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Weakest Link Theory and Multiaxial Criteria

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ABSTRACT: From the weakest link theory, the classical multiaxial criteria, i.e. the principal normal stress criterion, the maximum shear stress criterion and the von Mises criterion, have been derived as special cases. On the basis of this analysis, a general fatigue criterion is formulated for multiaxial stresses. The existing multiaxial criteria of integral approach and of the critical plane approach can be derived as special cases from the general fatigue criterion. On this basis, a new modification of the shear stress intensity hypothesis SIH which provides satisfactory agreement between experimental and calculated results is proposed.

Introduction

A multiaxial stress state which varies with time is generally present at the most severely stressed point in a structural component. As a rule, the multiaxial stress state is of a very complex nature. The individual stress components may vary in a mutually independent manner or at different frequencies, for instance, if the bending and torsional stresses on a shaft are derived from two vibrational systems with different natural frequencies.

For assessing this multiaxial stress, the classical multiaxial criteria, such as the von Mises criterion or the maximum shear stress criterion, are not directly applicable. This is illustrated in **figure 1** for the example of two load cases. In the first case, an alternating normal stress occurs in combination with an alternating shear stress with a phase shift of 90°, **figure 1a**. The second case involves a pulsating normal stress, σ_x, and a compressively pulsating normal stress, σ_y, **figure 1b**. In both load cases, the principal stresses exhibit the same variation with time. In accordance with the classical multiaxial criteria, the same equivalent stresses are calculated in both cases. The endurance limits are very different, however, as shown by experiments [1]. This is explained by the fact that the principal direction can vary in the case of multiaxial fatigue stress. A variable principal direction is not taken into account by the classical fatigue criteria.

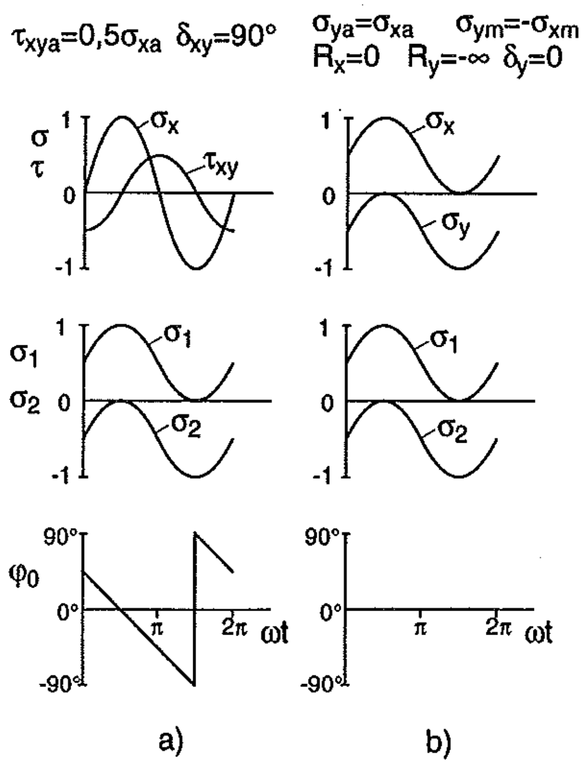

Fig. 1: Two examples of multiaxial stresses and the time histories of principal stresses and principal directions

For calculating the endurance limit in the case of multiaxial stresses, a number of multiaxial criteria have been developed during the past decades [2-12]. These developments are even more comprehensive, as indicated by recent studies [13-17]. The multiaxial criteria differ considerably in formulation, in the range of applicability, and in the reliability of prediction. Furthermore, they also involve highly different physical interpretations, if such an interpretation has been considered and indicated at all in formulating the hypothesis.

As a matter of principle, the known multiaxial fatigue criteria can be subdivided into hypotheses of the critical plane approach, hypotheses of integral approach, as well as empirical criteria. In the case of integral approach, the equivalent stress is calculated as an integral of the stresses over all intersection planes of a volume element; compare with the hypothesis of the effective shear stress [2] and the shear stress intensity hypothesis SIH [3, 11]. In the case of the critical intersection plane approach, only the intersection plane with the critical stress combination is considered, for instance, with the modified shear stress hypothesis proposed by McDiarmid [5].

In the present publication, the weakest link theory is first analysed. Subsequently, the relationship between the weakest link theory and the classical multiaxial criteria is explained. The multiaxial fatigue criteria of the critical intersection plane approach and of integral approach prove to be limiting cases of the weakest link theory. On the basis of this analysis, a general multiaxial criterion is formulated for arbitrary multiaxial stresses. From the general criterion, the known multiaxial criteria can be derived as special cases. Finally, the fatigue behaviour under multiaxial stress is described for a few load cases as examples. The further developed shear stress intensity hypothesis is verified on the basis of test results.

