation near the crack tip. We have, cf. [24]

$$\begin{aligned}
\sigma_{rr} &= \frac{1}{4\sqrt{2\pi r}}\left[K_I\left(5\cos\frac{\vartheta}{2} - \cos\frac{3}{2}\vartheta\right) - K_{II}\left(5\sin\frac{\vartheta}{2} - 3\sin\frac{3}{2}\vartheta\right)\right] + T_\sigma\cos^2\vartheta, \\
\sigma_{\vartheta\vartheta} &= \frac{1}{4\sqrt{2\pi r}}\left[K_I\left(3\cos\frac{\vartheta}{2} + \cos\frac{3}{2}\vartheta\right) - 3K_{II}\left(\sin\frac{\vartheta}{2} + 3\sin\frac{3}{2}\vartheta\right)\right] + T_\sigma\sin^2\vartheta, \\
\tau_{r\vartheta} &= \frac{1}{4\sqrt{2\pi r}}\left[K_I\left(\sin\frac{\vartheta}{2} + \sin\frac{3}{2}\vartheta\right) + K_{II}\left(\cos\frac{\vartheta}{2} + 3\cos\frac{3}{2}\vartheta\right)\right] - T_\sigma\sin\vartheta\cos\vartheta.
\end{aligned} \tag{19}$$

where r, ϑ are the polar coordinates with the origin at the crack tip.

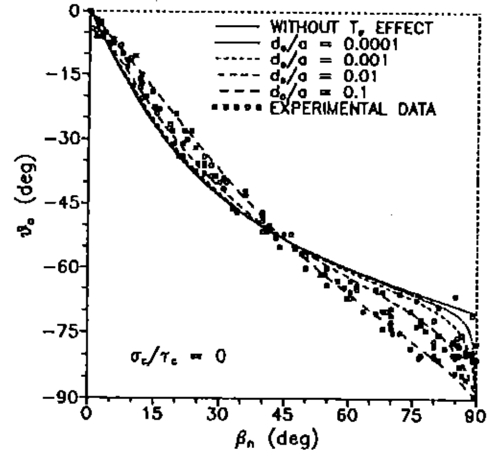

Fig. 7. The effect of non-singular stress T_σ on the crack propagation direction for an inclined central crack. The prediction was obtained from a non-local stress condition (4) for $\sigma_c/\tau_c = 0$. Experimental data from [24]

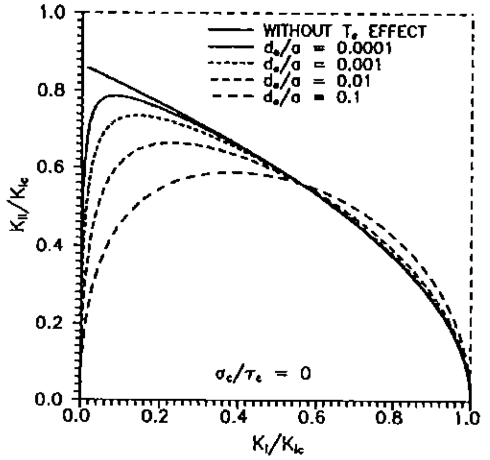

Fig. 8. The effect of non-singular stress T_σ on critical values of stress intensity factors for central crack. The prediction was obtained from a non-local stress condition (4) for $\sigma_c/\tau_c = 0$.

Consider now a wide plate with the central crack of length $2a$ inclined at the angle $\pi - \beta_n$ to the tensile stress direction, Fig. 6. Now the values of K_I, K_{II} and T_σ are

$$\begin{aligned} K_I &= p_o\sqrt{\pi a}\cos^2\beta_n, \\ K_{II} &= p_o\sqrt{\pi a}\sin\beta_n\cos\beta_n, \\ T_\sigma &= -p_o\cos 2\beta_n\ , \end{aligned} \qquad (20)$$

where p_o denotes the remote tensile stress. Experimental data related to crack propagation in a plate with an inclined central crack [24,25], indicate that the minimal value of critical loading occurs for $\beta_n = 20 \div 30°$. When $\beta_n \to 90°$, then $K_I \to 0$ and $K_{II} \to 0$, moreover $K_I/K_{II} \to 0$ which indicates the shear mode and the propagation angle $\vartheta_0 = -80° \div -70°$.

However, the experimental data indicate that this angle equals $-90°$ which is in agreement with the classical normal stress condition (tension along the crack line).

Figure 7 presents the predicted dependence of crack propagation direction ϑ_0 on the inclination angle β_n for $\sigma_c/\tau_c = 0$ and different length of the averaging zone d_0.

