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A Crack Terminating at a Nonideal Interface under Multiaxial Loading

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ABSTRACT: *Local stress field near tip of a plane crack terminating at a bimaterial interface is considered. Instead of the "ideal contact" interfacial conditions, usually done, we have introduced an adhesive interlayer of infinitesimal thickness between the materials, and we model it as a thin elastic region the width of which changes according to an exponential law. It is shown that the geometry of the thin region influences essentially the stress not only qualitatively (the character of the stress singularity near the crack tip), but also quantitatively (the increase of a number of singular terms in the asymptotics).*

Notation

$u_k^{(j)}$	- components of displacement vector $u^{(j)}$ in j-material
$\sigma_{kl}^{(j)}$	- components of stress tensor $\sigma^{(j)}$ in j-material
$\sigma_n\|_\Gamma$	- vector of tractions along boundary Γ
d_j	- characteristic damage segment of j-material
$\tau_k \quad (\tau_k > 0)$	- components of diagonal matrix τ defining mechanical properties of nonideal interface
$\alpha \quad (\alpha \geq 0)$	- parameter defining geometry of intermediate zone
K_1, K_2, K_3	- Generalized Stress Intensity Factors (SIF) for Mode I, Mode II, Mode III, respectively
$\gamma_m - 1$, $[\omega_m - 1]$	- exponents of the main singular terms of stress under Mode m, (m=1, 2, 3) for nonideal [ideal] interface, respectively.

Introduction

When strength of a nonhomogeneous body with a crack is investigated two different problems should be solved. The first one is to obtain displacement and stress fields by a convenient model of the body taking into account its geometry and physical feature of the materials. The second problem is to analyze the obtained strain-stress distribution near the crack tip by an appropriate fracture mechanics criterion.

Both the problems are connected themselves. Thus the first problems, as a rule, are discussed in the case of so-called "ideal contact" conditions. They consist of the continuity of displacement and the traction vectors along the interface. It is known from papers [18,19] that in such problems the exponent of the stress singularity near the crack tip, is not equal to -0.5. Moreover, in the case of an interfacial crack, the model of the "ideal contact" leads to such inconsistencies as oscillating character of stress and strain components and overlapping of the crack surfaces in vicinity of the crack tips; both unacceptable from the standpoint of physics. Besides, the usually applied Griffith-Irwin's criterion cannot be directly employed. For extensive literature on this topic see [4,14].

To eliminate such inconsistencies, in the case of an interfacial crack *a modification of the contact conditions* along the crack surfaces near the crack tip has been proposed [4]. Another way to investigate the problem when the crack tip is situated on the bimaterial interface is to use so-called "kinked crack approach" [7]. A third way is to assume that there exists a thin *it interfacial zone*, mechanical properties of which vary between the different materials [1,6,10]. Then the fracture mechanics analysis can be done in terms of the stress intensity factor (SIF) with Griffith-Irwin's criterion.

In contradiction to the other models, the "intermediate region" allows us to take into account the influence of the real adhesive region not only at the stage of strength analysis by the theory of adhesion [3], but also at the stage of finding the displacements and stress fields near the crack tip.

Let us discuss now models of "nonideal contact" along the interface being under investigation in this paper. In Fig.1 as an example a bimaterial rod with a plane crack terminating at the interface is presented in the Cartesian coordinate system x_*, y_*, z_* (OZ_*-axis coincides with the rod axis). We assume that characteristic thickness h_* of the adhesive interlayer between the different materials is lesser than length l_* of the crack and than characteristic size of the solid D_*, i.e. $h_* \ll l_*$, $h_* \ll D_*$.

If nonlocal strain-stress state of the solid is of interested (outside of the neighbourhood of the crack tip), then using the "ideal contact" interaction between the materials is justified. However, if we are interested in the local fields near the crack tip, then the "ideal contact" conception is not satisfactory. Let us normalize variables x_*, y_*, z_* by the value $d_* = \min\{d_0, d_1\}$ (d_0, d_1 are characteristic damage segments of the materials (see [13,16])) in the following manner: $x = x_* / d_*, y = y_* / d_*, z = z_* / d_*$. It can be naturally assumed that $d_* \ll l_*$. Then in the variables x, y, z we obtain the respective modelling problem in the bimaterial infinite space with plane semiinfinity crack terminating at the adhesive layer of the characteristic thickness $h = h_* / d_*$. In this modelling problem we are interested in the displacement and stress fields on the unit distance from the crack tip. Thus if the value of h is of the same order or higher ($h \sim 1$ or $h > 1$) then the adhesive layer has to be considered as a separate structural element with its own geometry and specific mechanical properties [1,6,10].

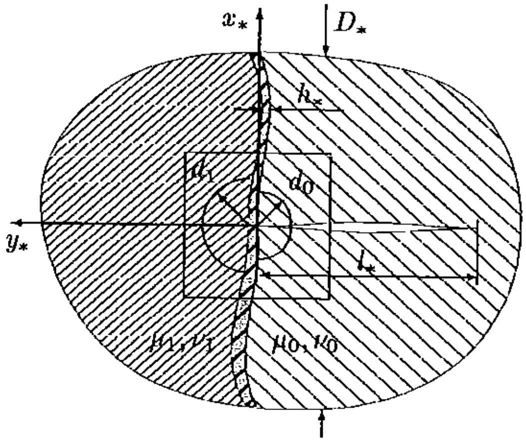


Fig.1 Bimaterial solid with plane crack terminating at nonideal interface

In the opposite case, when the thin interlayer is of infinitesimal thickness, the influence of the adhesive layer should be taken into account in the other way. It is clear that even under assumption $h \ll 1$, the "ideal contact" model cannot satisfactory. (It is enough to imagine that the shear modulus μ_* of the elastic adhesive layer is far less than the moduli of the materials being in contact).

In this case ($h \ll 1$) the intermediate zone can be considered as a thin elastic inclusion. Usually we do not have any additional information on exact form of the adhesive interlayer. To investigate an arbitrary geometry of the intermediate zone we assume that the thickness of the elastic inclusion near the crack tip is described by the relation: $h(r) = hr^{\alpha}$, $(0 \le h << 1,\ 0 \le \alpha < \infty)$ where r is a distance from the crack tip along the interface, but h, α are dimensionless values. The corresponding square region near the crack tip from Fig.1 is presented in Fig.2a-Fig.2d depending on the value of parameter α. Here μ_0, μ_1 and ν_0, ν_1 are the shear moduli and Poisson's ratios of the elastic materials being in contact.

