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A Numerical Method to Study the Full Elastic-Plastic Stress Field of a Tensile Crack

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ABSTRACT: In this paper, an energy method is proposed to study the full elastic-plastic field of a tensile crack. A statically admissible stress field is established by developing an expansion of the stress function in separable form. The unknown parameters are determined by minimizing the complementary energy of the structure. The results obtained by this method are compared with the fine finite element analysis in previous literature. This method shows some advantages for studying the elastic-plastic cracks, in the capacity to find out an algebraic expression of the stress field connecting the plastic near-tip field to the elastic far field, in the highly accurate representation of the full elastic-plastic field surrounding the crack-tip and in the economical calculation, etc.. This method is also used to evaluate the amplitudes in the analytical asymptotic expansion. The results show that even the four-term analytical solution can not describe the near-tip for some loading cases. More higher order terms are needed. In these cases, the present solutions may be more advantageous in engineering applications.

Introduction

The asymptotic behaviour near the crack tip has been well understood since the publication of Hutchinson [1] and Rice and Rosengren [2]. They showed that the asymptotic fields close to the crack-tip, known as HRR solution, could be described only by a single parameter for material showing a power-law hardening stress-strain response. This parameter was a path-independent line integral (J-integral) developed by Rice [3]. Afterwards, the numerical and experimental works of Begley and Landes [4] have demonstrated that the HRR solution was a good characterization of the near-crack-tip stress and strain fields. But in certain cases, the original HRR singular field significantly deviates from the fine results of the finite element solution as demonstrated by Shih *et al.* [5].

To improve the accuracy of the single parameter solution, multi-parameter solutions have been described by several authors. Li and Wang [6] suggested to use a second parameter, which is the magnitude of the second term in the expansion of the HRR solution, together with the J-integral in the fracture criterion. O'Down and Shih [7] suggested to use a hydrostatic stress, Q, as the higher order term for engineering applications. Xia *et al.* [8] developed the higher order asymptotic fields which were taken out to 4 or 5 terms. Similar studies were carried out by Yang *et al.* [9]. They found that two parameters, J and A_2, control at least three terms in the analytical asymptotic expansion and suggested to use them to characterize the crack-tip stress field. Recent study by Wei and Wang [10] showed that the J-Q solution is suitable for most specimen geometries from small-scale yielding to large-scale yielding. However, for cracked bend bar specimens from medium-scale yielding to large-scale yielding, the J-Q two-parameter solution will deviate gradually from finite-element solution.

It is reasonable to think that the development of higher-order terms in an elastic-plastic crack tip solution of HRR-type will give a more accurate asymptotic stress field. However, the amplitudes in the analytical expansion can not be self-determined except that of the first term under small-scale yielding conditions. The convergence of such an approach on a full elastic-plastic solution has not been explicitly demonstrated so far. Consequently, the complete elastic-plastic solutions for mode I crack are highly needed. Edmunds and Willis [11-13] have developed a matched asymptotic expansion method to study the full elastic-plastic stress fields for the mode III and mode I cracks in elastic-plastic materials. The linear elastic fracture mechanics was extended into the non-linear regime. The advantage of this approach is that the full elastic-plastic stress field can be described by an expansion of finite terms even though the HRR singularity did not appear in their results.

In this paper, we try to describe the full stress field of an elastic-plastic crack by an approximate energy method. By proceeding a few simple matrix operations, a statically admissible stress field has been established to connect the plastic zone near the crack-tip to the elastic far field. The undetermined parameters have been adjusted by minimizing the complementary energy of the structure. The results obtained by using this method for values of hardening exponents n=3 and 10 have been verified by the finite element calculations. The comparison shows that the present method gives satisfactory accuracy not only in the near crack-tip zone, but also in the far field region.

This method can be used to determine the amplitudes in asymptotic expansion which are usually estimated by matching the truncated asymptotic solutions with the results of the finite-element modeling at a few points specially chosen. It is clear that this "point matching" technique needs a rationalization, because the values obtained by applying this technique to a truncated solution are only an approximation of the "true" values.

In this paper, the amplitudes in asymptotic expansion are examined by using the present variational method. It is shown that this method can provide quite accurate values of the amplitudes in analytical solutions. The "point matching" technique is also examined. It is observed that this technique could fail in certain cases. The results show that the amplitudes calculated by these two methods may be quite different. In some cases. even the four-term analytical expansion can not reproduce correctly the stress field in the vicinity of the crack. In these cases, certain approximate solutions, such as the *J-Q* theory or the full field solution proposed in the present work, may be more advantageous due to their accuracy and simplicity in engineering applications.

General Formulations

We suppose that far enough from the crack-tip, the elastic singularity dominates the stress and the strain distribution. The elastic far stress field (the outer field) under mode I loading can be described, for example, by the first k+1 terms of Williams' [14] expansion of the stress function, namely:

$$\phi_e = r^{3/2}\tilde{\varphi}_{e0}(\theta) + r^2\tilde{\varphi}_{e1}(\theta)+\ldots+r^{(k+3)/2}\tilde{\varphi}_{ek}(\theta) \tag{1}$$

where r and θ are the polar coordinates whose origin is the crack-tip and $\tilde{\varphi}_{e0}(\theta)$, $\tilde{\varphi}_{e1}(\theta)$, ... are angular dependent functions which have been determined analytically by Williams [14] . In the plastic zone, the constitutive behaviour of the continuum can be described by the Ramberg-Osgood's stress-strain relation, namely:

$$\varepsilon_{ij} = \frac{1+\nu}{E}s_{ij} + \frac{1-2\nu}{3E}\sigma_{kk}\delta_{ij} + \frac{3\alpha}{2E}\left(\frac{\sigma_e}{\sigma_0}\right)^{n-1} s_{ij} \tag{2}$$

where ε_{ij} are the strain components, s_{ij} are the deviatoric stress components, E is the elastic modulus, ν is Poisson's ratio, δ_{ij} is the Kronecker delta, α is a material constant, n is the hardening exponent, σ_0 is the yielding stress, and σ_e is the Mises equivalent effective stress defined as follows:

$$\sigma_c = (3/2\, s_{ij}\, s_{ij})^{1/2} \tag{3}$$

The first two terms of (2) represent the elastic deformation, and the third term represents the plastic deformation, which can be neglected for rather large distances from the crack-tip. From (2), Hutchinson [1] and Rice and Rosengren [2] showed that the stress field near the crack tip (the inner field) can be derived from the following expansion of the stress function:

$$\phi = r^{s_0}\tilde{\varphi}_0(\theta) + r^{s_1}\tilde{\varphi}_1(\theta) + \ldots + r^{s_p}\tilde{\varphi}_p(\theta) + \ldots \tag{4}$$

where $\tilde{\varphi}_0(\theta)$, $\tilde{\varphi}_1(\theta)$, ... are angular dependent functions. s_0 , s_1 ... are undetermined exponents and $s_0 < s_1 < \ldots < s_p < \ldots$. For more convenience, the first $p+1$ terms in (4) are taken into consideration in this work.

Between the inner stress field and the outer one, there must exist an intermediate stress field connecting them continuously. To connect these two stress fields, a statically admissible stress field has been established in this work, the undetermined parameter can be adjusted by minimizing the complementary energy of the structure.

We know that the first term of (4) corresponds to the HRR solution. The angular distributions of the higher order terms in (4) can also be developed by analytical approaches (In general, the magnitudes of higher order terms in (4) can not be self-determined by these approaches). If the first $m+1$ terms of ϕ are supposed known , the stress function ϕ can be written in two parts:

$$\phi = [r^{s_0}\tilde{\varphi}_0(\theta) + r^{s_1}\tilde{\varphi}_1(\theta) + \ldots + r^{s_m}\tilde{\varphi}_m(\theta)] + [r^{s_{m+1}}\tilde{\varphi}_{m+1}(\theta) + r^{s_{m+2}}\tilde{\varphi}_{m+2}(\theta) + \ldots + r^{s_p}\tilde{\varphi}_p(\theta)]$$

$$= \{r_1\}^t\{\tilde{\psi}_1\} + \{r_2\}^t\{\tilde{\psi}_2\} \tag{5}$$

where $\{r_1\} = \{\, r^{s_0}\ r^{s_1} \ldots\ r^{s_m}\}^t$, $\{\tilde{\psi}_1\} = \{\, \tilde{\varphi}_0(\theta)\ \tilde{\varphi}_1(\theta) \ldots \tilde{\varphi}_m(\theta)\,\}^t$

and $\{r_2\} = \{\, r^{s_{m+1}}\ r^{s_{m+2}} \ldots\ r^{s_p}\}^t$, $\{\tilde{\psi}_2\} = \{\, \tilde{\varphi}_{m+1}(\theta)\ \tilde{\varphi}_{m+2}(\theta) \ldots \tilde{\varphi}_p(\theta)\,\}^t$

If $\tilde{\varphi}_0(\theta)$, $\tilde{\varphi}_1(\theta)$, ... and $\tilde{\varphi}_{e0}(\theta)$, $\tilde{\varphi}_{e1}(\theta)$, ... are supposed to be approached by polynomial functions of order q,

$$\tilde{\varphi}_i(\theta) = \sum_{j=0}^{q} a_{ij}\theta^j \tag{6}$$

$$\tilde{\varphi}_{ei}(\theta) = \sum_{j=0}^{q} b_{ij}\theta^j$$

equation (5) can be rewritten as follows:

$$\phi = \{r_1\}^t[A_1]\{\Theta\} + \{r_2\}^t[A_2]\{\Theta\} \tag{7}$$

where $[A_1]$ is a constant matrix which can be determined by polynomial approach (6) and $[A_1]=[a_{ij}]$ $(i=0,1,\ldots,m;\ j=0,1,\ldots,q)$;

$[A_2]$ is a unknown constant matrix and $[A_2]=[a_{ij}]$ $(i=m+1,m+2,\ldots,p;\ j=0,1,\ldots,q)$;

$\{\Theta\} = \{\ 1\ \ \theta\ \ \theta^2\ \ldots\ \theta^q\}^t$.

Similarly to (7), the stress function of the elastic far field (1) can also be developed as follows:

$$\phi_e = \{r_3\}^t[B]\{\Theta\} \tag{8}$$

where $\{r_3\} = \{\ r^{3/2}\ \ r^2\ \ldots\ r^{(k+3)/2}\}^t$;

$[B]$ is a coefficient matrix and can be determined in polynomial approach (6) and $[B]=[b_{ij}]$ $(i=0,1,\ldots,k;\ j=0,1,\ldots,q)$.

The unknown constant matrix $[A_2]$ can be obtained by considering the continuity between the inner and the outer fields at large distances from crack tip. One can suppose that at a circle $r=R$ far enough from the crack tip, the inner elastic-plastic field coincides with the outer elastic one. Under this condition, the stress function of the inner field ϕ and its partial derivatives with respect to r and θ of different orders must equal to, at $r=R$, their corresponding quantities of the outer field. It is clear that C^2 is the minimum order to ensure the continuity of all the stress components. If more computation accuracy is required, higher order continuities are needed. In fact, if the stress functions are assumed to be developed in polynomial form as in (7) and (8), one can easily demonstrate that all the continuity conditions above-mentioned can be satisfied only if the corresponding partial derivatives with respect to r are continuous, i.e.

$$\varphi_e(R)=\varphi(R) \qquad \frac{\partial\varphi_e(R)}{\partial r}=\frac{\partial\varphi(R)}{\partial r} \qquad \frac{\partial^2\varphi_e(R)}{\partial r^2}=\frac{\partial^2\varphi(R)}{\partial r^2} \qquad \ldots\ldots (9)$$