When $d_0 \to 0$, the effect of T_σ is negligible. Let us note that when $\beta_n = 90°$, $d_0 \neq 0$ and $\sigma_c/\tau_c = 0$, the crack propagation direction follows the line $\vartheta_0 = -90°$. The critical stress values are finite and reach minimum for $\beta_n = 20 \div 30°$.

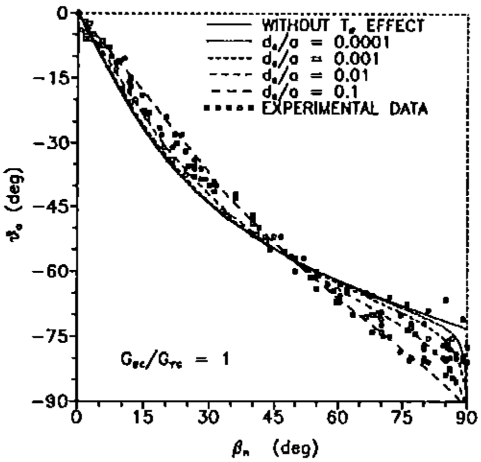

Fig. 9. The effect of non-singular stress T_σ on the crack propagation direction for an inclined central crack. The prediction was obtained from the non-local energy condition (13) for $G_{\sigma c}/G_{\tau c} = 1$. Experimental data from [24]

For the plate with an inclined crack, the effect of T_σ-stress is quite significant, especially for the inclination angle close to −90°. Hence, the limit diagram $f(K_I/K_{Ic}, K_{II}/K_{IIc}) = 1$ cannot be accurately specified. Figure 8 presents such diagram obtained from the non-local stress condition for different values of d_0/a. It is seen that the effect of T_σ-stress becomes predominant when Mode II prevails.

Figure 9 presents the predicted crack growth orientation obtained from the non-local energy condition for $G_{\sigma c}/G_{\tau c} = 1$. The prediction can be compared with the experimental data and also with the non-local stress failure condition, Fig. 7.

Damage accumulation for varying multiaxial stress state

Stress damage accumulation rule for high-cycle fatigue

Consider a variable loading programme within macroscopically elastic domain of response. Assume the damage growth on the physical plane Δ to depend on traction components and the damage state, so that

$$d\omega_{n\sigma} = d\omega_{n\sigma}\left(\Sigma, d\Sigma, \Omega_{n\sigma}\right). \tag{21}$$

It is assumed that the damage accumulation occurs for stress paths lying outside of the domain of no damage accumulation. This domain is specified by the condition

$$R_{f\sigma o} = \max_{(n,x_0)} R_{\sigma o}\left(\frac{\sigma_n}{\sigma_o}, \frac{\tau_n}{\tau_o}\right) < 1. \tag{22}$$

Here σ_o and τ_o correspond to fatigue limit stresses in pure tension and shear. The shapes of the damage initiation surface $R_{\sigma o} = 1$ and of the failure surface $R_\sigma(\sigma_n/\sigma_c, \tau_n/\tau_c) = 1$ are assumed to be the same.

The failure stresses σ_c, τ_c are now replaced by fatigue limit stress σ_o, τ_o, where $\sigma_o < \sigma_c$ and $\tau_o < \tau_c$. The fatigue damage accumulation occurs when the fatigue initiation stress function $R_{\sigma o}(\sigma_n/\sigma_o, \tau_n/\tau_o)$ on any physical plane Δ exceeds the critical value 1, thus

$$R_{f\sigma o} = \max_{(n,x_0)} R_{\sigma o}\left(\frac{\sigma_n}{\sigma_o}, \frac{\tau_n}{\tau_o}\right) > 1,$$

where $R_{f\sigma o}$ is the fatigue damage initiation factor.

Consider now the fatigue damage accumulation domain Ω_f in the plane (σ_n, τ_n) bounded by the curves $R_{\sigma o} = 1$ and $R_\sigma = 1$ corresponding to damage initiation and failure, Fig. 3. The damage accumulation occurs when the stress trajectory traverses the domain Ω_f. Introduce the non-dimensional function

$$R_{\sigma oc} = R_\sigma / R_{\sigma o} \tag{23}$$

and express the damage increment by the relation

$$\mathrm{d}\omega_{n\sigma} = \Psi_\sigma(R_\sigma)\mathrm{d}\hat{R}_\sigma, \tag{24}$$

where $\Psi_\sigma(R_\sigma)$ is the damage accumulation function and $\mathrm{d}\hat{R}_\sigma$ is expressed as follows