Weakest Link Theory and Classical Multiaxial Criteria

The weakest link theory was originally developed by Weibull for describing the static strength of brittle materials [18]. It was extended by Batdorf for considering the probability of failure of ceramic materials under multiaxial static load [19-21]. The weakest link theory is frequently applied for calculating the probability of failure of ceramic components under multiaxial load [22-24].

In accordance with the weakest link theory, the probability of survival of a component can be described as follows:

$$P_{\ddot{U}} = \exp\left[-\frac{1}{4\pi}\int_V\int_\Omega\left(\frac{\sigma_{\gamma\varphi e}}{\sigma_0}\right)^{\kappa}\cdot d\Omega\cdot dV\right] \qquad (1)$$

where denotes

$\sigma_{\gamma\varphi e}$ - local equivalent stress in the intersection plane of the defect
κ - Weibull's exponent
Ω - the spherical surface area, **figure 2**
V - the volume of the machine component.

The equivalent stress can be calculated as follows:

$$\sigma_V = \left[\frac{1}{4\pi} \int_\Omega \left(\sigma_{\gamma\varphi e} \right)^\kappa \cdot d\Omega \right]^{\frac{1}{\kappa}} \quad (2)$$

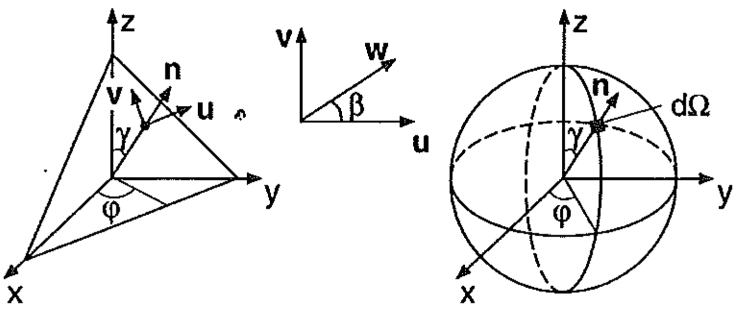

Fig. 2: intersection plane and spherical surface

For an inhomogeneous stress distribution in the volume, a stress integral is employed for calculating the statistical size effect:

$$I = \int_V \left(\frac{\sigma_V}{\sigma_{V\max}} \right)^\kappa \cdot dV \quad (3)$$

The probability of survival can thus be described for the overall system:

$$P_{\ddot{U}} = \exp\left[-I \cdot \left(\frac{\sigma_{V\max}}{\sigma_0} \right)^\kappa \right] \quad (4)$$

The statistical size effect is described by means of equations (3) and (4) [25-27]. In deriving the equations, it has been assumed that failure originates at the interior of the volume with the same probability as for the surface. As a rule, however, failure occurs at the surface; consequently, the surface area A must be inserted into the preceding equations, instead of the volume V.

The local failure criterion, that is, the equivalent stress $\sigma_{\gamma\varphi e}$, must be selected in correspondence with materials. A distinction must be made between ductile and brittle materials. In the case of brittle materials such as ceramics, the defect can be considered to be a crack as a first approximation. The normal stress, which is perpendicular to the crack plane, is decisive for the failure. The normal stress $\sigma_{\gamma\varphi}$ is selected as equivalent stress if the crack is not sensitive to shear stress. The equivalent stress is then given by

$$\sigma_V = \left[\frac{1}{4\pi} \int_{\gamma=0}^{\pi} \int_{\varphi=0}^{2\pi} \left(\sigma_{\gamma\varphi}\right)^{\kappa} \cdot \sin\gamma \cdot d\varphi \cdot d\gamma \right]^{\frac{1}{\kappa}} \tag{5}$$

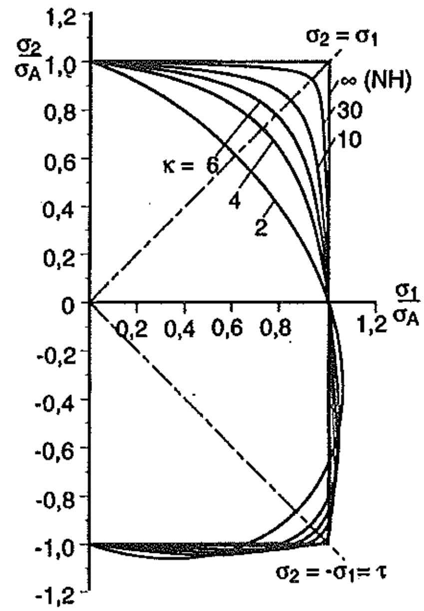

Fig. 3: failure limits with various Weibull exponents by equation (5)

In **figure 3**, the failure limits are plotted with the use of equation (5) for various Weibull exponents κ in the range from 2 to ∞ in an σ_1-σ_2 diagram for the plane stress state. For an infinitely large Weibull exponent, $\kappa \rightarrow \infty$, the same failure limit is obtained as with the principal normal stress criterion. In accordance with the maximum norm of the algebra [28] the equivalent stress given by equation (5) is the major principal stress.