Applying a standard technique for thin inclusion (see [2,8]), we can integrate respective equilibrium equations of the intermediate zone by the parameter determining normal direction to surfaces of the thin region. Then the following interfacial conditions arise:

$$[\sigma_n]_{|\Gamma} = 0, \quad ([u] - r^{\alpha}\tau\sigma_n)_{|\Gamma} = 0 \tag{1}$$

along the nonideal bimaterial interface. Here $[u]$, $[\sigma_n]$ are jumps of displacements and tractions, respectively, along the interface, while τ is a diagonal matrix of the components [2]:

$$\tau_1 = h_* / \mu^*_{xy}, \quad \tau_2 = h_* / E_*, \quad \tau_3 = h_* / \mu^*_{yz} \tag{2}$$

where E_* and μ^*_{xy}, μ^*_{yz} are the Young's and the shear moduli of the elastic inclusion. Conditions (1) can be considered independently of an assumed model of the thin interlayer. Then parameters $\tau_j > 0$ should be experimentally determined.

Such an approach allows us to investigate different forms of the intermediate zone near the crack tip. Thus, if $\tau = 0$, we have the usual "ideal contact". When $\alpha = 0, \tau \neq 0$ there is a thin interlayer of constant thickness between the materials (Fig.2a). The case where $0 < \alpha < 1$ can be interpreted as a thin adhesive zone with a damage near the crack tip (Fig.2b). When $\alpha = 1$, the materials are contracted by the thin wedges (Fig.2c). Finally, $1 < \alpha < \infty$ can represent an "almost ideal contact" between materials near the crack tip (Fig.2d).

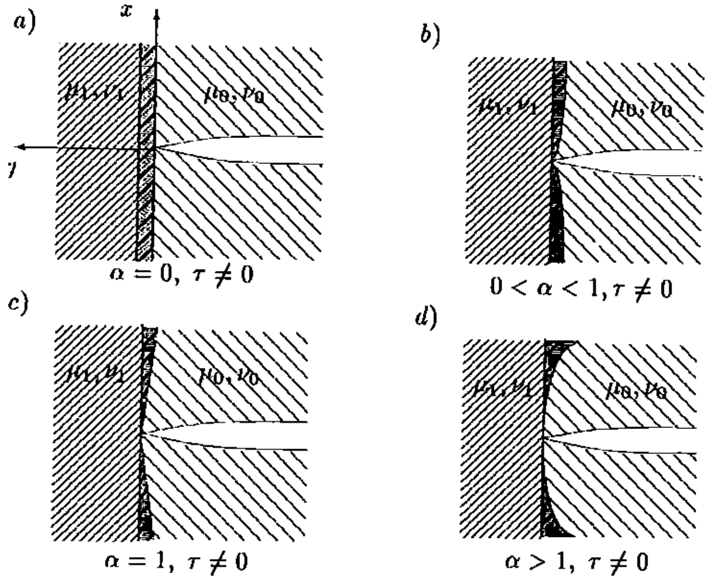


Fig.2 Geometry of the interfacial zone near the crack tip

As it is known, local fields near the tip of plane crack terminating normally at the interface are described in principal by combination of the Mode I, II, III states. We shown that in the most cases of nonideal interface $(\alpha > 0)$, the corresponding asymptctics can be written like this:

$$u_l(r,\theta) = w_{l0} + \sum_{m=1}^{m=3} \frac{K_m}{\gamma_m} f_l^{(m)}(\theta) r^{\gamma_m} + O(r^{\gamma^*}), \quad r \to 0, \quad (l = r,\theta,z),$$

$$\sigma_{kl}(r,\theta) = \sum_{m=1}^{m=3} K_m g_{kl}^{(m)}(\theta) r^{\gamma_m - 1} + O(r^{\gamma^* - 1}), \quad r \to 0, \quad (k,l = r,\theta,z), \tag{3}$$

where $f_l^{(m)}(\theta), g_{kl}^{(m)}(\theta)$ are cyclic functions, and r, θ, z are polar coordinates centered at the crack tip. Thus, for homogeneous material we have $w_{l0} = 0$, $\gamma_1 = \gamma_2 = \gamma_3 = 0{,}5$, $\gamma^* = 1$, but when crack terminates normally at the "ideal" interface: $w_{l0} = 0$, $\gamma_1 = \gamma_2 \neq \gamma_3 \in (0,1)$, $\gamma^* = 1$. In both cases, functions $f_l^{(m)}(\theta)$, $g_{kl}^{(m)}(\theta)$ are specified in any text book on fracture mechanics, e.g. [2].

Below it is proved that local strain-stress state depends essentially on the type of the nonideal contact interface. Moreover, we show that not only the quantitative modification of the asymptotics (change of the value of singularity exponents), but also the qualitative one (increase of a number of singular terms in the asymptotics) can arise. In the last case, γ^* is less than *1*, and the term $O\left(r^{\gamma^*-1}\right)$contains a number of singular terms of stress. Besides, situations can arise $0 \leq \alpha < 1$ when $w_{i0} \neq 0$ (the crack surfaces differ themselves near the crack tip). Finally, for $\alpha = 0$, logarithmic singularity of stress only arises.

Although in the considered models stress singularities are not equal to -0.5, such fracture mechanics criteria as the critical crack opening criteria [5,9,17] or the effective stress criteria [13,15,16], allow us to investigate arbitrary displacement and stress fields taking into account the corresponding (elastic, plastic) properties of the materials. Due to any of the mentioned criteria, the value of $K_m d_j^{\gamma_m} / \gamma_m$ has to be taken into account in fracture mechanics analysis, instead of the value of SIF K_m. Here the small parameter d_j of length dimension has different physical interpretation in frame of each of the criterion. The corresponding investigation is not a goal of this paper.

Problems Formulation

Let us consider modelling Mode m problems (m = 1, 2, 3) for a bimaterial plane with a semi-infinity crack terminating perpendicularly at the interface. We shall seek solutions of the problems satisfying equilibrium equations and the Hooke law in each of half-plane $(y > 0, y < 0)$. Along the crack surfaces $(x = 0, y < 0)$ the corresponding exterior boundary conditions are prescribed:

$$\sigma_n^{(0)}{}_{|\theta=-\pi/2\pm 0} = -g_m(r)e_m, \; e_1 = (1,0,0), \ldots, e_3 = (0,0,1). \tag{4}$$

We assume that functions $g_m(r)$ are sufficiently smooth and vanish at zero and infinity points (for example, $g_m \in C_0^\infty(\mathrm{R}_+)$), then any singularities of the solutions are connected with the interior properties of the problems only. In view of symmetry of the problems geometry, we can conclude that the following additional conditions are satisfied on the crack line ahead ($x = 0, y > 0$):

$$\begin{aligned}
&\text{Mode I}: && u_\theta^{(1)}{}_{|\theta=\pi/2} = 0, \; \sigma_{r\theta}^{(1)}{}_{|\theta=\pi/2} = 0,\\
&\text{Mode II}: && u_r^{(1)}{}_{|\theta=\pi/2} = 0, \; \sigma_\theta^{(1)}{}_{|\theta=\pi/2} = 0,\\
&\text{Mode III}: && u_z^{(1)}{}_{|\theta=\pi/2} = 0
\end{aligned} \tag{5}$$

Along the nonideal bimaterial interface ($y=0, x>0$) interfacial conditions (1) are assumed to be true. We shall seek the corresponding solutions of problems (1), (4) and (5) meeting with the following conditions at the singular points:

$$u=O(r^{\vartheta_m}),\quad \sigma=O(r^{\gamma_m-1}),\quad r\to 0,$$
$$u=O(r^{-\vartheta_\infty(m)}),\quad \sigma=O(r^{-\gamma_\infty(m)-1}),\quad r\to\infty, \tag{6}$$

where unknown constants $\vartheta_m, \vartheta_\infty(m)\geq 0$, $\gamma_m, \gamma_\infty(m)>0$ $(\vartheta_m+\vartheta_\infty(m)>0)$ depend on the values of mechanical parameters and the strain-stress state (Mode I, II, III). They will be calculated while the problems are solved.