$$\mathrm{d}\hat{R}_\sigma = \begin{cases} \mathrm{d}R_\sigma & \text{for } \mathrm{d}R_\sigma > 0 \text{ and } R_{\sigma o} > 1 \\ 0 & \text{for } \mathrm{d}R_\sigma \le 0 \text{ or } R_{\sigma o} \le 1 \end{cases}, \tag{25}$$

where

$$\mathrm{d}R_\sigma = \frac{\partial R_\sigma}{\partial(\sigma_n/\sigma_c)}\mathrm{d}\left(\frac{\sigma_n}{\sigma_c}\right) + \frac{\partial R_\sigma}{\partial(\tau_n/\tau_c)}\mathrm{d}\left(\frac{\tau_n}{\tau_c}\right) = \frac{\partial R_\sigma}{\partial\sigma_n}\mathrm{d}\sigma_n + \frac{\partial R_\sigma}{\partial\tau_n}\mathrm{d}\tau_n + \frac{\partial R_\sigma}{\partial\Omega_{n\sigma}}\mathrm{d}\Omega_{n\sigma}, \tag{26}$$

and the last term accounts for the effect of accumulated damage on function R_σ. The damage growth occurs for the stress path penetrating in the exterior of the domain bounded by the curve $R_\sigma = \text{const.}$, Fig. 3. Alternatively, the loading - unloading conditions can be specified as follows

$$\mathrm{d}\hat{R}_\sigma = \frac{\partial R_\sigma}{\partial\sigma_n}\mathrm{d}\hat{\sigma}_n + \frac{\partial R_\sigma}{\partial\tau_{n1}}\mathrm{d}\hat{\tau}_{n1} + \frac{\partial R_\sigma}{\partial\tau_{n2}}\mathrm{d}\hat{\tau}_{n2} + \frac{\partial R_\sigma}{\partial\Omega_{n\sigma}}\mathrm{d}\Omega_{n\sigma}, \tag{27}$$

where the increment of traction stresses $\mathrm{d}\hat{\sigma}_n, \mathrm{d}\hat{\tau}_{ni}$ (i = 1, 2) are given by the relation

$$\begin{aligned} \mathrm{d}\hat{\sigma}_n &= \mathrm{d}\sigma_n \quad \text{for} \quad \mathrm{d}\sigma_n \geq 0 \quad \text{and} \quad \sigma_n \geq 0, \\ \mathrm{d}\hat{\sigma}_n &= 0 \quad \text{for} \quad \mathrm{d}\sigma_n < 0 \quad \text{or} \quad \sigma_n < 0, \end{aligned} \tag{28a}$$

and

$$\begin{aligned} \mathrm{d}\hat{\tau}_{ni} &= \mathrm{d}\tau_{ni} \quad \text{for} \quad \tau_{ni}\mathrm{d}\tau_{ni} \geq 0, \\ \mathrm{d}\hat{\tau}_{ni} &= 0 \quad \text{for} \quad \tau_{ni}\mathrm{d}\tau_{ni} < 0. \end{aligned} \tag{28b}$$

In the traction plane (σ_n, τ_{ni}), the combined damage growth domain, shear damage domain, tension damage domain and unloading domain are specified by (27) - (28).

For large stress gradients a non-local increment of damage stress function can be applied, thus

$$\mathrm{d}\bar{\hat{R}}_\sigma = \frac{1}{d_0^{\,2}} \int_0^{d_0}\int_0^{d_0} \mathrm{d}\hat{R}_\sigma \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 . \tag{29}$$

The damage accumulation function $\Psi_\sigma(R_\sigma)$ was assumed by Seweryn and Mróz [5-9] in the form

$$\Psi_\sigma(R_\sigma) = A_\sigma \left(\frac{R_\sigma - R_{\sigma oc}}{1 - R_{\sigma oc}} \right)^{n_\sigma} \frac{1}{1 - R_{\sigma oc}}, \tag{30}$$

where n_σ and A_σ are the material parameters.

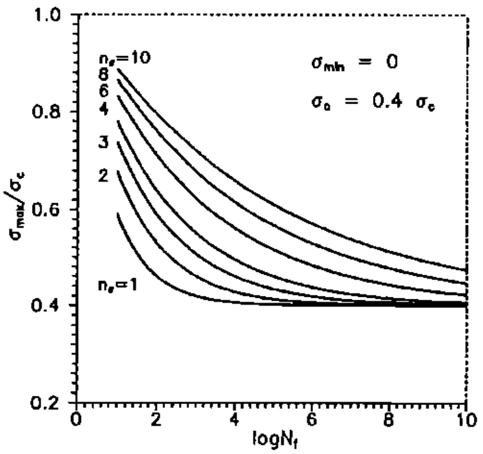

Fig. 10. Dependence of the maximal stress on number of cycles N_f at the crack initiation

The present model was applied to predict the number of cycles corresponding to crack initiation. For the tensile cyclic loading test between $\sigma = \sigma_{min} = 0$ and $\sigma = \sigma_{max}$, the following equation results

$$\log N_f = -(n_\sigma + 1)\log\left(\frac{\sigma_{max} - \sigma_o}{\sigma_c - \sigma_o} \right) \tag{31}$$

Figure 10 presents the fatigue initiation curves and their sensitivity with respect to the exponent n_σ occurring in (30).