$$\sigma_V \xrightarrow{\kappa \rightarrow \infty} \sigma_{max} \tag{6}$$

For a Weibull exponent, κ=2, an elliptical limit curve is obtained.

$$\sigma_V = \frac{1}{2}\left(3\sigma_x^2 + 2\sigma_x \cdot \sigma_y + 3\sigma_y^2 + 4\tau_{xy}^2\right)^{\frac{1}{2}} \tag{7}$$

The failure limits for various Weibull exponents differ especially for the stress states with biaxial tension, $\sigma_2=\sigma_1$ and pure shear with $\sigma_2=-\sigma_1=\tau$. For the range $\kappa=10\sim30$, the difference from the principle stress criterion is small. At $\kappa=2$, the ratio of the tolerable stresses with reference to the tolerable uniaxial stress σ_A is only 0.61 for biaxial tension, $\sigma_2=\sigma_1$, and 0.866 for pure shear, $\sigma_2=-\sigma_1=\tau$.

For ductile materials, such as steel, the beginning of plastic deformation, that is, the beginning of slip motion of the slip system under shear stress, is usually employed as failure limit. The slip system comprises the direction most densely occupied by atoms (slip direction) in the most densely occupied intersection plane (slip plane). The local failure criterion depends on the orientation of the slip plane $\gamma\varphi$ as well as on the orientation of the slip direction. The shear stress, $\tau_{\gamma\varphi\beta}$, is selected as local equivalent stress. Thus, equation (2) must be extended by integration over the angular range $\beta=0$ to π. The equivalent stress is then given by

$$\sigma_V=\left[\frac{1}{4\pi^2}\int_{\gamma=0}^{\pi}\int_{\varphi=0}^{2\pi}\int_{\beta=0}^{\pi}\left(\tau_{\gamma\varphi\beta}\right)^{\kappa}\cdot\sin\gamma\cdot d\varphi\cdot d\gamma\cdot d\beta\right]^{\frac{1}{\kappa}} \tag{8}$$

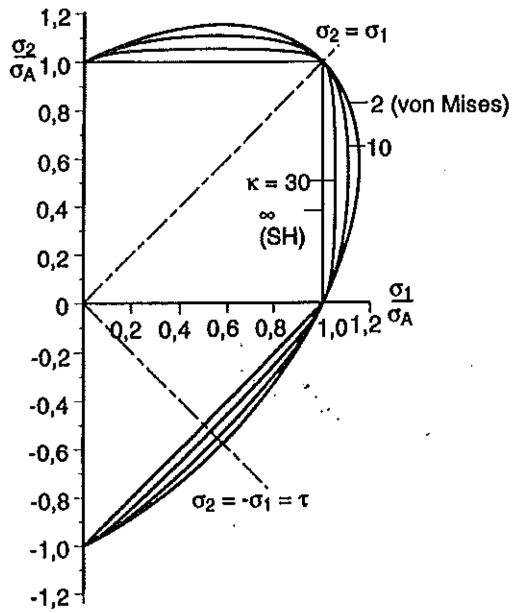

Fig. 4: failure limits with various Weibull exponents by equation (8)

In **figure 4**, the failure limits given by equation (8) for the Weibull exponents from $\kappa=2$ to ∞ are plotted in an σ_1-σ_2 diagram. For infinitely large Weibull exponents, $\kappa\rightarrow\infty$, the resulting failure limit, in accordance with the maximum norm of the algebra [27], is the same as that from the maximum shear stress criterion. At $\kappa=2$, the same failure limit is obtained as from the von Mises criterion. All failure limits with various Weibull exponents are situated between the two limiting curves, $\kappa=2$ and ∞. For the stress state $\sigma_2=\sigma_1$ (biaxial tension), the failure limit is independent of the Weibull exponent. For pure shear, $\sigma_2=-\sigma_1=\tau$, the ratio of the tolerable stress to the uniaxial tolerable stress, σ_A, varies from 0.5 at $\kappa=\infty$ (from the maximum shear stress criterion) to 0.577 at $\kappa=2$ (in correspondence with the von Mises criterion).

The von Mises criterion has been interpreted differently in the past:

- Distortion energy (Maxwell 1856, Hubèr 1904, Hencky 1924)
- Octahedral shear stress (Nadaj 1939)
- Root mean square of the principal shear stresses (Paul 1968)
- Root mean square of the shear stresses for all intersection planes (Novoshilov 1952)

Novoshilov [29] has shown that the root mean square of the shear stresses for all intersection planes is identical with that from the von Mises criterion:

$$\tau_{int} = \left[\frac{1}{4\pi} \int_{\gamma=0}^{\pi} \int_{\varphi=0}^{2\pi} \left(\tau_{\gamma\varphi}\right)^2 \cdot \sin\gamma \cdot d\varphi \cdot d\gamma \right]^{\frac{1}{2}} \cong \left(3I_2'\right)^{\frac{1}{2}} \tag{9}$$

The interpretation given by Novoshilov has led to the development of the hypothesis of effective shear stresses and the shear stress intensity hypothesis [2, 3].