Solution to Mode III problem

Applying the Mellin transform technique we obtain functional equation:

$$\overline{\tau}_3 p_3(s+\alpha-1)+F_3(s)p_3(s)=G_3(s), \tag{7}$$

with additional condition $p_3(0)=-\tilde{g}_3(0)$. Here an unknown function $p_3(s)$ and known function $\tilde{g}_3(s)$ are the Mellin transformations of the tractions along the interface and the crack surface, respectively:

$$p_3(s)=\int_0^\infty \sigma_{\theta z}(r,\theta)_{|\theta=0} r^s ds,\quad \tilde{g}_3(s)=\int_0^\infty g_3(r)r^s dr,$$

which are analytic in the strip $-\gamma_3<\mathrm{Re}\,s<\gamma_\infty(3)$ in view of *a priori* estimations (6). Besides, the following notations are introduced in (7):

$$F_3(s)=\frac{2(\kappa-\cos\pi s)}{(1-\kappa)s\sin\pi s},\quad G_3(s)=\frac{\tilde{g}_3(s)}{s\sin(\pi s/2)},\quad \kappa=\frac{\mu_0-\mu_1}{\mu_0+\mu_1},\quad \overline{\tau}_3=\mu_0\tau_3.$$

In the case of the "ideal contact" ($\overline{\tau}_3=0$), solution of equation (7) is found in a closed form, and is well known. Thus, function $p_3(s)$ is analytic in the strip $|\mathrm{Re}\,s|<\omega_3$, and has simple poles at points $s = \pm\omega_3$, where $\omega \in (0,1)$ is the first zero of the function $F_3(s)$ which is the nearest to the imaginary axis. Hence, it can be concluded that $\gamma_3 = \gamma_\infty(3) = \vartheta_\infty(3) = \vartheta_3 = \omega_3$.

In case $\alpha = 1$, equation (7) is also solved in a closed form. Function $p_3(s)$ is analytic in the strip $|\mathrm{Re}\, s| < \omega_3^*(\bar{\tau}_3)$, and has simple poles at points $s = \pm\omega_3^*(\bar{\tau}_3)$. Here $\omega_3^*(\bar{\tau}_3) \in (0,1)$ is the first zero of function $F_3(s) + \bar{\tau}_3$. The corresponding graphs of the values of $\omega_3^*(\bar{\tau}_3)$ are presented in Fig.3.

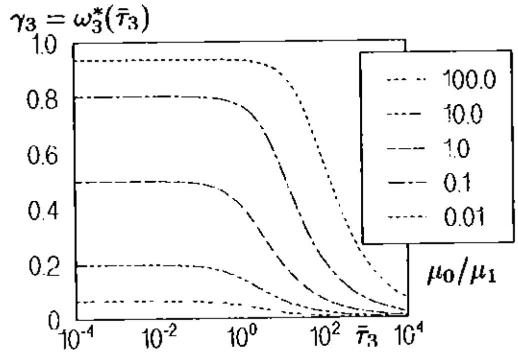


Fig.3 Graph of parameter γ_3 against $\bar{\tau}_3 = \tau_3\mu_0$ for $\alpha = 1$.

In case $\alpha \in [0,1)$, equation (7) is valid in the strip $0 < \mathrm{Re}\, s < \omega_3$, at least. Hence, function $p_3(s)$ should be analytic in the strip $\alpha - 1 < \mathrm{Re}\, s < \omega_3$, and has simple poles at points $s = \alpha - 1$ $(\alpha \neq 0)$ and $s = \omega_3$. But in case $\alpha = 0$ there exists a double pole at point $s = -1$. In case $\alpha > 1$, we can conclude that equation (7) holds true in the strip $-\omega_3 < \mathrm{Re}\, s < 0$. Hence, function $p_3(s)$ has simple poles at points $s = -\omega_3$ and $s = \alpha - 1$ and is analytic in the strip $-\omega_3 < \mathrm{Re}\, s < \alpha - 1$.

It can be proved that functional equation (7) has unique solution in both cases. They can be found from some singular integral equations with fixed point singularities. In case $\alpha = 1$, the corresponding results have been presented earlier [11].

Asymptotics of the solution for Mode III

Now we investigate asymptotics of displacement $u_z(r,\theta)$ near the crack tip. The corresponding relations for stress can be found according to Hooke law:

$$\sigma_{rz}^{(j)} = \mu_j \frac{\partial u_z^{(j)}}{\partial r}, \quad \sigma_{\theta z}^{(j)} = \mu_j \frac{1}{r}\frac{\partial u_z^{(j)}}{\partial \theta}.$$

When $\alpha = 1$, in paper [11] asymptotics is obtained:

$$\begin{aligned}
&u_z^{(0)}(r,\theta) = C_1 - C_1\bar{\tau}_3^{-1} r\sin\theta + O(r^{1+\omega_3}), \quad r \to 0,\\
&u_z^{(1)}(r,\theta) = -\frac{C_1\mu_0 r}{\pi\mu_1\bar{\tau}_3}[(\pi - 2\theta)\sin\theta + 2(C_2 + \ln r)\cos\theta] + O(r^{1+\omega_1}), \quad r \to 0,\\
&u_z^{(j)}(r,\theta) = O(r^{-\omega_3}), \quad r \to \infty,
\end{aligned} \qquad (8)$$

with some constants C_1, C_2.