The model can also be applied for the

stage of crack growth. For a tensile crack growth, the following growth rule was derived by Seweryn and Mróz [7]

$$\frac{da}{dN}=C\left[\left(\frac{K_{max}-K_{th}}{K_c-K_{th}}\right)^{n_\sigma+1}-\left(\frac{K_{min}-K_{th}}{K_c-K_{th}}\right)^{n_\sigma+1}\right], \tag{32}$$

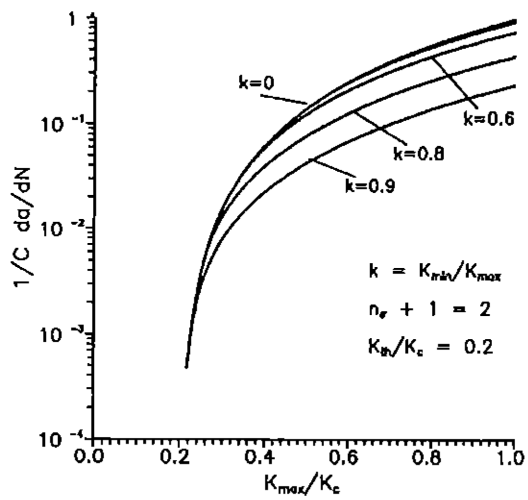

Fig. 11. Dependence of fatigue crack growth on the maximal value of stress intensity factors specified from the equation (32)

where C and n_σ are material constant, K_{max}, K_{min} are the maximal and minimal values of the stress intensity factors in each cycle and K_{th} is the threshold value of K. When $K_{min} < K_{th}$, then we set $K_{min} = K_{th}$. Let us note that the crack growth rule has a similar form to some previously formulated rules of fatigue crack growth, such as those proposed by Paris and Erdogan [26], Donahue et al. [27], Klesnil and Lukas [28], Cooke and Beevers [29] who available experimental data.

Experimental verification of non-local fatigue crack initiation condition for notched specimens under biaxial cyclic loading was presented in the paper by Molski and Seweryn [31].

Damage accumulation rule for low cycle fatigue

When macroscopic plastic deformations occur, the associated damage accumulation on the physical plane Δ can be related to the plastic strain components ε_n and γ_n, where ε_n is the normal strain and γ_n is the shear strain. The growth of plastic damage ω_{np} can be expressed similarly to (24), namely

$$\mathrm{d}\omega_{n\mathrm{p}} = \mathrm{d}\omega_{n\mathrm{p}}\left(\underline{E}, \mathrm{d}\underline{E}^{\mathrm{p}}, \omega_{n\sigma}, \omega_{n\mathrm{p}}\right). \tag{33}$$

where $\mathrm{E} = [\varepsilon_n, \gamma_n]^{\mathrm{T}}$, $\mathrm{dE^p} = [\,\mathrm{d}\hat{\varepsilon}_n^{\mathrm{p}}, \mathrm{d}\hat{\gamma}_n^{\mathrm{p}}\,]^{\mathrm{T}}$. The ductile damage function $R_{\mathrm{p}} = R_{\mathrm{p}}(\varepsilon_n/\varepsilon_{\mathrm{c}}, \gamma_n/\gamma_{\mathrm{o}})$ specifies the initiation of ductile failure, thus

$$R_{\mathrm{fp}} = \max_{(\mathbf{n}, \mathbf{x}_0)} R_{\mathrm{p}}\left(\varepsilon_n / \varepsilon_{\mathrm{c}}, \gamma_n / \gamma_{\mathrm{c}}\right) = 1. \tag{34}$$

Maximising function R_{p} with respect to $\mathbf{n}$ and $\mathbf{x}_0$, both orientation and location of ductile crack can be specified.