It can be proved that the integration over the angle β is proportional to the resultant shear stress, $\tau_{\gamma\varphi}$:

$$\left[\int_{\beta=0}^{\pi} \left(\tau_{\gamma\varphi\beta}\right)^{\kappa} d\beta \right]^{\frac{1}{\kappa}} \cong \tau_{\gamma\varphi} \tag{10}$$

Hence, the integration over the angle β can be omitted for static loads. The interpretation according to equation (9) can be regarded as a special case of equation (8) with the exponent $\kappa=2$.

The classical multiaxial criteria, the principal normal stress criterion as well as the maximum shear stress criterion and the von Mises criterion, can thus be considered as special cases of the weakest link theory, equation (2). This fact is utilised for formulating a general strength hypothesis in the following section.

General fatigue criterion for multiaxial stress

A multiaxial fatigue criterion must first satisfy the invariance condition; that is, the calculated equivalent stress must be independent of the selected fixed coordinate system with respect to the body. Moreover, for multiaxial fatigue stresses, the criterion must take into account the variable principal stress direction, see **figure 1**. In order to satisfy these conditions, a multiaxial criterion can basically be formulated in two ways:

- as a hypothesis of the integral approach, and
- as a hypothesis of the critical plane approach.

In the case of multiaxial fatigue stresses with a periodically varying stress tensor $\sigma_{ij}(t)$, the stress components can be calculated in an arbitrarily oriented intersection plane at any time. The normal and shear stresses in the intersection plane, which vary with time, are described by mean values and amplitudes. The amplitude and mean value of the normal stress, $\sigma_{\gamma\varphi}$, in the intersection plane and of the shear stress, $\tau_{\gamma\varphi\beta}$, in direction w can be simply calculated from the maximum and minimum during a period, **fig 2**. If the local failure criterion is selected independently of the direction w in the intersection plane, the maxima of $\tau_{\gamma\varphi\beta a}$ and $\tau_{\gamma\varphi\beta m}$ can be employed. Thus, four stress components, $\sigma_{\gamma\varphi a}$, $\tau_{\gamma\varphi a}$, $\sigma_{\gamma\varphi m}$, and $\tau_{\gamma\varphi m}$, are present in the intersection plane, **figure 5**. The calculation of the amplitudes and mean values of the stress components in the intersection plane is described in more detail in [1, 11].

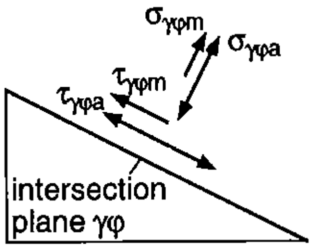

Fig. 5: stress components in an intersection plane

Let $\Sigma_{\gamma\varphi}$ and $T_{\gamma\varphi}$ be two stress components, or two arbitrary combinations of the four stress components in the intersection plane. In the following, the equivalent stress is formulated for $\Sigma_{\gamma\varphi}$ in the sense of the weakest link theory:

$$\Sigma_{\Sigma v} = \left[\int_{\Omega} \left(\Sigma_{\gamma\varphi} \right)^{\mu} \cdot d\Omega \right]^{\frac{1}{\mu}}; \qquad \Sigma_{\gamma\varphi} > 0 \tag{11}$$

In this form, the effect of the stress components which are decisive for damage can be described, for instance, the shear stress amplitude for ductile and "flawless" materials, and the normal stress amplitude for brittle and "defective" materials.

If the exponent μ approaches infinity, the resulting equivalent stress is the maximal stress Σ_{max} in accordance with the maximum norm. In this case, the formulation is applied for the multiaxial criterion of the critical intersection plane approach; accordingly, the stresses in the intersection plane of the maximal stress are decisive for fatigue failure.

If a defined real number is chosen as exponent, equation (11) corresponds to the formulation for the fatigue hypothesis of integral approach. For the sake of simplicity, the exponent is set equal to 2 for the shear stress intensity hypothesis.

Mean stresses alone cannot cause fatigue failure. In the presence of fatigue stress, however, they decrease or increase the tolerable stress amplitude. The effect of stress components such as the mean normal and mean shear stress can be assessed by means of the following formulation:

$$T_{\Sigma v} = \left[\frac{\int_{\Omega} \left(\Sigma_{\gamma\varphi} \right)^{\nu} \cdot \left(T_{\gamma\varphi} \right)^{\mu} \cdot d\Omega}{\int_{\Omega} \left(\Sigma_{\gamma\varphi} \right)^{\nu} \cdot d\Omega} \right]^{\frac{1}{\mu}}; \qquad \Sigma_{\gamma\varphi}, T_{\gamma\varphi} > 0 \tag{12}$$

If μ and ν approach infinity, and ν is much larger than μ, the formulation of the stress component T in the intersection plane corresponds to the maximal stress component Σ.