For all remaining cases ($\alpha > 0$), the corresponding relations can be rewritten in a common form like this ($\phi = \pi/2 - \theta$):

$$\begin{aligned}
&u_z^{(0)} = C_0 + \frac{K_3}{\mu_0\gamma_3} r^{\gamma_3} \operatorname{ctg}\frac{\gamma_3\pi}{2}\cos\gamma_3(\pi - \phi) + O(r^{\gamma_3^*}), \quad r \to 0,\\
&u_z^{(1)} = K_3(\mu_1\gamma_3)^{-1} r^{\gamma_3}\sin\gamma_3\phi + O(r^{\gamma_3^*}), \quad r \to 0,\\
&u_z^{(0)}(r,\theta) = C_\infty + O(r^{-\gamma_\infty(3)}), \quad u_z^{(1)}(r,\theta) = O(r^{-\gamma_\infty(3)}), \quad r \to \infty.
\end{aligned} \qquad (9)$$

where the constants from *a priori* estimations (6) are calculated taking into account the behaviour of function $p_3(s)$:

$$\begin{aligned}
&\gamma_3 = 1 - \alpha, \quad \gamma_\infty(3) = \vartheta_\infty(3) = \omega_3, \quad \vartheta_3 = 0, \quad 0 < \alpha < 1,\\
&\gamma_3 = \gamma_\infty(3) = \vartheta_\infty(3) = \vartheta_3 = \omega_3^*(\bar{\tau}_3), \quad \alpha = 1,\\
&\gamma_\infty(3) = 1 - \alpha, \quad \gamma_3 = \vartheta_3 = \omega_3, \quad \vartheta_\infty(3) = 0, \quad 1 < \alpha < \infty
\end{aligned} \qquad (10)$$

In equations (9), constant K_3 is calculated by a constant at the corresponding pole of function $p_3(s)$. Let us note that situations can arise where there is a number of singular terms of stress near the crack tip ($\gamma_3^* < 1$).

In Fig.4, a graph of the main exponent of stress singularity ($\gamma_3 - 1$) is presented with respect to the value of parameter α. Besides, a scheme demonstrating a distribution of the number of singular terms in the stress asymptotics in the neighbourhood of the crack tip is shown.

– For $\alpha = 0$, there is not an exponential singularity of stress near the crack tip for any values of the mechanical parameters μ_0, μ_1, $\bar{\tau}_3$. In this case, stress concentration appears only in the domain $y > 0$ (on the crack line ahead), and it has a logarithmic character (see [11]). In regards to the displacement field, there is displacement discontinuity near the crack tip along the bimaterial interfacial contact ($C_1 \neq 1$ in (8)).

– If $\alpha \in (0{,}0.5]$, only one singular term in asymptotics of stress in the neighbourhood of the crack tip appears ($\gamma_3^* \geq 1$). Corresponding exponent in interval (-1,0) is $\gamma_3 - 1 = -\alpha$, and does not depend on the values of μ_0, μ_1, $\bar{\tau}_3$.

– For case $\alpha \in (0.5{,}1)$, or more precisely $\alpha \in (\alpha_n^-, \alpha_{n+1}^-)$, ($n$=2,3,...), where $\alpha_n^- = 1 - 1/n$, there are exactly n singular terms in the asymptotics of stress with exponents: $\gamma_3 - 1 = -\alpha$, $\gamma_{3j}^- - 1 = (j+1)(1-\alpha) - 1 \in (-1{,}0)$, j=1,...,n (see the diagram in Fig.4). Moreover, if $\alpha \to 1$, then the number of singular terms tends to infinity $n \to \infty$! In the last two cases ($\alpha \in (0{,}1)$), the displacement discontinuity near the crack tip also appears ($C_0 \neq 0$ in (9)).

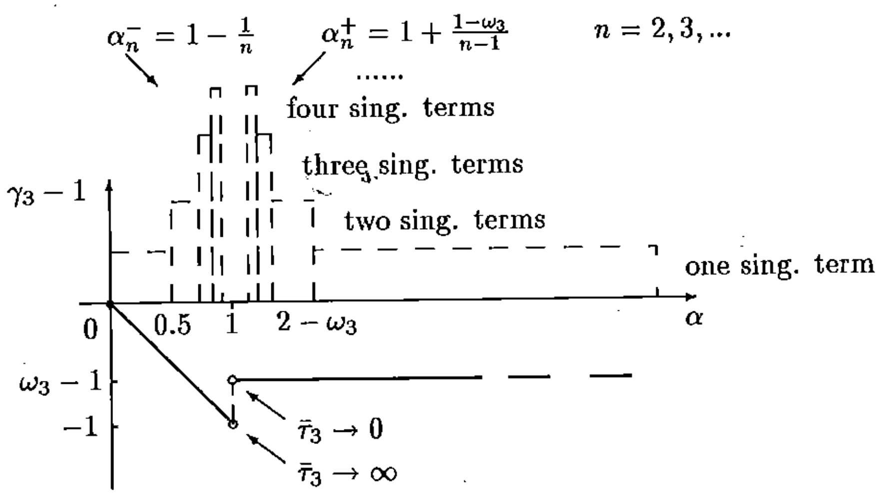


Fig.4 Stress singularity, and number of singular terms of stress

– If $\alpha = 1$ there is on singular term of asymptotics ($\gamma_3^* \geq 1$) with exponent $\gamma_3 - 1 = \omega_3^*(\bar{\tau}_3) - 1 \in (-1, \omega_3 - 1)$ depending essentially on the values of mechanical parameters μ_0, μ_1, $\bar{\tau}_3$ (see Fig.3). Thus, if $\bar{\tau}_3 \to 0$ then $\gamma_3 - 1 \to \omega_3 - 1$, which coincides with the result for the "ideal contact". In this case ($\alpha = 1$) and in the next one ($\alpha > 1$) the displacement field is continuous near the crack tip ($C_0 = 0$). However, it is discontinuous on any distance from the crack tip along the bimaterial contact in view of the condition $(1)_2$.

– For case $\alpha \in (1, 2 - \omega_3)$, or more precisely $\alpha \in (\alpha_{n+1}^+, \alpha_n^+)$, ($n$=2,...), where $\alpha_n^+ = 1 + (1 - \omega_3)/(n-1)$, there are accurately n singular terms in asymptotics of stress with exponents: $\gamma_3 - 1 = \omega_3 - 1$, $\gamma_{3j}^+ - 1 = -j(\alpha - 1) + \omega_3 - 1 \in (-1, 0)$, j=1,...,n (the diagram in Fig4). As above, $n \to \infty$ when $\alpha \to 1$.

– Finally, in case $\alpha \in [2 - \omega_3, \infty)$, one singular term of stress asymptotics appears ($\gamma_3^* \geq 1$). The corresponding exponent $\gamma_3 - 1 = \omega_3 - 1$ is similar to that for the "ideal contact" model, and does not depend on the remaining problem parameters.

Numerical results for SIF and the displacement discontinuity near the crack tip are presented in the papers [11,12] for different values of the mechanical parameters μ_0, μ_1, $\bar{\tau}_3$. In particular, it has been shown that for $0 < \alpha < 1$, estimations are true:

$$C_0 \sim \bar{\tau}_3^{\omega_3 + \alpha}, \quad K_3 \sim \bar{\tau}_3^{\omega_3 + \alpha - 1}, \quad \bar{\tau}_3 \to 0. \tag{11}$$

In the opposite case ($1 < \alpha$), the following relation is obtained ($C_0 = 0$):

$$K_3 = K_3^{id} + O(\bar{\tau}_3^{\beta}), \quad \bar{\tau}_3 \to 0, \tag{12}$$

where K_3^{id} is SIF in the case of the "ideal contact", but $\beta = \beta(\alpha) > 0$ is some constant. In particular, $\beta(1 + \omega_3) = 1$. Hence, in all of the cases, asymptotics (9) is rebuilt to those for the "ideal bimaterial contact" when $\bar{\tau}_3 \to 0$.