For monotonic loading we have

$$\varepsilon_n = \varepsilon_n^{\mathrm{e}} + \varepsilon_n^{\mathrm{p}}, \qquad \gamma_n = \gamma_n^{\mathrm{e}} + \gamma_n^{\mathrm{p}},$$

and for variable loading, the equivalent plastic strain $\hat{\varepsilon}_n^{\mathrm{p}}$, $\hat{\gamma}_n^{\mathrm{p}}$ can be used, thus

$$\varepsilon_n = \varepsilon_n^{\mathrm{e}} + \hat{\varepsilon}_n^{\mathrm{p}}, \qquad \gamma_n = \gamma_n^{\mathrm{e}} + \hat{\gamma}_n^{\mathrm{p}}, \tag{35}$$

where $\hat{\varepsilon}_n^{\mathrm{p}}$, $\hat{\gamma}_n^{\mathrm{p}}$ are specified for each strain reversal, thus

$$\hat{\varepsilon}_n^{\mathrm{p}} = \begin{cases} \int_0^t \dot{\varepsilon}_n^{\mathrm{p}} \mathrm{d}t & \text{for} \quad \dot{\varepsilon}_n^{\mathrm{p}} \hat{\varepsilon}_n^{\mathrm{p}} \geq 0 \\ 0 & \text{for} \quad \dot{\varepsilon}_n^{\mathrm{p}} \hat{\varepsilon}_n^{\mathrm{p}} < 0 \end{cases} \tag{36}$$

and analogously $\hat{\gamma}_n^{\mathrm{p}}$.

Similarly as for stress damage, the critical strain values ε_{c} and γ_{c} are functions of temperature T_{o} and accumulated damage, thus

$$\varepsilon_{\mathrm{c}} = \varepsilon_{\mathrm{c}}\left(T_{\mathrm{o}}, \omega_{n\sigma}, \omega_{n\mathrm{p}}, \omega_{n\mathrm{v}}, \omega_{n\mathrm{h}}\right). \quad \gamma_{\mathrm{c}} = \gamma_{\mathrm{c}}\left(T_{\mathrm{o}}, \omega_{n\sigma}, \omega_{n\mathrm{p}}, \omega_{n\mathrm{v}}, \omega_{n\mathrm{h}}\right)$$

and the specific forms are similar to (7):

$$\varepsilon_{\mathrm{c}} = \varepsilon_{\mathrm{co}}\left(1 - \Omega_{n\mathrm{p}}\right)^{\bar{p}}, \qquad \gamma_{\mathrm{c}} = \gamma_{\mathrm{co}}\left(1 - \Omega_{n\mathrm{p}}\right)^{\bar{p}}, \tag{37}$$

where $\Omega_{n\mathrm{p}}$ is the equivalent damage on the physical plane

$$\Omega_{n\mathrm{p}} = \omega_{n\mathrm{p}} + \Phi_{\mathrm{p}}\left(\omega_{n\sigma}, \omega_{n\mathrm{v}}, \omega_{n\mathrm{h}}\right). \tag{38}$$

The growth rule of $\omega_{n\mathrm{p}}$ is expressed as follows

$$d\omega_{np} = \Psi_p\left(R_p\right)d\hat{R}_p, \tag{39}$$

where the strain damage accumulation function Ψ_p takes the form

$$\Psi_p\left(R_p\right) = A_p R_p^{\,n_p} \tag{40}$$

and n_p, A_p are the material constants. Similarly as previously, the increment $d\hat{R}_p$ is specified in terms of plastic strain and damage increments

$$d\hat{R}_p = \frac{\partial R_p}{\partial \hat{\varepsilon}_n^p} d\hat{\varepsilon}_n^p + \frac{\partial R_p}{\partial \hat{\gamma}_n^p} d\hat{\gamma}_n^p + \frac{\partial R_p}{\partial \Omega_{np}} d\Omega_{np}. \tag{41}$$

More complex macroplastic damage accumulation condition could be formulated by following the multisurface plasticity hardening rules proposed by Mróz [32,33].

Consider, for example, the case of uniaxial tension of a cylindrical specimen. The deformation process can be divided into three stages. In the first stage the elastic deformation occurs and the associated damage $\omega_{n\sigma}$ develops. In the second stage both plastic and stress damage components develop. In the third stage, the strain localisation may occur with the localised damage growth inducing ductile rupture. Figure 12 presents the damage evolution and variation of stress and strain.

