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A Computational Investigation of Crack Growth Initiation in Linear Viscoelastic Material

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ABSTRACT : The increased use of viscoelastic materials for structural applications expected to operate for long periods of time, requires a better understanding of their mechanical behaviour and fracture properties. It has been observed that time dependence is of great importance in determining the rate of crack growth when the material strength becomes an important design parameter. Predicting the initiation and crack growth in such materials is an important step in developing numerical models and for predicting their long term performance. In this paper, a new formulation in the time domain is developed for the displacement and stress analysis of quasi-static linear viscoelastic fracture. We formulate a new constitutive equation in terms of stress and crack opening intensity factors, using correspondence principle by means of Volterra integral equation. The crack growth process is studied by means of a computational approach using a modified path independent integral J. The formulation is incorporated in a finite element software and the algorithm is suited for the time dependent behaviour of cracks in viscoelastic materials.

1. Introduction

The increasing use of viscoelastic materials for structural applications requires a better understanding of their mechanical behaviour including fracture characteristics. The general goal of this investigation is to apply fracture mechanical methods to characterise the long-term behaviour of viscoelastic material under static loads [1].

In the first section we use an incremental formulation in the time domain for the displacement and stress analysis of quasistatic, linear viscoelastic structures. The viscoelastic behaviour of the material is described by a discrete creep spectrum and an incremental formulation [2]. This allows to avoid the retaining of stress history in computer solutions [3,4]. The linear viscoelastic model is integrated into a finite element software. Highly accurate results are obtained.

In the next section we describe the kinematical and mechanical fields around a crack in a linearly viscoelastic material. Crack opening intensity factors [5] are presented in order to formulate a new constitutive equation in terms of stress and crack opening intensity factors using the correspondence principle. The solutions are general with respect to boundary conditions and material properties but quasistatic and isothermal conditions are assumed.

Finally, in the last sections crack growth initiation is presented. The formulation based on a spectral decomposition of the reduced viscoelastic compliance is detailed.
The numerical results are intended to demonstrate the validity of the finite element algorithm, and are compared with analytical solutions [6].

2. Viscoelastic finite element analysis

The deformation of viscoelastic materials are not only related to the instantaneous state of stress but depend on the loading history. According to Boltzmann's equation for a non ageing linear viscoelastic material, the constitutive equations between the components of the stress tensor $\sigma_{ij}(t)$ and the components of the strain tensor $\varepsilon_{ij}(t)$ can be written in terms of creep functions [7]:

$$\varepsilon_{ij}(t) = \int_{-\infty}^{t} J_{ijkl}(t-\tau).\frac{\partial \sigma_{kj}(\tau)}{\partial \tau} d\tau \tag{1}$$

J_{ijkl} designate creep tensor components. The constitutive equations can now be written under the plane stress or strain conditions:

$$\begin{pmatrix} \varepsilon_{11}(t) \\ \varepsilon_{22}(t) \\ 2\cdot\varepsilon_{12}(t) \end{pmatrix} = \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2\cdot(1+\nu) \end{bmatrix} \cdot \int_{0}^{t} \frac{1}{E}(t-\tau)\cdot\frac{\partial}{\partial \tau}\begin{pmatrix} \sigma_{11}(\tau) \\ \sigma_{22}(\tau) \\ \sigma_{12}(\tau) \end{pmatrix} d\tau \tag{2}$$

In equations (2), we have assumed that the Poisson coefficient ν is constant, and the Young's modulus depends on time. The solution procedure for the plane problem involves finding a displacement vector field and stress tensor field that simultaneously satisfy the condition of static equilibrium, the kinematic relation between strains and displacements, and the viscoelastic constitutive equations (2) for all t. To obtain the stresses and strains at any time t_n and to avoid the retaining of stress history in the computer solutions, we use the spectral decomposition technique [3] for the Young's modulus:

$$\frac{1}{E}(t)=\frac{1}{k^{(0)}}+\frac{t}{\eta^{(\infty)}}+\sum_{m=1}^{M}\frac{1}{k^{(m)}}\cdot\left(1-e^{-\lambda^{(m)}\cdot t}\right) \tag{3}$$

where $\lambda^{(m)}=k^{(m)}/\eta^{(m)}$ are strictly positives. This spectral decomposition correspond to a general Kelvin Voigt model as shown in figure 1, $k^{(m)}$ and $\eta^{(m)}$ designate modulus of elasticity and coefficients of viscosity. The solution process of a step-by-step nature is then obtained. Consider the time step $\Delta t_n = t_n - t_{n-1}$ ($n \in 1,..,N$). During each step the load is taken to be constant. The constitutive equations (3) can now be given in terms of increments of stresses and strains (for more details, see reference 3):

$$\begin{pmatrix}\Delta\varepsilon_{11}(t_n)\\ \Delta\varepsilon_{22}(t_n)\\ 2\cdot\Delta\varepsilon_{22}(t_n)\end{pmatrix}=\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2\cdot(1+\nu)\end{bmatrix}\cdot M_n\cdot\begin{pmatrix}\Delta\sigma_{11}(t_n)\\ \Delta\sigma_{22}(t_n)\\ \Delta\sigma_{12}(t_n)\end{pmatrix}+\begin{pmatrix}\tilde{\varepsilon}_{11}(t_n-1)\\ \tilde{\varepsilon}_{22}(t_n-1)\\ 2\cdot\tilde{\varepsilon}_{22}(t_n-1)\end{pmatrix} \tag{4}$$

M_n is the viscoelastic compliance which reflect the amount of creep deformation, where $\tilde{\varepsilon}_{ij}(t_{n-1})$ represent the influence of the complete past history of strains on the components of stresses, and reflect the memory effect of the material.

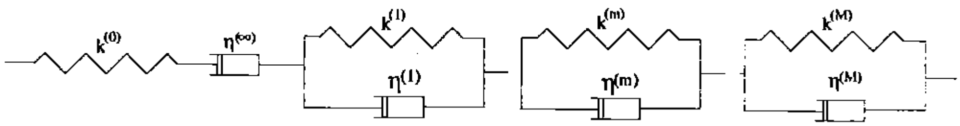

Fig. 1 : Generalised Kelvin Voigt model

The solution procedure in the context of a finite element method is obtained using the principle of virtual work for viscoelasticity; the equilibrium equations can be rewritten as:

$$[K_T]_n\cdot\{\Delta U\}_n=\{\Delta F\}_n+\{\Delta F_{vis}\}_{n-1} \tag{5}$$

$\{\Delta U\}_n$ and $\{\Delta F\}_n$ designate the increment displacement field and the incremental nodal force vector between t_{n-1} and t_n, [B] is the strain-displacement transformation matrix and $[K_T]_n$ is the tangent stiffness matrix, it is given by:

$$\left[K_T\right]_n = \int_\Omega \frac{1}{M_n} \cdot [B]^T \cdot \left[A_o\right] \cdot [B]\, d\Omega \text{ with } \left[A_o\right] = \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2\cdot(1+\nu) \end{bmatrix}^{-1}$$

$\left\{\Delta F_{vis}\right\}_{n-1}$ is the viscous load vector increment corresponding to the complete past history of strains and stresses. It is given by:

$$\left\{\Delta F_{vis}\right\}_{n-1} = \int_\Omega [B]^T \cdot \left\{\underline{\underline{\tilde{\sigma}}}\right\}_{n-1} d\Omega \text{ with } \left\{\underline{\underline{\tilde{\sigma}}}\right\}_{n-1} = \frac{1}{M_n} \cdot \left[A_o\right] \cdot \left\{\underline{\underline{\tilde{\varepsilon}}}\right\}_{n-1}$$