Solutions to Mode I and Mode II

Applying the Mellin transform we obtain the following systems of functional equations:

$$\Lambda p_m(s+\alpha-1)+K_m(s)p_m(s)=G_m(s), \tag{13}$$

where m = 1, 2 for the Mode I and Mode II, respectively. Here, unknown vector-functions $p_m(s)$ are the Mellin transformations of the tractions along the interface

$$p_m(s)=(\tilde{\sigma}_\theta(s,0),\tilde{\sigma}_{r\theta}(s,0))^{\mathrm{T}}. \tag{14}$$

are analytic in the strip $-\gamma_m<\operatorname{Re}s<\gamma_\infty(m)$, and should satisfy additional conditions

$$p_m(0)=\begin{cases}(0,-\tilde{g}_1(0))^{\mathrm{T}}, & m=1,\\ (-\tilde{g}_2(0),0)^{\mathrm{T}}, & m=2.\end{cases}$$

Vectors $G_m(s)$ and matrices Λ, $F_m(s)$ are defined like this:

$$G_1=\frac{-2\tilde{g}_1(s)}{X(s)}\begin{pmatrix}s\cos(\pi s/2)\\ (s+1)\sin(\pi s/2)\end{pmatrix},\quad G_2=\frac{2\tilde{g}_2(s)}{X(s)}\begin{pmatrix}(1-s)\sin(\pi s/2)\\ s\cos(\pi s/2)\end{pmatrix}.$$

$$\Lambda=\begin{pmatrix}\bar{\tau}_\theta & 0\\ 0 & \bar{\tau}_r\end{pmatrix},\quad F_2(s)=EF_1(s)E,\quad E=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad X(s)=s[2s^2-1+\cos\pi s].$$

The corresponding components $\phi_{ij}(s)$ of matrix-function $F_1(s)$ are calculated by the relations

$$\phi_{11}(s)=\frac{\sin\pi s}{X(s)}-\frac{b}{s}\operatorname{ctg}\frac{\pi s}{2},\quad \phi_{12}(s)=\frac{2s-1+\cos\pi s}{X(s)}-\frac{a+b}{s}$$

$$\phi_{22}(s)=\frac{\sin\pi s}{X(s)}+\frac{b}{s}\operatorname{tg}\frac{\pi s}{2},\quad \phi_{21}(s)=\frac{2s+1-\cos\pi s}{X(s)}+\frac{a+b}{s}$$

Here dimensionless parameters a, b, $\bar{\tau}_{r(\theta)}$ are introduced like this:

$$a=\frac{\mu_1-\mu_0}{2\mu_1(1-\nu_0^*)},\quad b=\frac{\mu_0(1-\nu_1^*)}{\mu_1(1-\nu_0^*)},\quad \overline{\tau}_{r(\theta)}=\frac{\mu_0}{1-\nu_0^*}\tau_{1(2)},$$

where $\nu_j^*=\nu_j$ under plane strain deformations, but $\nu_j^*=\nu_j/(1-\nu_j)$ under plane stress conditions.

For the "ideal contact" ($\Lambda=0$), solutions of the Mode I and Mode II are found in closed forms [18,19]. The corresponding vector-functions $p_m(s)$ in (13) are analytic in the strip $|\mathrm{Re}\, s|<\omega_1$, and have simple poles at points $s=\pm\omega_1$. Here $\omega_1\in(0,1)$ is the first zero of the determinant $\det F_m(s)$ ($\det F_1(s)=\det F_2(s)$) which is the nearest to the imaginary axis. So, in this case we can conclude that for both Mode I and Mode II $\gamma_m=\gamma_\infty(m)=\vartheta_\infty(m)=\vartheta_m=\omega_1$.

When $\alpha=1$, systems (13) are solved in closed forms also. Vector-functions $p_m(s)$ are analytic in the strip $|\mathrm{Re}\, s|<\omega_m^*(\overline{\tau}_\theta,\overline{\tau}_r)$, and have simple poles at points $s=\pm\omega_m^*$. Here $\omega_m^*\in(0,1)$ are the first zeros of functions $\det\Psi_m$ ($\Psi_m(s,\overline{\tau}_\theta,\overline{\tau}_r)=F_m(s)+\Lambda(\overline{\tau}_\theta,\overline{\tau}_r)$). Consequently, in this case ($\alpha=1$) for Mode I and Mode II problems, parameters determining stress singularity are calculated like this: $\gamma_m=\gamma_\infty(m)=\vartheta_\infty(m)=\vartheta_m=\omega_m^*(\overline{\tau}_\theta,\overline{\tau}_r)$.

Let us note that

$$\det\Psi_2(s,\overline{\tau}_\theta,\overline{\tau}_r)=\det\Psi_1(s,\overline{\tau}_r,\overline{\tau}_\theta) \tag{15}$$

Hence, exponents ω_1^*-1, ω_2^*-1 of the stress singularity are different for the Mode I and Mode II problems! Moreover, as it will be seen below, in this case ($\alpha=1$) situations can arise when two singular terms of stress asymptotics exist in contradiction to the Mode III problem.

Taking into account relation (15) we can present numerical results for exponents of stress singularity for Mode I only. Exponents of the stress singularity for Mode II problem γ_2-1 can be calculated using the value $\omega_2^*(\overline{\tau}_\theta,\overline{\tau}_r)=\omega_1^*(\overline{\tau}_r,\overline{\tau}_\theta)$.