Since the dimension of the unknown constant matrix $[A_2]$ is of $(p\text{-}m,q+1)$, a continuity of order $p\text{-}m\text{-}1$ is sufficient to determine $[A_2]$. This condition leads to the following relation by introducing (7), (8) into (9):

$$[\boldsymbol{R}_2][A_2]\{\Theta\}= [\boldsymbol{R}_3][\boldsymbol{B}]\{\Theta\} - [\boldsymbol{R}_1][A_1]\{\Theta\} \qquad (10)$$

where :

$$[\boldsymbol{R}_1] = \left\{1 \quad \frac{d}{dr} \quad \frac{d^2}{dr^2} \cdots \frac{d^{(p-m-1)}}{dr^{(p-m-1)}}\right\}^t \{\boldsymbol{r}_1(r=R)\}^t$$

$$[\boldsymbol{R}_2] = \left\{1 \quad \frac{d}{dr} \quad \frac{d^2}{dr^2} \cdots \frac{d^{(p-m-1)}}{dr^{(p-m-1)}}\right\}^t \{\boldsymbol{r}_2(r=R)\}^t$$

$$[\boldsymbol{R}_3] = \left\{1 \quad \frac{d}{dr} \quad \frac{d^2}{dr^2} \cdots \frac{d^{(p-m-1)}}{dr^{(p-m-1)}}\right\}^t \{\boldsymbol{r}_3(r=R)\}^t$$

The unknown coefficient matrix $[A_2]$ can easily by found be resolving (10):

$$[A_2]=[\boldsymbol{R}_2]^{-1}([\boldsymbol{R}_3][\boldsymbol{B}]- [\boldsymbol{R}_1][A_1]) \qquad (11)$$

By substituting (11) into (7), the stress function of the inner field can be found. The stress field derived from the stress function constructed this way satisfy the equilibrium equations and can connect the near crack-tip field to the elastic far field, whatever the exponents s_i. So these field can be considered as statically admissible.

To determine the $p\text{-}m$ exponents s_i , the theorem of the minimum complementary energy is used. The complementary energy U_c of a structure which obeys the stress-strain relationship defined by (2) for either plane stress or plane strain is:

$$U_C = \int_{\Omega} \left[\frac{1+\nu}{3E}\sigma_e^{\,2} + \frac{1-2\nu}{6E}\sigma_{kk}^{\,2} + \frac{\alpha\sigma_0^{\,2}}{E(n+1)}\left(\frac{\sigma_e}{\sigma_0}\right)^{n+1} \right] d\Omega \qquad (12)$$

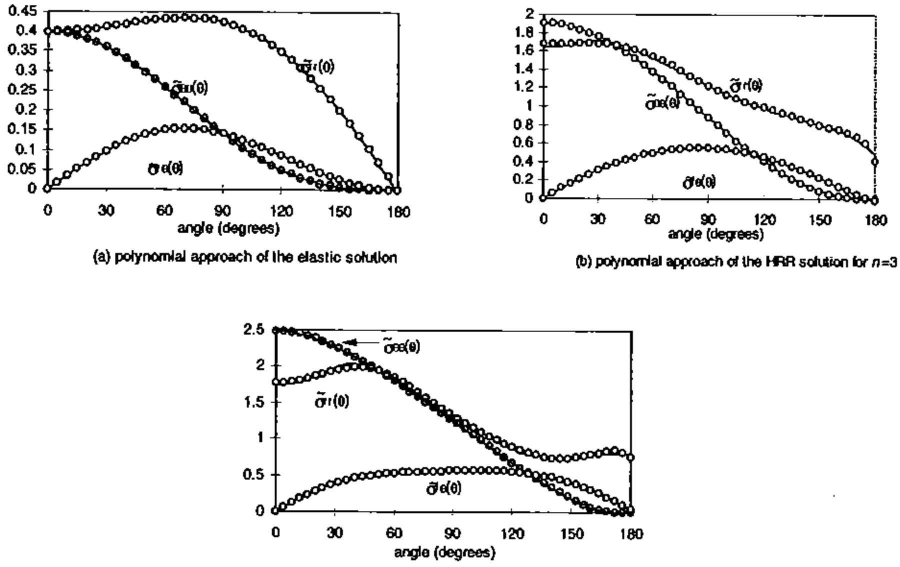
The integration is calculated in the circle area $\Omega \in r < R$. The variation of the statically admissible stress can be completed by varying the exponents $(s_{m+1}, s_{m+2} \dots s_p)$. The problem then consists in finding out the parameters $(s_{m+1}, s_{m+2} \dots s_p)$ so that the variation of the complementary energy of the structure becomes zero, or in optimizing the parameters $(s_{m+1}, s_{m+2} \dots s_p)$ so that the complementary energy becomes stationary and minimum The minimization problem can be solved by using numerical methods which will be discussed in the next section.

Numerical Procedure

The boundary conditions of the statically admissible stress field thus determined are the elastic solution for the far field and the HRR expansion for the near crack-tip field as indicated before. The angular distributions of these stress fields are needed to be approached by polynomial functions. This approximation can be highly accurate and does not present any numerical difficulty.

Figure 1 illustrates the comparison between the analytic angular stress distributions and their polynomial approximations. One can note that the accuracy of this numerical approach is quite good whether for the elastic solution or for the HRR solution.

Gauss-Legendre numerical integration method is used to calculate the complementary energy in the area $r<R$ according to (12). The minimization of the functional (12) can be resolved numerically. The numerical procedure used in this work is the downhill simplex method. This method is easy to use in finding the minimum of a function of more than one independent variables. With this method, the exponent vector $\{s_{m+1}, s_{m+2} \dots s_p\}$ can be found. The convergence is guaranteed by this method for any initial simplex guessed. However, it is possible that a local minimum point is found. So it is preferable to proceed several downhills with different initial simplexes. The optimal point can be chosen after survey of the results.



(a) polynomial approach of the elastic solution

(b) polynomial approach of the HRR solution for n=3

(c) polynomial approach of the HRR solution for n=10

Fig. 1: Polynomial approaches of the analytical stress fields. The continuous lines represent the analytical solutions; the open circles represent the approaching functions.