3. Definition of crack tip parameters

3. 1. Mechanical fields

Here we use Brincker's ideas to describe the mechanical fields around the crack tip for a viscoelastic material in plane problems. As pointed by Brincker [8], the state of stress and strain in the vicinity of a crack, in an elastic material, is completely determined by two constants; the stress intensity factors K_γ^e ($\gamma = 1,2$) and are given by:

$$\sigma_{\alpha\beta}^e(r,\theta,t) = K_\gamma^e(2\mu,3k,t) \cdot \frac{f_{\alpha\beta}(\theta)}{\sqrt{2\pi r}}$$

and the displacement field around the crack tip, in the elastic case, is given by:

$$u_\alpha^e(r,\theta,t) = \sqrt{\frac{r}{2\pi}} \cdot \left\{ \frac{1}{2\mu} \cdot g_{\alpha\beta}(\theta) \cdot K_\beta^e + \frac{\lambda(2\mu,3k)}{2\mu} \cdot h_{\alpha\beta}(\theta) \cdot K_\beta^e \right\}$$

where (r,θ) is the polar co-ordinate system, μ is the shear modulus, k is the modulus of compression, $f_{\alpha\beta}$, $g_{\alpha\beta}$ and $h_{\alpha\beta}$ represent the well-known angular functions and λ is defined as:

$$\lambda = (3-4\nu) \text{ in plane strains}$$

$$\lambda = (3-\nu)/(1+\nu) \text{ in plane stresses}$$

The most common way to generalise this development to a viscoelastic material is to derive the viscoelastic solution from the corresponding elastic problem by the so-called correspondence principle [9]. This approach makes use of the observation that the equations of viscoelasticity can be converted to the equations of elasticity by means of Laplace Carson transformation, and thus the viscoelastic solution is then obtained by inversion of the Laplace Carson transform. Note that the elastic constants 2μ and $3k$ must be replaced by the complex functions $R_1^*(p)$ and $R_2^*(p)$, which are Laplace Carson transform isotropic relaxation functions, Brincker [8] and are given by:

$$K_\gamma^{(\sigma)}(t) = \wp_{p\to t}^{-1}\left\{K_\gamma^{e*}\left(R_1^*(p), R_2^*(p)\right)\right\}$$

$$\text{with } K_\gamma^{e*}\left(R_1^*(p), R_2^*(p)\right) = \wp_{t\to p}\left\{K_\gamma^{e}\left(2\mu, 3k, t\right)\right\}$$

$\wp_{t\to p}$ and $\wp_{p\to t}^{-1}$ represent the Laplace Carson transformation and the inverse of Laplace Carson transformation. By applying this transformation to the displacement field, we have :

$$U_\alpha(t) = \wp_{p\to t}^{-1}\left\{U_\alpha^{e*}(p)\right\} \text{ such as } U_\alpha^{e*}(p) = \sqrt{\frac{r}{2\pi}}\cdot\left\{g_{\alpha\beta}\cdot C_\beta^*(p) + h_{\alpha\beta}\cdot D_\beta^*(p)\right\} \quad (6)$$

$$\text{with } C_\beta^* = \frac{K_\beta^{e*}}{R_1^*} \text{ and } D_\beta^* = \frac{K_\beta^{e*}}{R_2^*} \quad (7)$$

and the viscoelastic displacement field is then given by:

$$U_\alpha(t) = \sqrt{\frac{r}{2\pi}}\cdot\left\{g_{\alpha\beta}\cdot C_\beta(t) + h_{\alpha\beta}\cdot D_\beta(t)\right\} \quad (8)$$

C_β and D_β represent the four strain intensity factors.