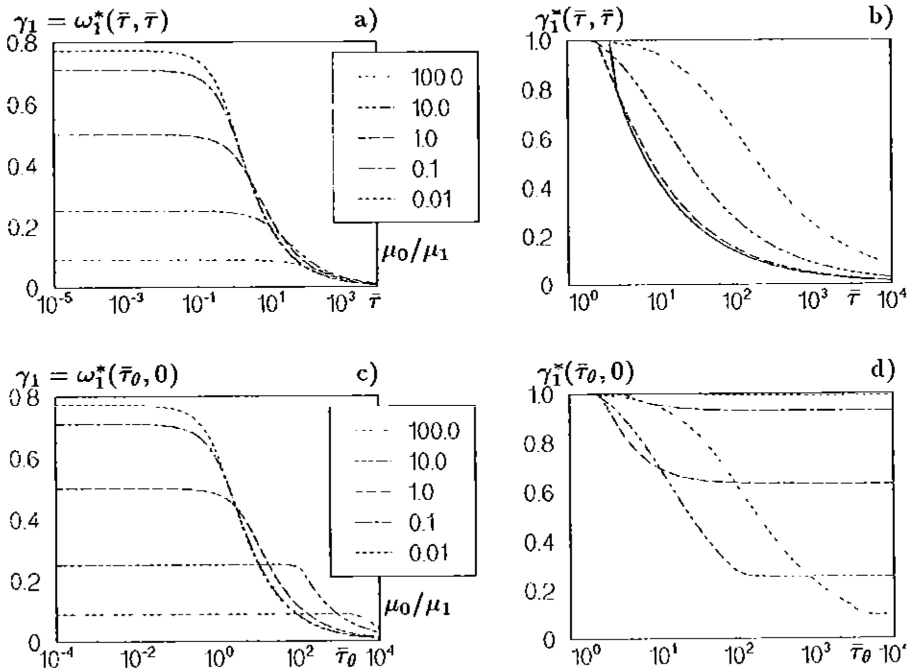
In Fig.5 - Fig.7 graphs of two first real zeros of function $\det\Psi_1(s,\overline{\tau}_\theta,\overline{\tau}_r)$ belonging to interval (0,1) and determining parameters γ_1, γ_1^* in asymptotics (3) are presented. In comparison with Mode III problem where the first zero $\omega_3^*(\overline{\tau}_r)$ depends on two parameters

$\bar{\tau}_r$ and μ_1/μ_0, for Mode I problem the value of the first zero $\omega_1^*(\bar{\tau}_\theta,\bar{\tau}_r)$ depends on five parameters: $\nu_0, \nu_1, \bar{\tau}_\theta, \bar{\tau}_r$ and μ_1/μ_0.

Influence of the Poisson ratios ν_0, ν_1 is shown by the following selections: $\nu_0 = \nu_1 = 0.2$ in Fig.5; $\nu_0 = 0, \nu_1 = 0.5$ in Fig.6, and $\nu_0 = 0.5, \nu_1 = 0$ in Fig.7.

In each of figures Fig.5 - Fig.7, graphs marked by letters a), b) present two first zeros when parameters $\bar{\tau}_\theta, \bar{\tau}_r$ have the same values ($\bar{\tau}_\theta = \bar{\tau}_r = \bar{\tau}$).

Graphs marked by letters c), d) correspond to situation when $\bar{\tau}_r = 0$, but $\bar{\tau}_\theta \neq 0$.



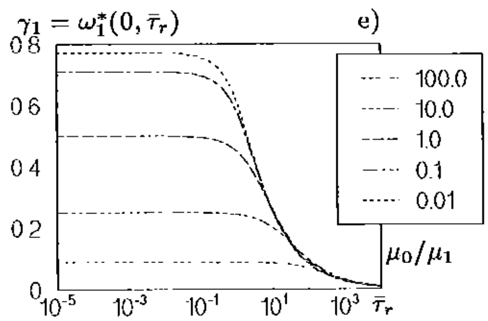
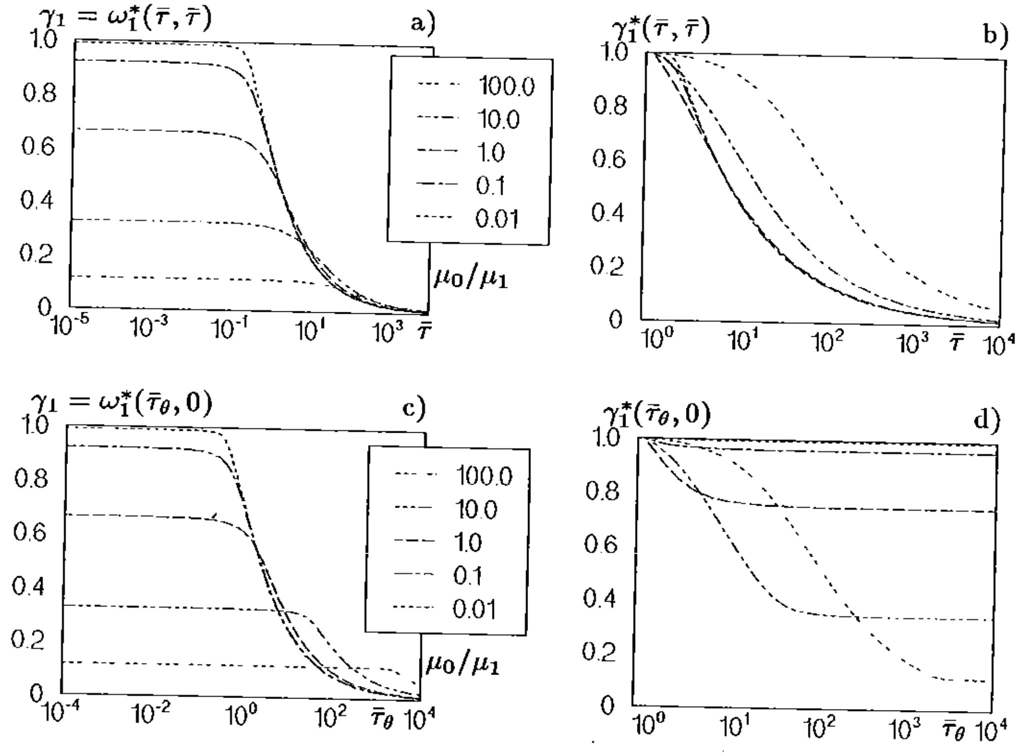


Fig.5 Graphs of parameters γ_1, γ_1^* determining the stress singularity in the case $\alpha = 1$, for different values of μ_0/μ_1 and $\nu_0 = \nu_1 = 0.2$.

Finally, in graphs marked by letter e) the values of the contact parameters are defined like this: $\bar{\tau}_r \neq 0, \bar{\tau}_\theta = 0$. Then only one real zero of function $\det \Psi(s,0,\bar{\tau}_r)$ appears in interval (0,1). As it can be seen from graphs in Fig.5 - Fig.7, behaviour of the first zero in the Mode I problem is similar to that for the Mode III problem. However, the values of $\omega_1^*(\bar{\tau}_\theta, \bar{\tau}_r)$ depend essentially



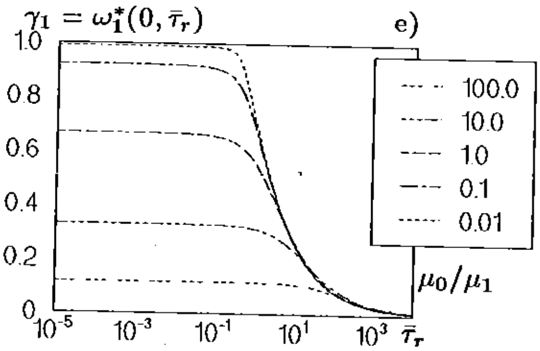


Fig.6 Graphs of parameters γ_1, γ_1^* determining the stress singularity in the case $\alpha = 1$, for different values of μ_0/μ_1 and $\nu_0 = 0.0, \nu_1 = 0.5$.

not only on magnitudes of $\bar{\tau}_\theta, \bar{\tau}_r$, but in the most degree they depend on the values of Poisson ratios ν_0, ν_1.