With the equivalent stresses thus formulated for the stress components, or combinations of same, equations (11) and (12), the failure criterion for multiaxial stress can be established. This fatigue criterion is generally applicable, since all known multiaxial criteria can thus be derived. In the following, this is illustrated for the example of the critical shear stress criterion, as indicated by Nøkleby [6].

In accordance with the criterion of critical shear stress after Nøkleby [6], the critical intersection plane is defined as that with the maximal equivalent stress:

$$\tau_{\gamma\varphi v} = 2\tau_{\gamma\varphi a} + 2\alpha \cdot \sigma_{\gamma\varphi a} + 2\beta \cdot \sigma_{\gamma\varphi m} \tag{13}$$

The failure criterion is given by

$$\begin{aligned}\sigma_V &= c \cdot \max\left\{\tau_{\gamma\varphi v}\right\} \\ &= \sigma_W\end{aligned} \tag{14}$$

where c is a compensation factor which depends on the tensile-compressive fatigue strength. In accordance with the general multiaxial fatigue criterion, the failure criterion can be expressed as:

$$\begin{aligned}\sigma_V &= c \cdot \left[\int_{\Omega} \left(\tau_{\gamma\varphi v}\right)^{\mu} \cdot d\Omega\right]^{\frac{1}{\mu}} \\ &= \sigma_W\end{aligned} \tag{15}$$

with $\mu \rightarrow \infty$.

From the general fatigue criterion, arbitrary fatigue criteria can be formulated. For this purpose, only the exponents must be defined differently, or the stress components and combinations of stress components must be selected differently. For the sake of simplicity, the exponents are set equal to 1, 2, or ∞ in equations (11) and (12).

Further development of the shear stress intensity hypothesis SIH

In the sense of the general fatigue criterion, the shear stress intensity hypothesis, SIH [1, 3, 12], is modified in the following. For the modification, the shear stress amplitude and the normal stress amplitude are evaluated as the integral of the stresses over all intersection planes. The mean shear stress is weighted over the shear stress amplitude, and the mean normal stress over the normal stress amplitude. Thus, an equivalent stress is formed for each of the four stress components in the intersection plane.

$$\tau_{va} = \left\{\frac{15}{8\pi} \int_{\gamma=0}^{\pi} \int_{\varphi=0}^{2\pi} \tau_{\gamma\varphi a}^{\mu_1} \cdot \sin\gamma \cdot d\varphi \cdot d\gamma\right\}^{\frac{1}{\mu_1}} \tag{16a}$$

$$\sigma_{va} = \left\{\frac{15}{8\pi} \int_{\gamma=0}^{\pi} \int_{\varphi=0}^{2\pi} \sigma_{\gamma\varphi a}^{\mu_2} \cdot \sin\gamma \cdot d\varphi \cdot d\gamma\right\}^{\frac{1}{\mu_2}} \tag{16b}$$

$$\tau_{vm} = \left\{ \frac{\int\limits_{\gamma=0}^{\pi} \int\limits_{\varphi=0}^{2\pi} \tau_{\gamma\varphi a}^{\mu_1} \cdot \tau_{\gamma\varphi m}^{\nu_1} \cdot \sin\gamma \cdot d\varphi \cdot d\gamma}{\int\limits_{\gamma=0}^{\pi} \int\limits_{\varphi=0}^{2\pi} \tau_{\gamma\varphi a}^{\mu_1} \cdot \sin\gamma \cdot d\varphi \cdot d\gamma} \right\}^{\frac{1}{\nu_1}} \tag{16c}$$

$$\sigma_{vm} = \left\{ \frac{\int\limits_{\gamma=0}^{\pi} \int\limits_{\varphi=0}^{2\pi} \sigma_{\gamma\varphi a}^{\mu_2} \cdot \sigma_{\gamma\varphi m}^{\nu_2} \cdot \sin\gamma \cdot d\varphi \cdot d\gamma}{\int\limits_{\gamma=0}^{\pi} \int\limits_{\varphi=0}^{2\pi} \sigma_{\gamma\varphi a}^{\mu_2} \cdot \sin\gamma \cdot d\varphi \cdot d\gamma} \right\}^{\frac{1}{\nu_2}} \tag{16d}$$

In the shear stress intensity hypothesis, μ_1 and μ_2 are again set equal to 2. For simplicity, a value of 2 is also selected for ν_1. For evaluating the mean normal stress, the exponent ν_2 is set equal to unity; hence, the difference between a positive and a negative mean stress can be taken into consideration. The failure condition can then be formulated by a combination of equivalent stresses.