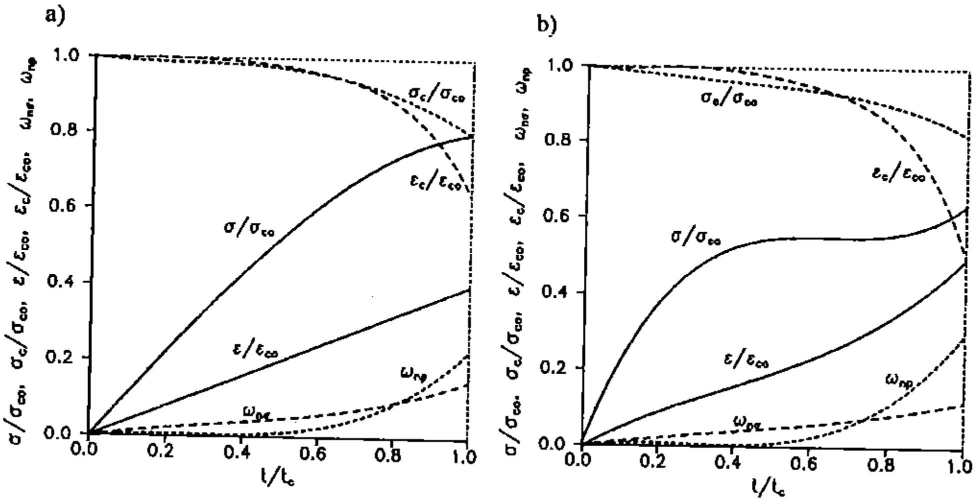

Fig. 12. The interaction of loading with the rupture stress and strain evolution due to damage growth for a) brittle b) ductile failure

For uniaxial fatigue problems, the accumulated plastic strains are used in the damage model equations. The plastic damage ω_{np} can be expressed in the form

$$\omega_{np} = N_f C_1\left(\Delta\varepsilon^{n_1} - \left(\Delta\varepsilon^{e}\right)^{n_1}\right) = N_f C_1\left(\left(\Delta\varepsilon^{e} + \Delta\varepsilon^{p}\right)^{n_1} - \left(\Delta\varepsilon^{e}\right)^{n_1}\right), \tag{42}$$

where $\Delta\varepsilon$ is the total strain amplitude and $\Delta\varepsilon^{e}$, $\Delta\varepsilon^{p}$ are the elastic and plastic strain amplitudes, n_1 is material constants ($n_1 = n_p + 1$), and $C_1 = C_1(\vartheta)$ depends on the orientation of physical plane.

For high cycle fatigue, the plastic strain can be neglected and we have (for $\sigma_o = 0$):

$$\omega_{n\sigma} = N_f C_2 \Delta\sigma^{n_2} = N_f C_3\left(\Delta\varepsilon^{e}\right)^{n_2}, \tag{43}$$

where $\Delta\sigma$ is the stress amplitude, $n_2 = n_\sigma + 1$; $C_2 = C_2(\vartheta)$, $C_3 = C_3(\vartheta)$.

For the intermediate case, the coupling process between $\omega_{n\sigma}$ and ω_{n_p} occurs, so the equivalent damage measures can be assumed as follows

$$\begin{aligned} \Omega_{np} &= \omega_{np} + \Phi_p\left(\omega_{n\sigma}\right) = \omega_{np} + A_{p\sigma}\omega_{n\sigma}, \\ \Omega_{n\sigma} &= \omega_{n\sigma} + \Phi_\sigma\left(\omega_{np}\right) = \omega_{n\sigma} + A_{\sigma p}\omega_{np}, \end{aligned} \tag{44}$$

where $A_{p\sigma}$ and $A_{\sigma p}$ are the coupling terms.

The crack initiation can be expressed as follows

$$\max\left(\Omega_{n\sigma}, \Omega_{np}\right) = 1. \tag{45}$$

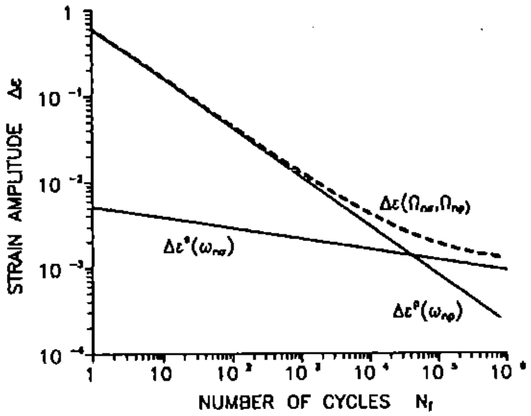

Fig. 13. The dependence of the strain amplitude on the number of cycles N_f corresponding to crack initiation

Figure 13 presents the dependence of strain amplitude on the number of cycles N_f at the crack initiation specified by the stress damage rule $\Delta\varepsilon^{e}(\omega_{n\sigma})$, plastic strain damage rule $\Delta\varepsilon^{p}(\omega_{np})$ and by the combined rule $\Delta\varepsilon(\Omega_{n\sigma}, \Omega_{np})$.