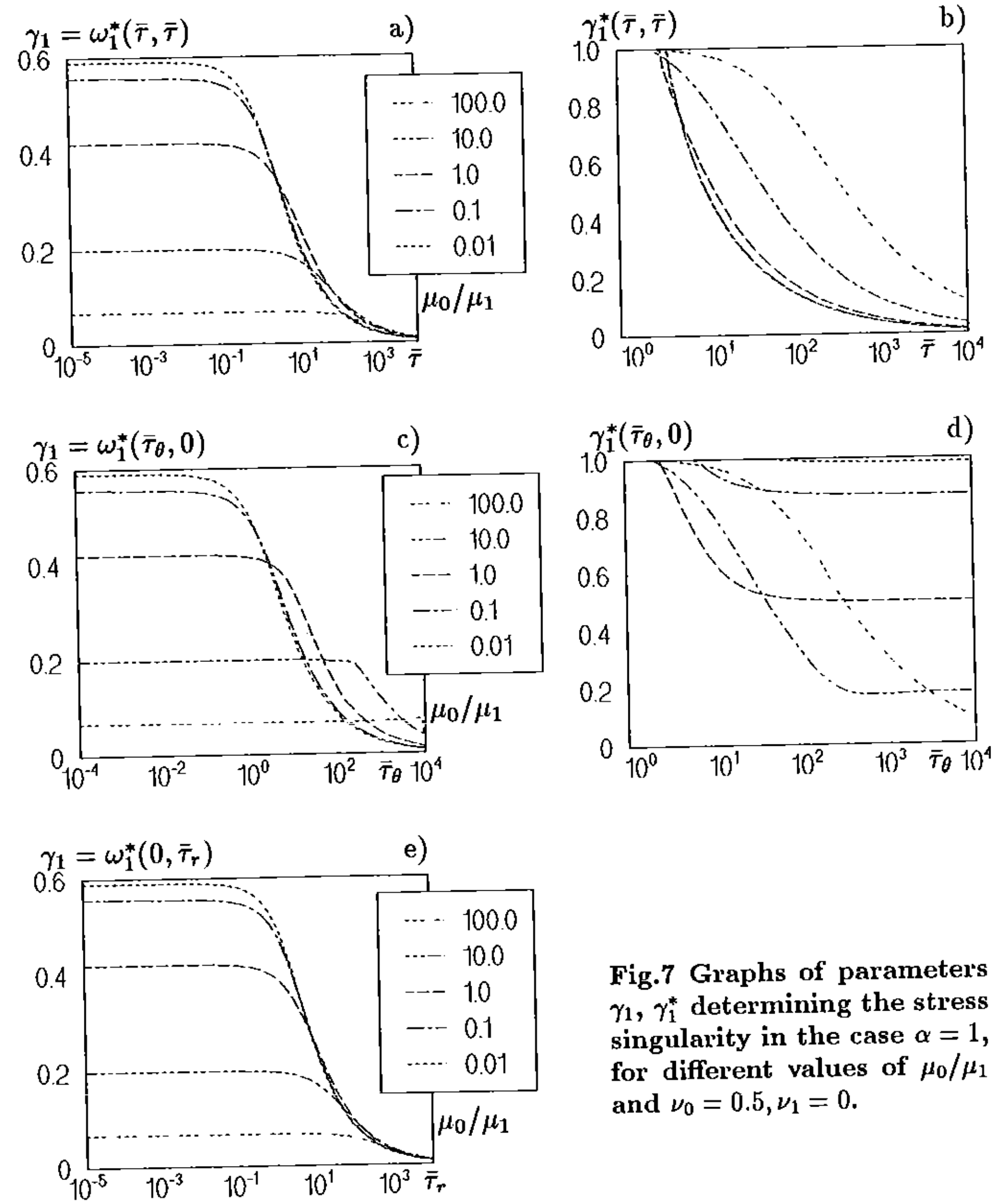


Fig.7 Graphs of parameters γ_1, γ_1^* determining the stress singularity in the case $\alpha = 1$, for different values of μ_0/μ_1 and $\nu_0 = 0.5, \nu_1 = 0$.

As regards to the second real zero γ_1^* of function $\det\Psi_1(s,\bar{\tau}_\theta,\bar{\tau}_r)$, situation changes. Namely, parameters $\bar{\tau}_\theta,\bar{\tau}_r$ influence in the most degree, than the values of Poisson ratios ν_0,ν_1. Thus, in the case $\bar{\tau}_\theta=0$ there exist only one real zero in interval (0,1), it means that $\gamma_1^*\geq 1$.

For small magnitudes of dimensionless parameters $\bar{\tau}_\theta$ and $\bar{\tau}_r$ $(\bar{\tau}_\theta,\bar{\tau}_r<0.1)$ stress singularity is closed to that for "ideal contact". Moreover, in this case only one singular term of stress apears. For large values of these parameters, the second singular term contributes significantly to the stress ditribution near the crack tip.

Let us investigate system of equations (13) in the both remaining cases of interfacial zone geometry ($0\leq\alpha<1$ and $\alpha>1$). We can conclude that vector-functions $p_m(s)$ should be analytic in the strips $\alpha-1<\mathrm{Re}\,s<\omega_1$, and $-\omega_1<\mathrm{Re}\,s<\alpha-1$, respectively. They have simple poles at points $s=\alpha-1$, $s=\omega_1$ $(\alpha\neq 0)$ and $s=-\omega_1$, $s=\alpha-1$, but in case $\alpha=0$ there exist a double pole at point $s=-1$. As for the Mode III problem, it can be proved that systems of functional equations (13) have also unique solutions for the Mode I and Mode II problems for all values of parameter α. They can be found from respective systems of singular integral equations with fixed point singularities.