3. 2 Criterion of crack initiation

The theory of Schapery considers the existence of a failure zone in the vicinity of the crack tip [9]. Consider the two following integrals (see figure 2):

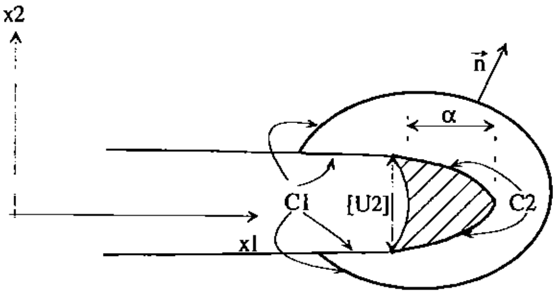

Fig. 2 : Integration domain and failure zone

$$Jv = \int_{C1} \left(W dx_2 - \sigma_{ij} n_j u_{i,1}\right) ds \text{ and } Jf = \int_{C2} \left(W dx_2 - \sigma_{ij} n_j u_{i,1}\right) ds \qquad (9)$$

$$\text{where } \sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$$

The properties of invariance of the line integral provides, taking into account the sense of integration along paths *C1* and *C2*, the relation, [10]:

$$Jv = Jf \qquad (10)$$

If σ_m represents the stress distribution, in an opening mode of crack in the failure zone, the singularity being neglected in this zone, J_f can be defined as being the necessary work W_f to create a supplementary surface crack of length a:

$$J_f = \sigma_m \cdot [U_2] = W_f \qquad (11)$$

$[U_2]$ designates the crack opening displacement. The contour *C1* being arbitrary, except that it starts and ends on the crack faces. Equation (10) is very important because it provides a means of determining indirectly W_f without more hypothesis on the behaviour of the material there (far field solution). So at time t_i, crack growth initiation is determined by:

$$\dot{J}_v(t_i) = W_f$$

In this case, J_v designates energy release rate changes.

3. 3 Crack opening intensity factors

Choose a contour CI such that along this path material points have a linear viscoelastic behaviour. By introducing stress and displacement viscoelastic fields in the expression (9), we demonstrate that :

$$J_v = \sum_{\beta} \frac{K_\beta^{(\sigma)} \cdot \left(C_\beta + D_\beta\right)}{4} \tag{12}$$

The crack opening displacement is given by the difference between the displacement of the top crack face and the bottom one.

$$[U_\alpha](\xi, t) = U_\alpha(\xi, \theta = \pi, t) - U_\alpha(\xi, \theta = -\pi, t) \tag{13}$$

x designates the distance from the considered point to the apparent crack tip. Introducing equation (8) into the relationship (13), the crack opening displacement is determined by:

$$[U_1](\xi, t) = K_2^{(\varepsilon)}(t) \cdot \sqrt{\frac{\xi}{2\pi}} \text{ and } [U_2](\xi, t) = K_1^{(\varepsilon)}(t) \cdot \sqrt{\frac{\xi}{2\pi}} \tag{14}$$

$$\text{with } K_\beta^{(\varepsilon)}(t) = 2 \cdot \left(C_\beta(t) + D_\beta(t)\right) \tag{15}$$

$K_\beta^{(\varepsilon)}$ represent the two opening crack intensity factors, [5]. Using equations (12) and (15), one can find:

$$J_v = \sum_{\beta} \frac{K_\beta^{(\varepsilon)} \cdot K_\beta^{(\sigma)}}{8} \tag{16}$$

By coupling equations (7) and (15), we obtain, using the Laplace Carson transformation, the complex strain intensity factors:

$$K_\beta^{(\varepsilon)*}(p) = C^*(p) \cdot K_\beta^{e*}(p) \tag{17}$$

$$\text{with } C^*(p) = \frac{2 \cdot \left(1 + \lambda\left(R_1^*(p), R_2^*(p)\right)\right)}{R_1^*(p)} = \wp_{t \to p}\{C(t)\}$$

$C(t)$ designates the reduce viscoelastic compliance defined by :

$$C(t) = \frac{1 + \lambda\left(2\mu(t), 3k(t)\right)}{\mu(t)}$$

the inverse transform of equation (17) permits to define the relationship between stress and crack opening intensity factors, [11]:

$$K_\beta^{(\varepsilon)}(t) = \int_{-\infty}^{t} C(t-\tau) \cdot \frac{\partial K_\beta^{(\sigma)}(\tau)}{\partial \tau} \, d\tau \tag{18}$$

4. Spectral decomposition

4. 1 Constitutive equation

To study crack problems in complex structures, it is necessary to use an incremental formulation for the constitutive law. In order to determine the mechanical fields near the crack tip and the crack initiation critical times, it is necessary to use the same spectral decomposition technique but applied on the reduce viscoelastic compliance *C(t)* [11]. Consider the generalised Kelvin Voigt model as shown in figure 3.