By substituting the parameters obtained ($s_{m+1}, s_{m+2} \dots s_p$) into (11), the unknown coefficient matrix $[A_2]$ can easily be calculated. The stress field connecting the far elastic field is then completely defined.

Full Stress Fields

In order to verify the energy method developed in the preceding sections, we carry out detailed calculations of the full stress field under small-scale yielding conditions. We consider the near-tip region of a mode I plane strain crack in a homogeneous elastic-plastic material. The near-tip field is supposed dominated by the HRR solution. The stress corresponding to the first term of expansion (1) is applied to a remote circular boundary. The values of the material properties used in the calculation are E/σ_0=300, ν=0.2, and α=0.1. The computations are carried out for two different values of the hardening exponent n=3 and 10. The numerical results obtained by the present method have been compared with those of the fine finite element modelling. We compare our results essentially with those obtained by Sharma and Aravas [14].

To study the influence of the radius of the integration area on the accuracy of the method, different values of R are chosen. Table 1 lists the values of the exponents s_1, s_2 ... s_p obtained for n=10 and for different values of R from $45J/\sigma_0$ to $600J/\sigma_0$. The analytical values of s_1, s_2 ... s_p obtained in [8] and [9] are also listed for comparison. Table 1 shows that with a small R, few exponents s_i are needed to describe the near-tip stress field, the exponents of higher orders become very large and have nearly any influence on the near-tip stress distribution . However, if it is of interest to study the stress field far away from the crack-tip, a larger value of R can be chosen. In which case more exponents s_i must be calculated to ensure satisfactory accuracy. detailed calculations demonstrate that for a large range of values of R, the results calculated by the present method are very stable for both near-tip and far stress fields.

Figure 2 shows the variation of the stress components for n=3 and 10 along the radial lines θ=41.3°. The finite element results obtained by Sharma and Aravas [14], when using the same material parameters, are plotted for comparison. The HRR solution and the elastic solution are also illustrated in the figure. Figure 3 shows the angular variation of the stress components for n=10 at three different radial distances, namely $r/(J/\sigma_0)$ = 0.8, 2, and 5. The HRR solution and the results of the finite element solution are also illustrated.

Figures 2-3 show that the results obtained by the present energy method fit well with the results of the fine finite element analysis in the whole region considered, whether for the near-tip asymptotic field or for the far elastic field. This demonstrates the high accuracy of the energy method applied in the elastic-plastic crack problem.

If the present method is able to give the same accuracy as that of the fine finite element analysis, its main advantage is that an algebraic expression can be obtained to describe the stress field of the whole elastic-plastic region around the crack-tip. It makes it all the easier to select the criteria which characterize the behaviour of the fracture process.

It is observed that the exponents obtained by the present energy method are much higher than those deduced from the analytical approaches [8,9]. This means that the expansion in present work converges more quickly to the full solution than the analytical expansion without loose of accuracy. By taking the HRR solution as the known near-tip boundary condition, only 4 to 7 terms are sufficient to describe accurately the full elastic-plastic field. Another advantage of this method is that all the amplitudes can be self-determined. This is not the case in the analytical approach.

Table 1: Exponents for different radii of integration area (n=10)

$R/(J/\sigma_0)$	s_0	s_1	s_2	s_3	s_4	s_5	s_6	
45	1.909	2.41	4.5	80	–	–	–	p = 3
78	1.909	2.58	3.22	70	–	–	–	p = 3
130	1.909	2.59	2.70	60	–	–	–	p = 3
220	1.909	2.72	2.78	2.91	–	–	–	p = 3
	1.909	2.66	2.87	3.00	50	–	–	p = 4
360	1.909	2.70	2.79	3.19	3.41	–	–	p = 4
	1.909	2.62	2.98	3.14	3.24	60	–	p = 5
600	1.909	2.76	2.83	2.95	3.07	4.50	–	p = 5
	1.909	2.71	2.87	2.97	3.14	5.14	30	p = 6
analytical	1.909	2.06977	2.2304	2.2695	–	–	–	