The resemblance to the Manson - Coffin model [32,33] is seen from the diagram.

Compliance of the damaged material

The effect of accumulated damage ω_n on the material compliance can be specified by assuming the normal and shear strain components $\varepsilon_n^{\mathrm{d}}$, γ_n^{d} due to damage to occur at each physical plane (cf. Seweryn and Mróz [9]), so that

$$\varepsilon_n^{\mathrm{d}} = \varepsilon_n^{\mathrm{d}}(\sigma_n, \omega_n), \qquad \gamma_n^{\mathrm{d}} = \gamma_n^{\mathrm{d}}(\tau_n, \omega_n). \tag{46}$$

Consider for simplicity, only the damage strain $\varepsilon_n^{\mathrm{d}}$ related the tensile stress in the form

$$\varepsilon_n^{\mathrm{d}} = A_{\mathrm{n}} \left(\frac{\omega_n}{1-\omega_n} \right)^q \langle \sigma_n \rangle, \tag{47}$$

where A_{n}, q are the material parameters and $\langle \sigma_n \rangle = \sigma_n$ for $\sigma_n > 0$ and $\langle \sigma_n \rangle = 0$ for $\sigma_n \leq 0$. The related compliance variation is

$$C_n^{\mathrm{d}} = A_{\mathrm{n}} \left(\frac{\omega_n}{1-\omega_n} \right)^q H(\sigma_n), \tag{48}$$

where $H(\sigma_n) = \langle \sigma_n \rangle / \sigma_n$.

The increment of damage strain is

$$\dot{\varepsilon}_n^{\mathrm{d}} = \frac{\partial \varepsilon_n^{\mathrm{d}}}{\partial \sigma_n} \dot{\sigma}_n + \frac{\partial \varepsilon_n^{\mathrm{d}}}{\partial \omega_n} \dot{\omega}_n = A_{\mathrm{n}} \left(\frac{\omega_n}{1-\omega_n} \right)^q H(\sigma_n) \left(\dot{\sigma}_n + \frac{q \sigma_n \dot{\omega}_n}{\omega_n (1-\omega_n)} \right). \tag{49}$$

Introduce the scalar $C_{\mathrm{o}}^{\mathrm{d}*}$, and tensors $C_{ij}^{\mathrm{d}*}$, $C_{ijkl}^{\mathrm{d}*}$, specified by the relations

$$C_{\mathrm{o}}^{\mathrm{d}*} = \int_{4\pi} C_n^{\mathrm{d}}(\mathbf{n})\, \mathrm{d}\Omega, \quad C_{ij}^{\mathrm{d}*} = \int_{4\pi} C_n^{\mathrm{d}}(\mathbf{n})\, n_i n_j \, \mathrm{d}\Omega, \quad C_{ijkl}^{\mathrm{d}*} = \int_{4\pi} C_n^{\mathrm{d}}(\mathbf{n})\, n_i n_j n_k n_l \, \mathrm{d}\Omega, \tag{50}$$

where the integration is performed for all plane orientations. The compliance variation due to accumulated damage can now be expressed as follows (cf. Lubarda and Krajcinowic [36,37])

$$C_{ijkl}^{\mathrm{d}} = \frac{315}{32\pi} \left(C_{ijkl}^{\mathrm{d}*} - \frac{2}{3} A_{ijkl} + \frac{C_{\mathrm{o}}^{\mathrm{d}*}}{21} I_{ijkl} \right), \tag{51}$$

where

$$A_{ijkl} = \frac{1}{6}\left(\delta_{ij} C_{kl}^{d*} + \delta_{kl} C_{ij}^{d*} + \delta_{ik} C_{jl}^{d*} + \delta_{il} C_{jk}^{d*} + \delta_{jk} C_{il}^{d*} + \delta_{jl} C_{ik}^{d*}\right),$$
$$I_{ijkl} = \frac{1}{3}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{52}$$

For plane problems, we have respectively

$$C_o^{d*} = \int_{2\pi} C_n^d(\vartheta)\, d\vartheta, \quad C_{ij}^{d*} = \int_{2\pi} C_n^d(\vartheta)\, n_i n_j\, d\vartheta, \quad C_{ijkl}^{d*} = \int_{2\pi} C_n^d(\vartheta)\, n_i n_j n_k n_l\, d\vartheta \tag{53}$$

and the compliance variation due to damage is

$$C_{ijkl}^{d} = \frac{8}{\pi} C_{ijkl}^{d*} - \frac{6}{\pi} A_{ijkl} + \frac{C_o^{d*}}{2\pi} I_{ijkl}. \tag{54}$$