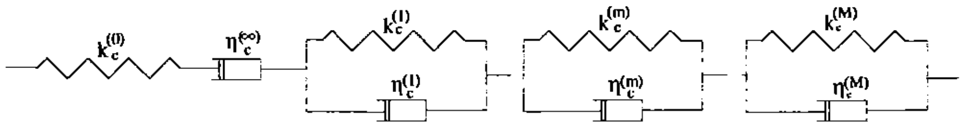

Fig. 3 : Spectral decomposition of C(t)

The input solicitation to this mechanical model is the stress intensity factors $K_\beta^{(\sigma)}$ while the output is the strain intensity factors $K_\beta^{(\varepsilon)}$. Note that the viscoelastic compliance *C(t)* is given in a discrete spectrum such that, [11]:

$$C(t) = \frac{1}{k_c^{(0)}} + \frac{t}{\eta_c^{(\infty)}} + \sum_{m=1}^{M} \frac{1}{k_c^{(m)}} \cdot \left(1 - e^{-\lambda_c^{(m)} \cdot t}\right) \quad \text{with } \lambda_c^{(m)} = \frac{k_c^{(m)}}{\eta_c^{(m)}}$$

In each time step Dt_n ,the solution of the differential equation , obtained by the use of the mechanical model shown above, allows to relate the increment of crack opening intensity factors $\Delta\left(K_\beta^{(\varepsilon)}\right)_n$ to the increment of stress intensity factors $\Delta\left(K_\beta^{(\sigma)}\right)_n$:

$$\Delta\left(K_\beta^{(\varepsilon)}\right)_n = C_n' \cdot \Delta\left(K_\beta^{(\sigma)}\right)_n + \left(\tilde{K}_\beta^{(\varepsilon)}\right)_{n-1} \tag{19}$$

where C_n' represents a reduced compliance viscoelastic function, it depends on the spectral decomposition of the compliance viscoelastic $C(t)$ and the time increment Dt_n :

$$C_n' = \frac{1}{k_c^{(0)}} + \frac{t}{\eta_c^{(\infty)}} + \sum_{m=1}^{M} \frac{1}{k_c^{(m)}} \cdot \left(1 - \frac{1}{\lambda_c^{(m)} \cdot \Delta t_n} \cdot \left(1 - e^{-\lambda_c^{(m)} \cdot \Delta t_n}\right)\right)$$

$\left(\tilde{K}_\beta^{(\varepsilon)}\right)_{n-1}$ reflects the influence of the complete past history of crack opening intensity factors, it is given by:

$$\left(\tilde{K}_\beta^{(\varepsilon)}\right)_{n-1} = \left\{\frac{\Delta t_n}{\eta_c^{(\infty)}} + \sum_{m=1}^{M} \frac{1}{k_c^{(m)}} \cdot \left(1 - e^{-\lambda_c^{(m)} \cdot \Delta t_n}\right)\right\} \cdot \left(K_\beta^{(\sigma)}\right)_{n-1} - \sum_{m=1}^{M} \left(1 - e^{-\lambda_c^{(m)} \cdot \Delta t_n}\right) \cdot \left(K_\beta^{(m)}\right)_{n-1}$$

where :

$$\left(K_\beta^{(m)}\right)_{n+1} = \left(K_\beta^{(m)}\right)_n + \left\{\frac{1}{k_c^{(m)}} \cdot \left(1 - e^{-\lambda_c^{(m)} \cdot \Delta t_n}\right)\right\} \cdot \left(K_\beta^{(\sigma)}\right)_n - \left(1 - e^{-\lambda_c^{(m)} \cdot \Delta t_n}\right) \cdot \left(K_\beta^{(m)}\right)_{n-1} + \left\{\frac{1}{k_c^{(m)}} \cdot \left(1 - e^{-\lambda_c^{(m)} \cdot \Delta t_n}\right) \cdot \left(1 - \frac{1}{\lambda_c^{(m)} \cdot \Delta t_n}\right)\right\} \cdot \Delta\left(K_\beta^{(\sigma)}\right)_n$$