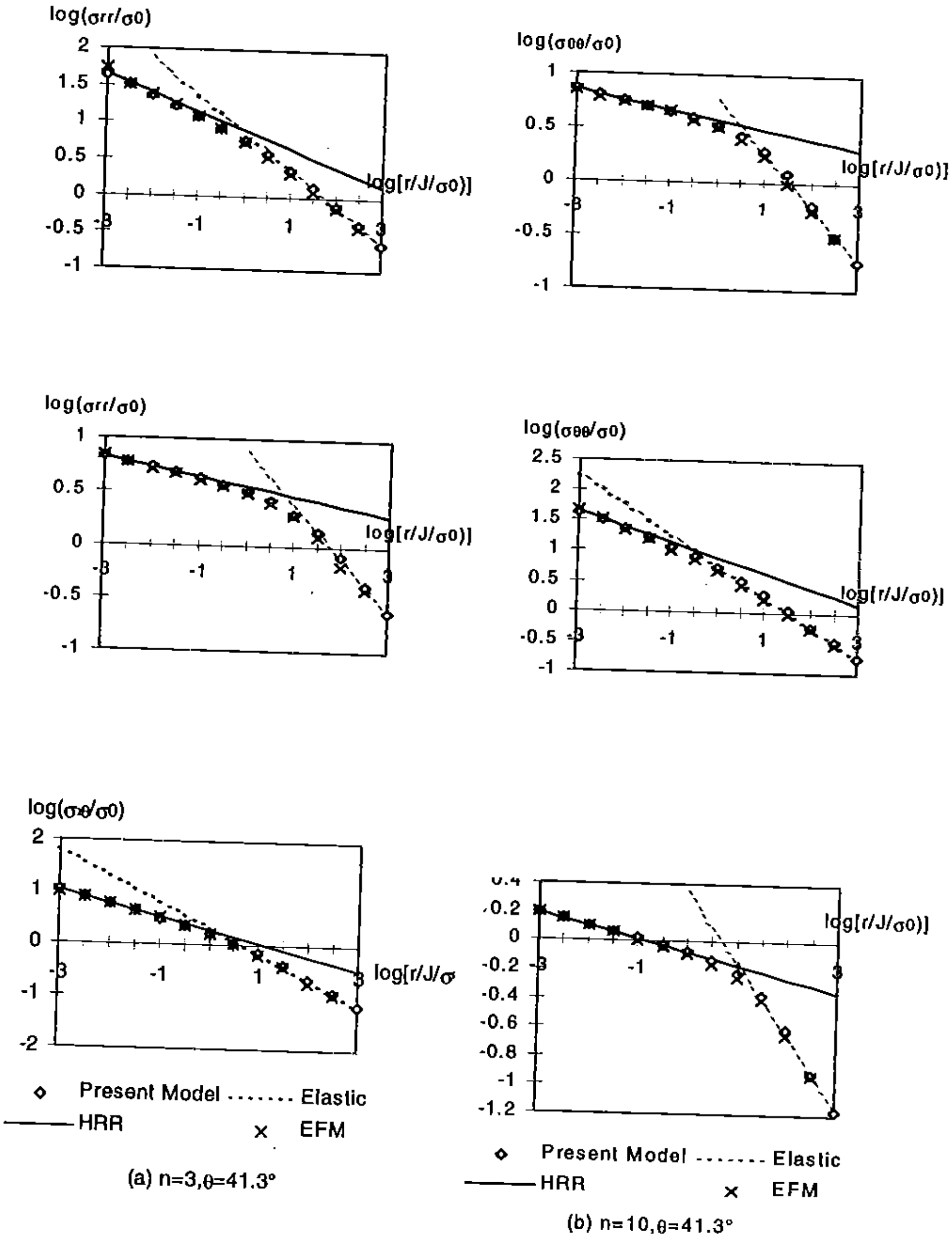


Fig. 2: Radial variation of the normalized stress components for n=3 and 10.

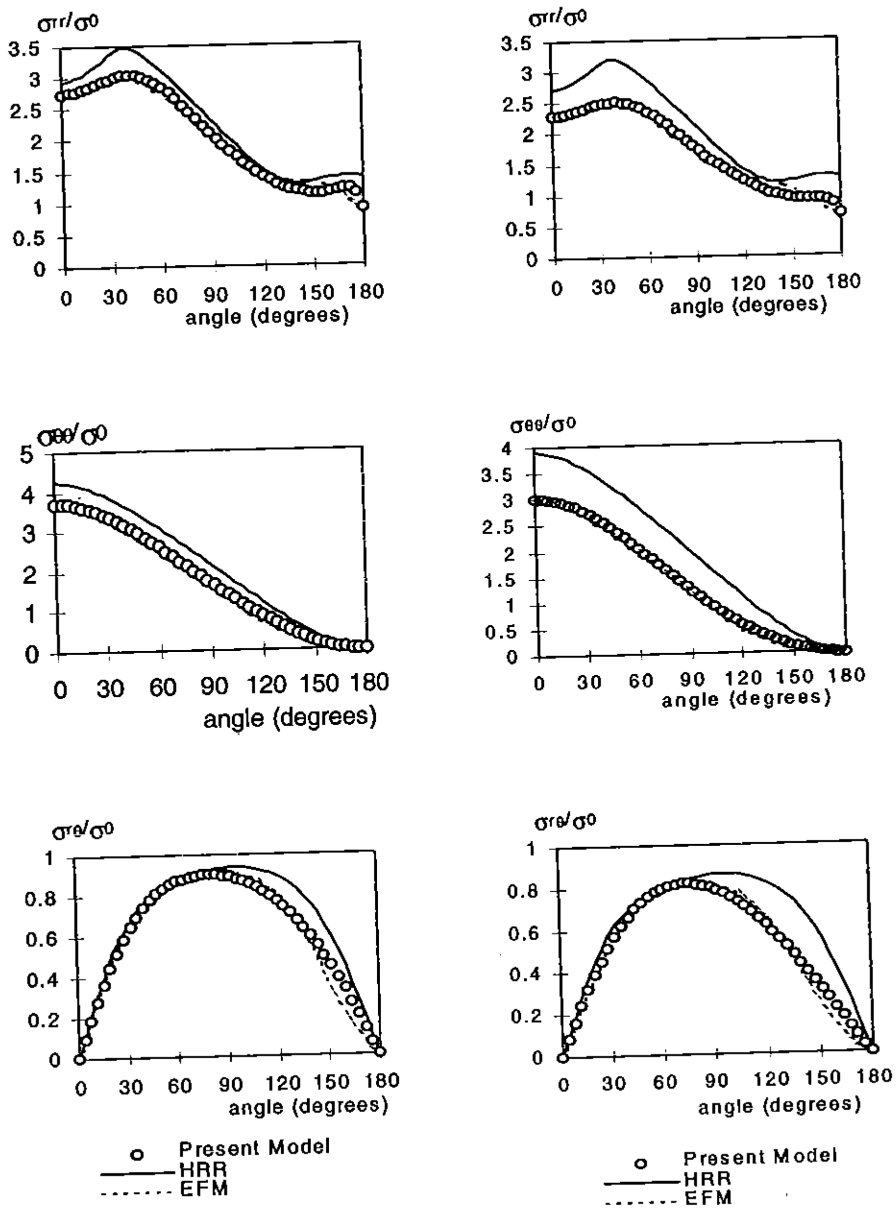


Fig. 3: Angular variation of the normalized stress components for n=10

Magnitudes of the Asymptotic Expansion

The magnitudes in the analytical asymptotic expansion can be calculated by using this method without difficulty. It is just needed to take the magnitudes as unkown constants in the minimization of the complementary energy.

In this work, we carry out detailed calculations to determine the magnitudes in asymptotic expansion. The values of the material properties used are: $E/\sigma_0=300$, $\nu=0.3$ and $\alpha=1$. The computation are carried out for two hardening exponents $n=5$ and 10.