Let us apply the present model to a tensile loading test. The stress damage function is assumed to depend only on normal stress ($\sigma_c/\tau_c \to 0$). The increment $d\hat{R}_\sigma$ on the physical plane can be expressed by the relation

$$d\hat{R}_\sigma = \frac{\langle d\sigma_n \rangle}{\sigma_{co}\left(1-\omega_n\right)^p} + \frac{p\langle \sigma_n \rangle\, d\omega_n}{\sigma_{co}\left(1-\omega_n\right)^{p+1}}. \tag{55}$$

and the damage increment is given by

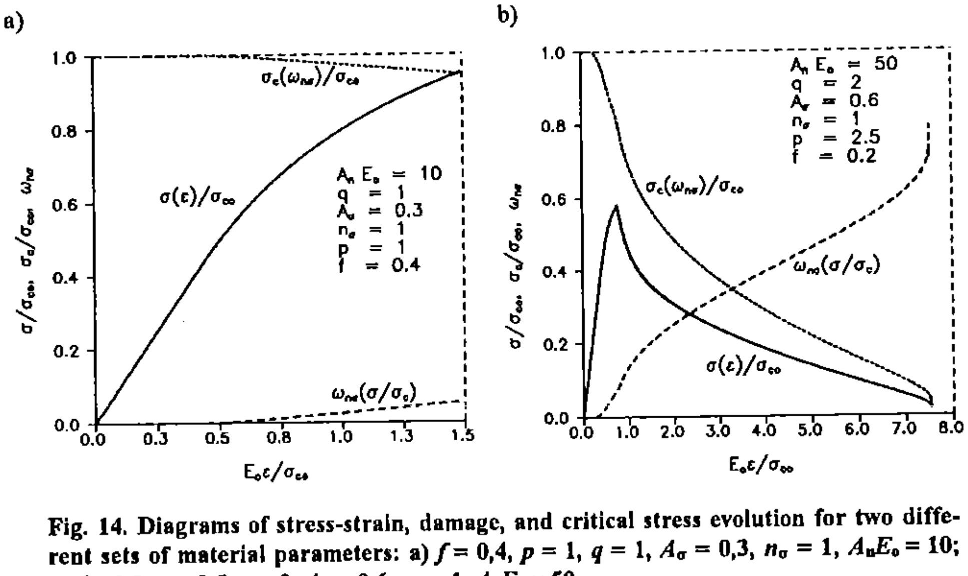

Fig. 14. Diagrams of stress-strain, damage, and critical stress evolution for two different sets of material parameters: a) $f = 0{,}4$, $p = 1$, $q = 1$, $A_\sigma = 0{,}3$, $n_\sigma = 1$, $A_n E_o = 10$; b) $f = 0{,}2$, $p = 2{,}5$, $q = 2$, $A_\sigma = 0{,}6$, $n_\sigma = 1$, $A_n E_o = 50$

$$\mathrm{d}\omega_n = \frac{\Psi_\sigma \langle \mathrm{d}\sigma_n \rangle}{\sigma_c \left(1 - \Psi_\sigma \frac{p\langle \sigma_n \rangle}{\sigma_c (1 - \omega_n)} \right)}, \tag{56}$$

where

$$\Psi_\sigma = A_\sigma \left(\frac{\langle \sigma_n - \sigma_o \rangle}{\sigma_c (1 - f)} \right)^{n_\sigma} \frac{1}{1 - f}, \qquad f = \frac{\sigma_o}{\sigma_c} = \frac{\tau_o}{\tau_c} = \mathrm{const} \tag{57}$$

The resulting stress-strain relation with progressing damage is

$$\sigma = \frac{\varepsilon E_o}{1 + A_n E_o \left(\frac{\omega_n}{1 - \omega_n} \right)^q H(\sigma)}, \tag{58}$$

where E_o is the initial Young modulus.

Figure 14 presents the damage evolution and stress-strain relation for two sets of material parameters. Both stable and unstable responses are predicted by the model.

Concluding remarks

In the present paper, the damage was associated with the critical plane and the rules of damage evolution on each physical plane were specified. The damage distribution on all physical planes can thus be determined. The critical plane corresponds to maximal damage value. The crack initiation and propagation conditions follow from the critical plane concept. Both regular and singular stress regimes can be treated by using the non-local damage evolution rule proposed in the paper. When macroscopic plastic deformation occurs, the damage is decomposed into stress and strain dependent damage components $\omega_{n\sigma}$ and ω_{np}. They affect the strength or critical strain parameters on the physical plane, and also material compliance. The variation of material compliance can be specified if the damage distribution on all physical planes is known.

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