Equation (19) permits to determine the opening crack intensity factors in function of the stress intensity factors. However, for the complete solution of the mechanical problem in an opening crack mode I, it is necessary to calculate one of these two values. Local techniques are often used (static methods or cinematic methods) but they present the disadvantage to calculate displacement or stress fields in the vicinity of the crack tip (singular zone). An alternative method consists in determining the energy release rate using the equation (16). This will be detailed in the next section.

4.2 Energy release rate relation

The use of the integral J_v permits to evaluate the energy release rate over an area on which the material is considered viscoelastic. However, in the finite element method which is used in the solution, the mechanical and kinematical field solution are evaluated in the integration points. It is then necessary to realise an interpolation process so as to project these different fields on the integral line. To avoid this approximation, Destuynder [12], propose to employ another independent path integral, the so-called G_θ integral, and is defined by :

$$J_v = G_\theta = \int_V \left(-W \cdot \theta_{k,k} + \sigma_{ij} \cdot u_{i,k} \cdot \theta_{k,j} \right) \mathrm{dV} \tag{20}$$

The area of integration V is a crown delimited by two contours (see figure 4).

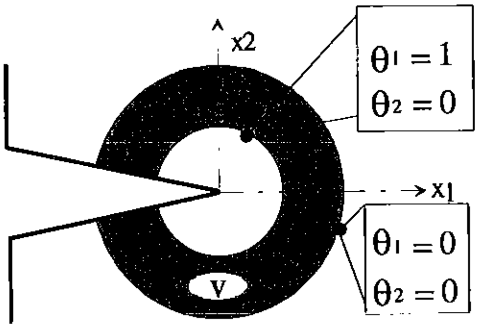

Fig. 4: Integration domain of G_θ

The field $\vec{\theta}$ is continuously derivable such as $(\theta_1 = 1,\ \theta_2 = 0)$ inside the crown and $(\theta_1 = 0,\ \theta_2 = 0)$ outside. Note that the vector $\vec{\theta}$ varies from (1,0) to (0,0) in the band of the crown as described in figure4. This fictitious field permits to evaluate the integral G_θ without using the mechanical fields near the crack tip (far field solution). We note that equation (20) permits to evaluate the integral G_θ in an opening mode. Mixed modes are not allowable.

5. Numerical application

We consider a viscoelastic plate CTT of 500 mm of length, 200 mm of width, under uniform tension of 10 Mpa with a central crack of length 80 mm perpendicular to the direction of loading, as indicated in figure 5.

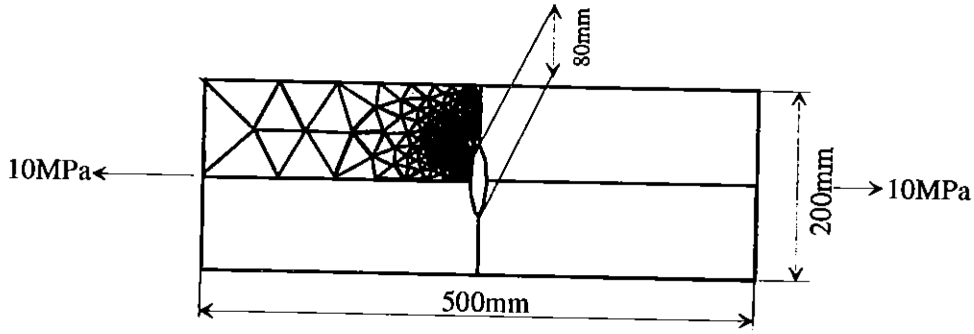

Fig. 5 : Center cracked plate mesh

We use a spectral decomposition of the Young modulus presented in figure 6 as well as a constant Poisson ratio ($\nu = 0.3$).