The outer field: Consider a boundary layer formulation in which the remote tractions are given by the first two terms of the linear elastic solution. They are the well known K stress and T stress defined by Rice [16]:

$$\varphi_e = \frac{K}{\sqrt{\pi}} r^{3/2}\left(\cos\frac{1}{2}\theta + \frac{1}{3}\cos\frac{3}{2}\theta\right) + \frac{T}{4} r^2\left(1-\cos 2\theta\right) \qquad (13)$$

In the present work, the loadings applied at the circle boundary include the following combinations:

$K = \frac{\sigma_0\sqrt{\pi}}{2}$, $T=0$; $K = \frac{\sigma_0\sqrt{\pi}}{2}$, $T=-0.56\sigma_0$; and $K = \frac{\sigma_0\sqrt{\pi}}{2}$, $T=0.34\sigma_0$

The inner field: The solution of Xia *et al.* is used in this work. The first four terms of the asymptotic expansion of the stress function for $n= 5$ and 10 can be written as follows:

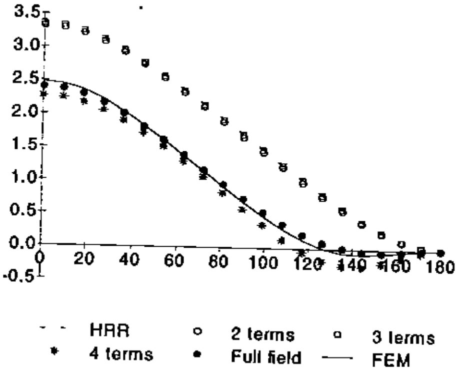
$$\phi = K_1 r^{s1}\phi_1(\theta) + K_2 r^{s2}\phi_2(\theta) + K_2\,(K_2/K_1)\, r^{s3}\phi_3(\theta) + K_4 r^{s4}\phi_4(\theta) \qquad (14)$$

Finite-element model: Each calculation of such an approach are verified by a finite-element modeling. The corresponding geometries, loadings and material properties in energy minimization approach are reconstructed by finite elements. A general-purpose finite-element program developed by Centre d'Energy Atomique de France, named CASTEM 2000, is used in the present study.

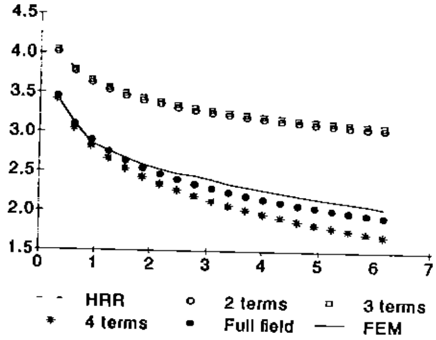
First, the four-term solution of Xia et *al.* is included in the statically admissible full stress field for $n=10$. The three independent amplitudes J, K_2 and K_4 are calculated by the minimization of the complementary energy of the structure. Knowing these amplitudes, the full stress field and the differently truncated asymptotic stress fields are obtained. Fig.4 shows these stress fields in the vicinity of the crack tip, The full field solution and the finite-element solution are also illustrated for comparison. In order to save space, only the stress components $\sigma_{\theta\theta}$ are plotted, circumferentially at distances r~2 to $4J/\sigma_0$ and radially ahead of the crack tip. The results of the other stress components will give same conclusions. From Fig 4, one can note that the full solutions approach best the finite-element solutions. However, the two-term and three-terms solutions deviate significantly from the full stress solution. In general, the four-term solutions agrees well the finite-element solutions in the near-tip region $r<5J/\sigma_0$.

The same approach is made for n=5. The components $\sigma_{\theta\theta}$ of the different stress fields are shown in Fig.5. Fig.5 shows some different results from those illustrated in Fig.4. From Fig 5, one can note that under certain loadings, none of the above-mentioned truncated asymptotic fields approaches the finite-element solution in the vicinity of the crack tip. This means that in certain cases, even the four-term asymptotic solution can not reproduce adequately the stress field near the crack tip. More terms in analytical asymptotic solution are needed.

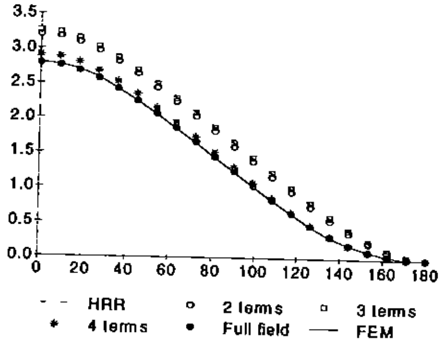
For comparison, the "point matching" technique is also used in this work. The matched point was chosen at ($r=2J/\sigma_0$,θ=0). The stress component $\sigma_{\theta\theta}$ is used to determine K_2 in the three-term asymptotic solution. The stress components σ_{rr} and $\sigma_{\theta\theta}$ are used to determine K_2 and K_4 in the four-term asymptotic solution. However, it is found that this technique is not always applicable when matching the results of the finite-element analysis. For example, the truncated four-term solution for n=10 of Xia *et al.* can match the stress field of the finite-element modeling successfully, however, this technique fails with the truncated three-term solution of the same authors. On the contrary, the point matching works well with the three-term solution of Yang *et al.*, but does not with their four-term solution. This "anomaly" leads to two questions: first, are the truncated asymptotic solutions unique? second, does the "point matching" technique can be used to determine the amplitudes of the truncated asymptotic solution in general cases? Anyway, clarifications will be necessary for the analytical asymptotic solutions to explain such an anomaly.