Analysis of the solutions

Now we discuss asymptotics of displacements and stress fields near the crack tip. Corresponding relations in all cases ($0<\alpha<\infty$) can be written in the following forms:

$$
\begin{aligned}
\sigma_\theta^{(j)} &= K_m f(\theta) r^{\gamma_m-1} + O(r^{\gamma_m^*-1}),\ \ r\to 0,\\
\sigma_{r\theta}^{(j)} &= -K_m(\gamma_m+1)^{-1} f' r^{\gamma_m-1} + O(r^{\gamma_m^*-1}),\ \ r\to 0,\\
\sigma_r^{(j)} &= \frac{K_m}{\gamma_m}\left[\frac{1}{\gamma_m+1}f''+f\right] r^{\gamma_m-1} + O(r^{\gamma_m^*-1}),\ \ r\to 0,\\
u_r^{(j)} &= C_{r0}^{(j)} + \frac{K_m r^{\gamma_m}}{2\gamma_m^2\mu_j}\left[\left[1-\nu_j(1+\gamma_m)\right]f + \frac{1-\nu_j}{\gamma_m+1}f''\right] + O(r^{\gamma_m^*}),\ \ r\to 0,\\
u_\theta^{(j)} &= C_{\theta 0}^{(j)} - \frac{K_m(1-\nu_j)r^{\gamma_m}}{2\gamma_m^2\mu_j(\gamma_m^2-1)}\cdot\left[\left(\frac{\gamma_m(\gamma_m-1)}{1-\nu_j}+(1+\gamma_m)^2\right)f'+f'''\right] + O(r^{\gamma_m^*}),\ \ r\to 0,
\end{aligned}
\qquad (16)
$$

where $C_{r0}^{(1)} = C_{\theta 0}^{(1)} = 0$, but constants K_m $(m = 1, 2)$ are generalized SIF (for the Mode I and Mode II, respectively). In relations (16), $f' = f'_\theta(\theta)$ is the derivative of function $f(\theta) = f_m^{(j)}(\gamma_m, \theta)$ which has different form in each of the domain (j = 0; 1 for $-\pi/2 < \theta < 0$; $0 < \theta < \pi/2$, respectively). Thus, for j = 1 ($\phi = \pi/2 - \theta \in [-\pi/2, \pi/2]$) we obtain:

$$f_1^{(1)} = \cos[(1-\gamma_1)\phi] + B_1 \cos[(1+\gamma_1)\phi],\ f_2^{(1)} = \sin[(1-\gamma_2)\phi] + B_2 \sin[(1+\gamma_2)\phi],$$

$$B_m = (-1)^m \frac{\delta_m \overline{p}_1 + t_m \overline{p}_2}{\overline{p}_1 - t_m \overline{p}_2},\quad t_m = \mathrm{tg}\left[\frac{\pi}{2}(m + \gamma_m - 1)\right],$$

where $\delta_m = (1-\gamma_m)/(1+\gamma_m)$, but the values of $\overline{p}_1, \overline{p}_2$ are calculated by relation $(s+\gamma_m)p_m(s) \sim [\overline{p}_1, \overline{p}_2]^{\mathrm{T}}$, $s \to -\gamma_m$. We do not present here the forms of functions $f_m^{(0)}(\gamma_m, \theta)$ determining the asymptotics in domain $-\pi/2 < \theta < 0$, because they are cumbersome, and the volume of the paper is limited.

In equations (16), constants from *a priori* estimations (6) are calculated taking into account the behaviour of vector-functions $p_m(s)$ by relations (10) with the respective values of parameters: γ_m, $\gamma_\infty(m)$, ω_1, $\omega_m^*(\overline{\tau}_\theta, \overline{\tau}_r)$ instead of γ_3, $\gamma_\infty(3)$, ω_3, $\omega_3^*(\overline{\tau}_3)$.

For the Mode I and Mode II problems, all conclusions drawn for the Mode III problems are also true. Therefore, parameter $\overline{\tau}_3$ should be replaced by parameters $\overline{\tau}_\theta$ and $\overline{\tau}_r$ in the diagram of Fig.4. Besides, the value of the first zero $\omega = \omega_3$ of function $F_3(s)$ is different from the value $\omega = \omega_1 (= \omega_2)$ of the first zero of function $\det F_m(s)$. Differences in contradiction to the Mode III problem only appear in the case of the thin adhesive wedge ($\alpha = 1$), when two real singular terms in asymptotics of stress near the crack tip can arise for the Mode I and II problems, and $\omega_1^* \neq \omega_2^*$.

Besides, parameter B_m in the relation for function $f_m^{(j)}$ depends on tractions along the crack surfaces, in general. And only for the following two cases: 1) $\Lambda = 0$; 2) $\alpha = 1$, $\Lambda \neq 0$ its values do not connect with the loading, and are defined by the mechanical parameters of the problems. The last fact means that two parameters arise in the asymptotics of stress for the "nonideal contact" conditions (K_m, B_m), what contradicts to the situation for the "ideal contact", when only one parameter K_1 (K_2) (generalized SIF) exists for the Mode I (the Mode II) problem.

We do not write here unwieldy expressions for the asymptotics of displacements and stress for the Mode I and Mode II problems in case $\alpha = 0$, when the stress singularity has a logarithmic character.

Conclusions

As it could be expected, geometry of the thin intermediate zone between the different materials (the value of parameter α) essentially influences the stress singularity near the crack tip terminating at the interface. Moreover, this influence has not only qualitative character (different values of the stress singularity near the crack tip), but also a quantitative one (the increase of the number of singular terms in the asymptotics of stress). Besides, when thin adhesive zone is represented by thin wedge ($\alpha = 1$), different values of stress singularity appear for the Mode I and Mode II problems, and two or one singular terms of stress near the crack tip can present (see Figs 5-7). However, in the most of really possible situations ($\bar{\tau}_\theta, \bar{\tau}_r < 0.1$), there is one singular term and the corresponding singularity exponent is a little different with the exponent for "ideal contact".

The asymptotics in this case ($\alpha = 1$) is "not stable". If the value of α is closed, but not equal to 1, the main exponent of stress singularity rapidly changes, and the number of the singular terms of stress is very large (and tends to infinity as $\alpha \to 1$). The last fact can be considered as an additional argument to the opinion, that not only the first but the next terms of stress asymptotics near the defect tip should be taken into account in fracture mechanics analysis.

Let us remember, that in the most cases of the nonideal interface ($\alpha \neq 1$) two parameters (K_m, B_m) in the main singular term of stress appear depending on the loading. This is in contradiction to the situation for the ideal contact, where only one constant (SIF) exist.

With the other hand, stress singularity depends on contact parameters τ_j for the case $\alpha = 1$ only. However, magnitudes of τ_j influence essentially stress intensity factors K_m (and the coefficients in the next singular terms of stress).

We considered in the paper only symmetrical strain-stress states (see (4), (5)). As it is shown in [11] for Mode III, an additional singular term of stress near the crack tip terminating at the nonideal interface appears when loading is not symmetric. This fact contradicts with the situation for ideal interface, where nonsymmetric loading leds to bounded stress field near the crack tip. However, the mentioned above singularity is closed zero in the most important cases ($\bar{\tau}_j < 1$).

Of course, these theoretical results should be experimentally verified. Let us note in this connection that for a real adhesive intermediate zone the value of parameter τ_j is very small, as a rule. Hence, on the distance from the crack tip $r \sim \tau_j$, the asymptotics is rebuilt to that for the "ideal bimaterial contact". This fact should be taken into account when the corresponding experimental results are interpreted.

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