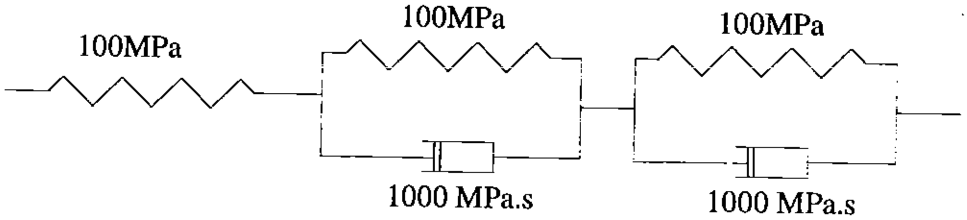

Fig. 6 : Spectral decomposition of 1/E(t)

Numerical results are compared with results obtained by Masuero [6] as well as with the analytical solution. In the case of a creep plane stress state, the stress intensity factor $K_I^{(\sigma)}$ is constant. The elastic value of $K_I^{(\sigma)}$ is $118.7\, N \cdot mm^{-3/2}$. Using equation (18), we can evaluate the variation of the crack opening intensity factor, such that:

$$K_I^{(\varepsilon)} = C(t) \cdot K_I^{(\sigma)}$$

By replacing the spectral decomposition of *C(t)* into the above expression, one find:

$$K_I^{(\varepsilon)} = 10.44 \cdot \left(2 - e^{-t/10}\right) mm^{1/2} \tag{21}$$

As can be seen from equations (16), (21) and (22), the value of the integral J_v is then given by :

$$J_v = 154.84 \cdot \left(2 - e^{-t/10}\right) N / mm$$

Figures 7 and 8 show the evolution of stress and opening crack intensity factors. We can note a good agreement between the numerical results and the analytical solution. Figure 9 shows the variation of the energy release rate. The numerical results, based on G_θ , behave better than the results of Masuero.