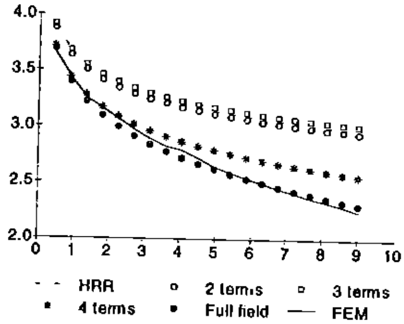
a: $\sigma_{\theta\theta}$ at r=2.46J/σ_0, T= -0.56σ_0



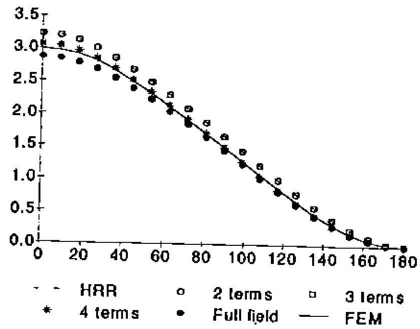
b: $\sigma_{\theta\theta}$ ahead of crack-tip, $x=r/(J/\sigma_0)$, T=-0.56σ_0



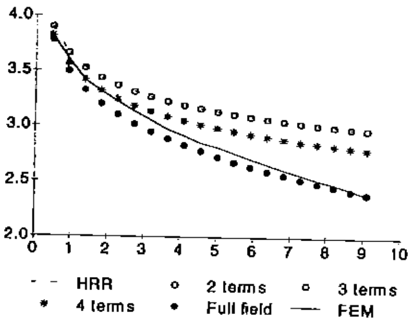
c: $\sigma_{\theta\theta}$ at r=3.6J/σ_0, T= 0



d: $\sigma_{\theta\theta}$ ahead of crack-tip, $x=r/(J/\sigma_0)$, T= 0

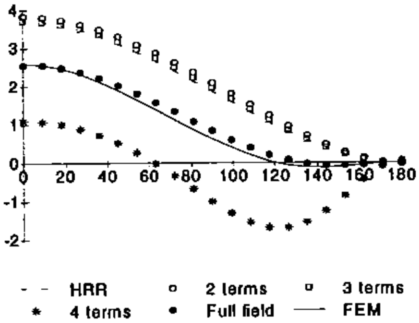


e: $\sigma_{\theta\theta}$ at r=3.63J/σ_0, T= 0.34σ_0

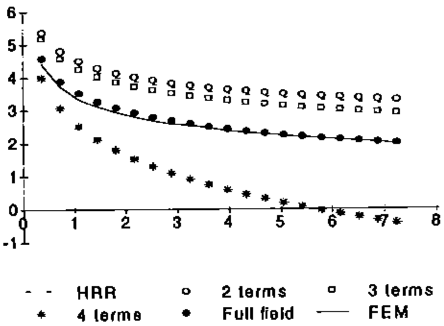


f: $\sigma_{\theta\theta}$ ahead of crack-tip, $x=r/(J/\sigma_0)$, T= 0.34σ_0

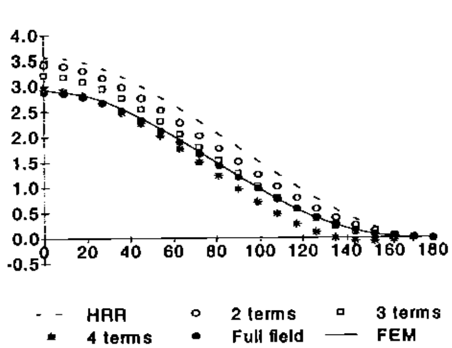
Fig.4: Comparison of truncated asymptotic solutions with FEM and full field solutions. n=10.



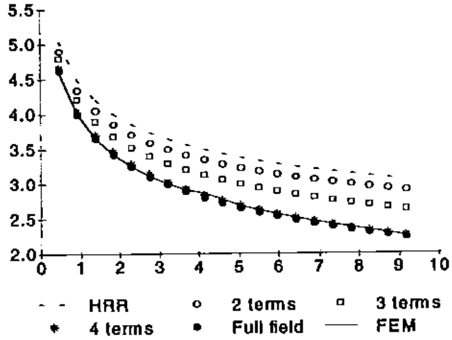
a: $\sigma_{\theta\theta}$ at r=2.89J/σ_0, T= -0.56σ_0



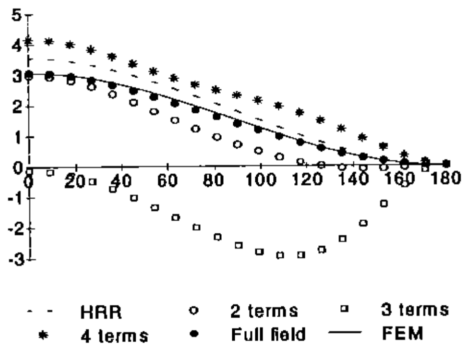
b: $\sigma_{\theta\theta}$ ahead of crack-tip, x=r/(J/σ_0), T=-0.56σ_0



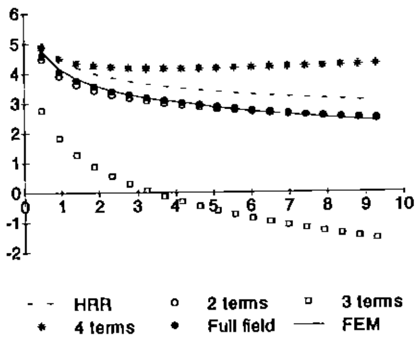
c: $\sigma_{\theta\theta}$ at r=3.67J/σ_0, T= 0



d: $\sigma_{\theta\theta}$ ahead of crack-tip,x=r/(J/σ_0), T= 0



e: $\sigma_{\theta\theta}$ at r=3.72J/σ_0, T= 0.34σ_0



f: $\sigma_{\theta\theta}$ ahead of crack-tip,x=r/(J/σ_0), T= 0.34σ_0

Fig.5: Comparison of truncated asymptotic solutions with FEM and full field solutions. n=5.

In this work, two methods, the virtual crack extension method implemented in the finite-element program CASTEM 2000 and the present variational method, are used to calculate the value of the J-integral. Based on the three-term and four-term truncated analytical solutions of Xia et al. , both the point matching method and the variational method are used to calculate the parameter K_2 and K_4. These approaches allow to estimate the accuracy of the point matching method and the variational method.