$$a\tau_{va}^2 + b\sigma_{va}^2 + m \cdot \tau_{vm}^2 + n \cdot \sigma_{vm} = \sigma_W^2 \tag{17}$$

The coefficients a, b, m, and n are determined by the requirement that the failure criterion can be satisfied for the uniaxial stress state.

$$a = \frac{1}{5}\left[3\left(\frac{\sigma_W}{\tau_W}\right)^2 - 4\right] \tag{18a}$$

$$b = \frac{1}{5}\left[6 - 2\left(\frac{\sigma_W}{\tau_W}\right)^2\right] \tag{18b}$$

$$m = \frac{\sigma_W^2 - \left(\frac{\sigma_W}{\tau_W}\right)^2 \cdot \left(\frac{\tau_{Sch}}{2}\right)^2}{\frac{4}{7} \cdot \left(\frac{\tau_{Sch}}{2}\right)^2} \tag{18c}$$

$$n = \frac{\sigma_W^2 - \left(\frac{\sigma_{Sch}}{2}\right)^2 - \frac{4m}{21}\cdot\left(\frac{\sigma_{Sch}}{2}\right)^2}{\frac{5}{7}\left(\frac{\sigma_{Sch}}{2}\right)} \tag{18d}$$

For this purpose, the characteristic parameters for alternating strength σ_W, pulsating tensile strength σ_{Sch}, alternating torsional strength τ_W, and pulsating torsional strength τ_{Sch} are required.

Since the coefficients a and b cannot be negative, the shear stress intensity hypothesis is applicable in the following range of fatigue strength ratio with respect to the materials:

$$\sqrt{\frac{4}{3}} < \frac{\sigma_W}{\tau_W} < \sqrt{3} \tag{19}$$

For extending the range of applicability, the exponents, μ_1 and μ_2, can be taken larger than 2.

For calculating the equivalent stress in accordance with the SIH, an integration is necessary for the general case. For synchronous multiaxial stresses,

$$\begin{aligned} \sigma_x &= \sigma_{xm} + \sigma_{xa}\cdot\sin\omega t \\ \sigma_y &= \sigma_{ym} + \sigma_{ya}\cdot\sin\omega t \\ \tau_{xy} &= \tau_{xym} + \tau_{xya}\cdot\sin\omega t \end{aligned} \tag{20}$$

the equivalent stresses can be calculated analytically:

$$a\tau_{va}^2 + b\sigma_{va}^2 = \sigma_{xa}^2 + \sigma_{ya}^2 + \left[2 - \left(\frac{\sigma_W}{\tau_W}\right)^2\right]\cdot\sigma_{xa}\cdot\sigma_{ya} + \left(\frac{\sigma_W}{\tau_W}\right)^2\cdot\tau_{xya}^2 \tag{21a}$$

$$\begin{aligned} \tau_{vm}^2 = \frac{1}{21}\cdot\big[&A_{11}\cdot\sigma_{xm}^2 + A_{12}\cdot\sigma_{ym}^2 + A_{13}\cdot\sigma_{xm}\cdot\sigma_{ym} \\ &+ A_{21}\cdot\tau_{xym}^2 + A_{22}\cdot\sigma_{xm}\cdot\tau_{xym} + A_{23}\cdot\sigma_{ym}\cdot\tau_{xym}\big] \end{aligned} \tag{21b}$$

$$\sigma_{vm} = \frac{3}{7}\cdot\left[A_{31}\cdot\sigma_{xm} + A_{32}\cdot\sigma_{ym} + A_{33}\cdot\tau_{xym}\right] \tag{21c}$$

The coefficients A_{ij} depend only on the mutual ratios of the stress amplitudes, σ_{xa}, σ_{ya}, and τ_{xya}. These coefficients are summarised in **table 1**.

i \ j	1	2	3
1	$\frac{4x^2+3y^2-4x\cdot y+7z^2}{x^2+y^2-x\cdot y+3z^2}$	$\frac{3x^2+4y^2-4x\cdot y+7z^2}{x^2+y^2-x\cdot y+3z^2}$	$\frac{-4x^2-4y^2+6x\cdot y-6z^2}{x^2+y^2-x\cdot y+3z^2}$
2	$\frac{7x^2+7y^2-6x\cdot y+36z^2}{x^2+y^2-x\cdot y+3z^2}$	$\frac{10x\cdot z-6y\cdot z}{x^2+y^2-x\cdot y+3z^2}$	$\frac{-6x\cdot z+10y\cdot z}{x^2+y^2-x\cdot y+3z^2}$
3	$\frac{5x^2+y^2+2x\cdot y+4z^2}{3x^2+3y^2+2x\cdot y+4z^2}$	$\frac{x^2+5y^2+2x\cdot y+4z^2}{3x^2+3y^2+2x\cdot y+4z^2}$	$\frac{8(x+y)\cdot z}{3x^2+3y^2+2x\cdot y+4z^2}$

Table 1: coefficients for the estimation of the equivalent stresses according to equation (21); with $x=\sigma_{xa}$, $y=\sigma_{ya}$ and $z=\tau_{xya}$

Endurance behaviour under multiaxial stress

On the basis of a few load cases, the endurance behaviour is analysed and compared with calculation. More detailed descriptions are presented in [12] with the older formulation of the shear stress intensity hypothesis SIH.