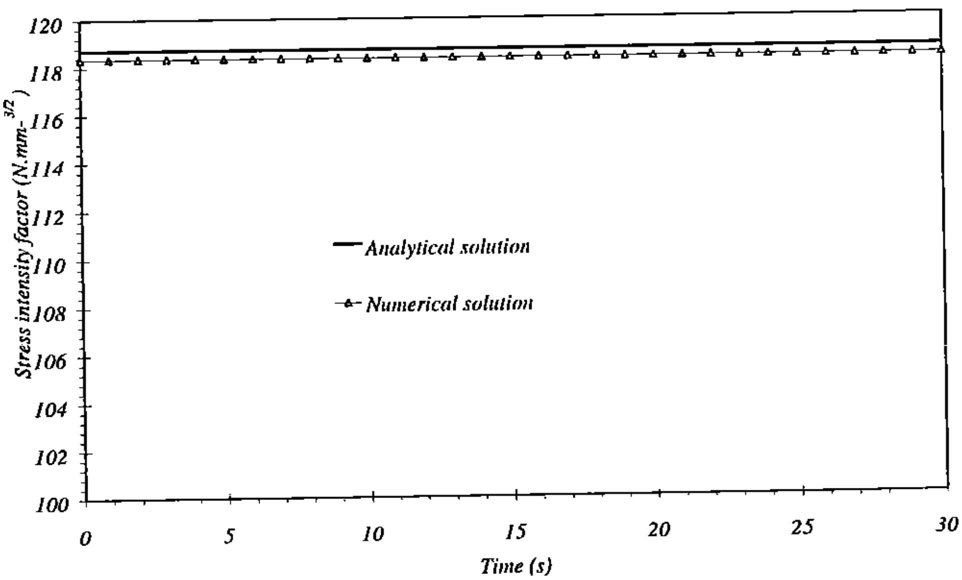

Fig. 7 : Variation of the stress intensity factor $K_I^{(\sigma)}$

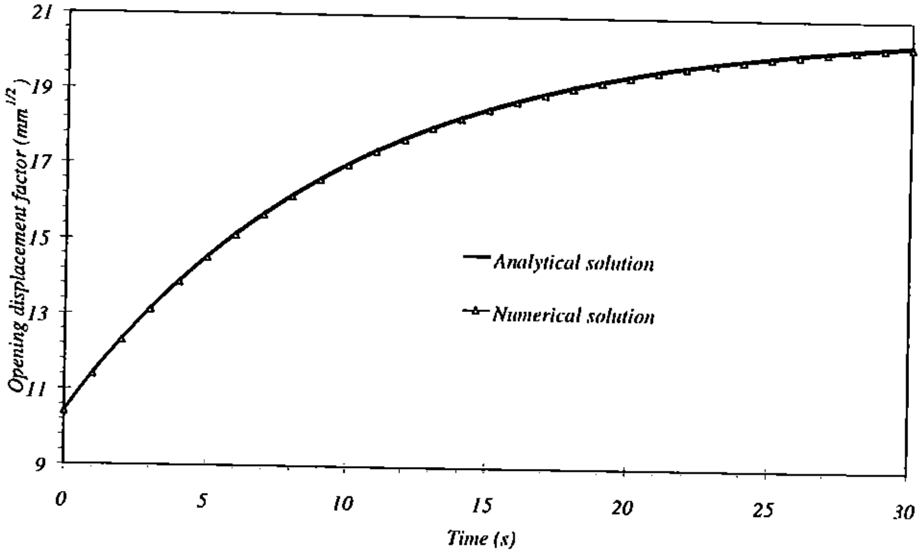

Fig. 8 : Variation of the opening displacement intensity factor $K_I^{(\varepsilon)}$

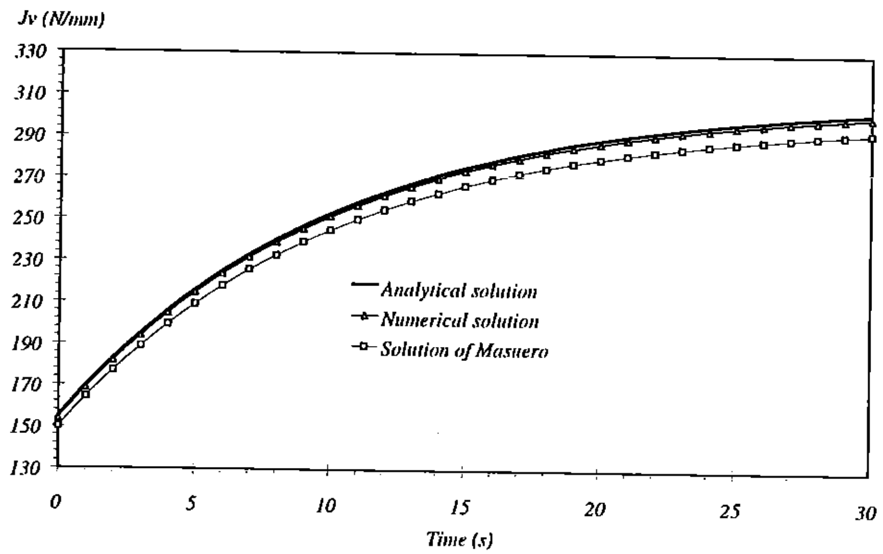

Fig. 9 : Variation of *Jv*

6. Conclusions and perspectives

We present an efficient numerical technique in order to study fracture parameters in a linear viscoelastic material. The model, which is implemented in a finite element program for the analysis of plane structures, appears to be very efficient. This approach, based on a spectral decomposition, can be generalised to anisotropic materials. The incremental solution can be extended to deal with the propagation of cracks in complex structures.

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