Table 2: Amplitudes in analytical asymptotic expansion

loading conditions	J *VCE	J *VM	K_2 *PM3	K_2 * PM4	K_2 VM	K_4 PM4	K_4 VM
(n=10)							
$T=-.56\sigma_0$	2.742	2.98	*–	0.061	0.013	1.81	1.33
T=0	1.853	2.03	–	0.024	0.017	0.51	0.43
$T=.34\sigma_0$	1.837	2.02	–	-0.006	0.002	0.22	0.17
(n=5)							
$T=-.56\sigma_0$	2.307	2.16	0.179	0.22	-0.073	-0.63	4.06
T=0	1.817	1.89	0.12	0.08	0.22	0.60	-2.48
$T=.34\sigma_0$	1.795	1.85	0.094	0.018	0.35	0.71	-6.68

*VCE: virtual crack extension method; *VM: variational method; *PM3: point matching method with 3-term analytical solution;*PM4: point matching method with 4-term analytical solution; *–: point matching method fails

Table 2 provides the values of J, K_2 and K_4 calculated for n=5 and 10 and for several boundary layer loadings. Table 2 shows that the values of J-integral calculated by the variational method agree well with those obtained by the virtual crack extension method in finite-element analysis. The differences between J obtained by using these two methods are within 10% for n=10, and 5% for n=5. These represent about 1% of uncertainty in the amplitude of the first term. The amplitude K_2 can not be obtained for n=10 by matching the three-term solution with the result of FEM. For n=5, the values of K_2 obtained by using the three-term solution are somewhat different from those obtained by using the four-term solution. This means that the amplitudes of the lower-order terms determined by the point matching method may change when adding some higher-order terms. One can also observe that the values of K_2 and K_4 provided by the point matching method agree approximately with those obtained by the variational method for n=10. However, for n=5, the amplitudes obtained by these two methods are quite different.

It is to note that the values obtained by applying the point matching technique to a truncated asymptotic solution are only an approximation of the "true" values. By using this method, the amplitudes of lower-order terms change when adding or removing a higher-order term in the expansion. The variational method allows to bypass this difficulty. In this method, the amplitudes in analytical solution are optimized by the minimization of the complementary energy of the structure. Therefore, this method can be considered as a "full field matching " method. It is evident that the "true" amplitudes make the complementary energy of the structure minimum. Moreover, following factors show that the variational method can provide quite correct values of the amplitudes:

1: the values of the J-integral , which is the amplitude of the first term in the analytical solution, obtained by using the variational method agree well with those obtained by using other existing techniques;

2: the modifications of higher order fields do not influence much the results of the amplitudes of lower order fields;

3: the stress fields calculated by using this method agree well with those existing in the structures under all loading cases and for all power-law hardening materials;

4: the amplitudes converge always to the same values when different initially guessed amplitudes are used in the minimization procedure of the complementary energy.

Conclusions

An energy method is proposed in this paper to describe the full elastic-plastic stress field of a plane strain mode I crack. By establishing a statically admissible stress field and by minimizing the complementary energy of the structure, an algebraic expression can be found to represent the full stress field near the crack tip. This method is pretty easy to apply to the elastic-plastic crack problem. The numerical results show that the present method is highly accurate and can be used as an alternative to the fine finite element analysis In this work, the amplitudes in analytical asymptotic expansion of a mode I elastic-plastic crack are calculated by using both the variational method and point-matching method. It is found that the point matching technique may fail in certain cases for different versions of the truncated analytical solutions. However, the variational method can provide convergent amplitude values in analytical expansion. The results obtained by using these two methods are quite different in some cases. It is shown that even the four-term analytical asymptotic solution can not reproduce adequately the near-tip stress field. More terms in asymptotic expansion are needed. In this case, some approximate approaches may be more advantageous in engineering applications.

References

[1] J.W.Hutchinson, *Journal of the Mechanics and Physics of Solids* **16**, 13 -31 (1968)

[2] J.R.Rice and G.F.Rosengren, *Journal of the Mechanics and Physics of Solids* **16**, 1-12 (1968)

[3] J.R.Rice, *Journal of applied Mechanics*, **35**, 379-386 (1968)

[4] J.A.Begley and J.D.Landes, *ASTM STP* **514**, 1-23,24-39 (1972)

[5] C.F.Shih, H.G..de Lorenzi and W.R.Andrews, *ASTM STP* **668**, 65-120 (1979)

[6] Y.Li and Z.Wang, *Scientia Sinica*, **A9** 941-955 (1986)

[7] N.P. O'Down and C.F.Shih, *Journal of the Mechanics and Physics of Solids* **39**, 989-1015 (1991)

[8] L. Xia, Wang and C.F.Shih, *Journal of the Mechanics and Physics of Solids* **41**, 665-687 (1993)

[9] S.Yang, Y.J.Chao and M.A. Sutton, *Engineering Fracture Mechanics* **45**, 1-20 (1993)

[10] Y.Wei and T. Wang, *Engineering Fracture Mechanics* **51**, 547-553 (1995)

[11] T.M. Edmunds and J.R. Willis, *Journal of the Mechanics and Physics of Solids* **24**, 205-223 (1976)

[12] T.M. Edmunds and J.R. Willis, *Journal of the Mechanics and Physics of Solids* **24**, 225-237 (1976)

[13] T.M. Edmunds and J.R. Willis, *Journal of the Mechanics and Physics of Solids* **25**, 423-455 (1977)

[14] M.L. Williams, *Journal of the Mechanics and Physics of Solids* **24**, 111-114 (1957)

[15] S.M.Sharma and N.Aravas, *Journal of the Mechanics and Physics of Solids* **39**, 1043-1072 (1991)

[16] J.R. Rice, *Journal of the Mechanics and Physics of Solids*, **22**, 17-26 (1974)