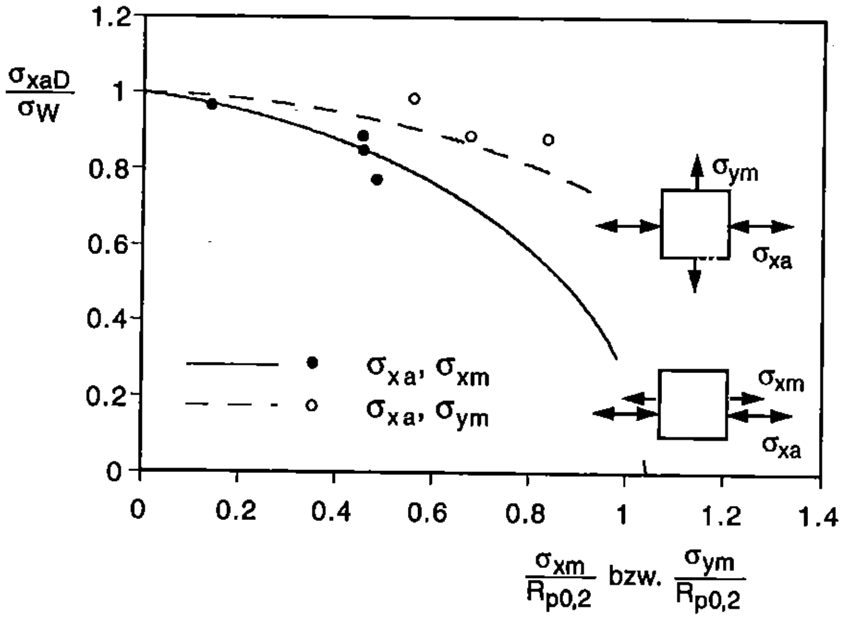

Fig 6: effect of a static normal stress on the endurance limit of an alternating normal stress

First, the effect of the mean stress is considered. This is described for the load case of an alternating normal stress with a static normal stress. If the direction of the static normal stress is the same as that of the alternating normal stress, the tolerable stress amplitude can be calculated from equations (20) and (21):

$$\sigma_{xaD} = \sqrt{\sigma_W^2 - \frac{4m}{21}\sigma_{xm}^2 - \frac{5n}{7}\sigma_{xm}} \tag{22}$$

If the static normal stress is perpendicular to the alternating normal stress, the tolerable stress amplitude is given by

$$\sigma_{xaD} = \sqrt{\sigma_W^2 - \frac{m}{7}\sigma_{ym}^2 - \frac{n}{7}\sigma_{ym}} \tag{23}$$

In **figure 6**, equations (22) and (23) are compared with the test results in the Haigh diagram. Accordingly, a mean normal stress which is perpendicular to the alternating normal stress is less detrimental than one which is parallel with the alternating normal stress.

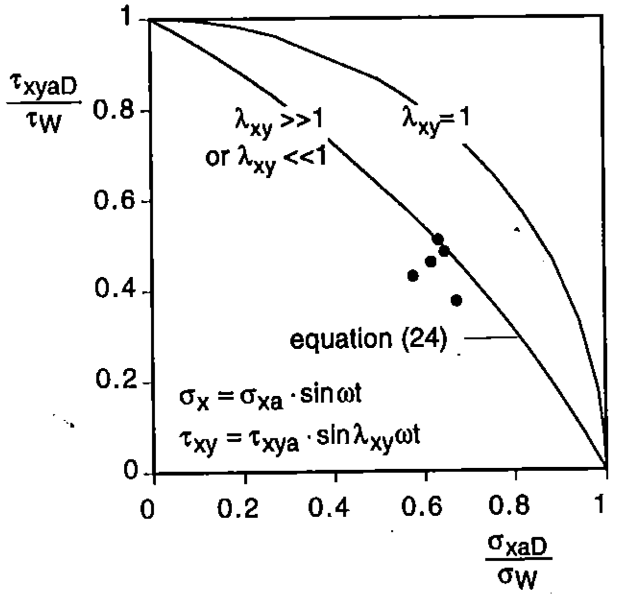

Fig. 7: fatigue limit curves for an alternating normal stress and an alternating torsional stress with different frequencies

An interesting load case is that of a normal stress and a shear stress of different vibrational frequencies. For instance, this case occurs with a shaft under bending and torsional load, if the natural frequencies of the two vibrations are different. For this case, a simple, explicit, approximate equation can be derived with the SIH [11], if the frequencies differ sufficiently.

$$\left(\frac{\sigma_{xa}}{\sigma_W}\right)^2 + \left(\frac{\tau_{xya}}{\tau_W}\right)^2 + \left(\frac{13}{\tau_W^2} - \frac{4}{\sigma_W^2}\right) \cdot \frac{\sigma_{xa} \cdot \tau_{xya}}{5\pi} = 1 \tag{24}$$

In **figure 7**, equation (24) is compared with corresponding test results. Accordingly, equation (24) is applicable for a frequency ratio $\lambda_{xy} > \text{or} < 2$.

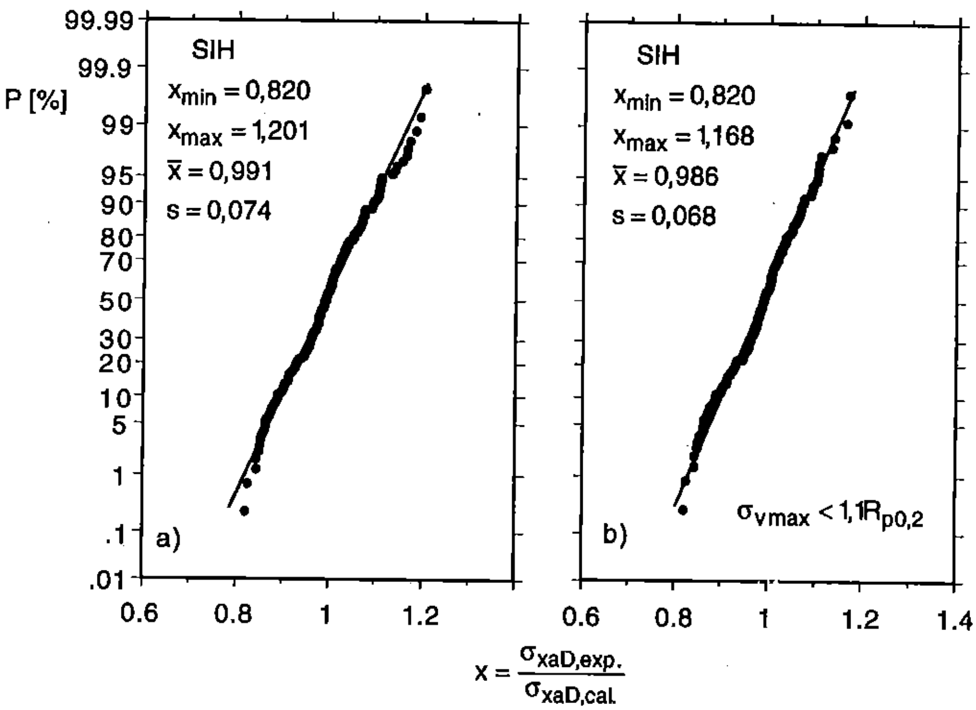

Fig. 8: statistical distributions of the ratio of experimentally determined fatigue limit and the calculated fatigue limit

Finally, the improved shear stress intensity hypothesis is verified on the basis of the data compiled in [11]. A total of 214 test series for multiaxial load cases with superimposed mean stresses, with phase shifts, with various vibration forms and with frequency differences among the stress components have been recorded in this data base. The ratio of the experimentally determined endurance limit to the value calculated with the SIH has been computed. The statistical distribution of this ratio is plotted on a Gaussian probability grid, **figure 8**. On the average, the ratio is close to unity and exhibits relatively low scatter (standard deviation: s=0.074), **figure 8a**. If the static failure limit is taken into account, and the maximal equivalent stress is limited to $\sigma_{vmax} < 1.1\ R_{p0,2}$, the scatter is decreased even further (to s=0.068), **figure 8b**.

As further modifications of the SIH, the equivalent stress of the shear stresses, equations (16a) and (16c), can be extended by integration over the angle ß, or larger values can be selected for the exponents μ_1, μ_2, ν_1, and ν_2. However, this does not appreciably affect the accuracy of prediction.

Conclusions

From the weakest link theory, the classical multiaxial criteria, the principal normal stress criterion, the maximum shear stress criterion and the von Mises criterion, have been derived as special cases. On the basis of this analysis, a general fatigue criterion is formulated for multiaxial stresses. The existing multiaxial criteria of integral approach and of the critical plane approach can be derived as special cases from the general fatigue criterion. On this basis, a new modification of the shear stress intensity hypothesis SIH which provides satisfactory agreement between experimental and calculated results is proposed.

An application of the proposed fatigue hypothesis to the service loading is also possible, but has not yet been attempted. For this purpose, an examination is necessary, especially to determine the extent to which the elastic-plastic cyclic deformation can and must be taken into consideration in the strength analysis.

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