Andrea CARPINTERI (*), Ewald MACHA (**),
Roberto BRIGHENTI (*),Rafat BRYŚ (**), Andrea SPAGNOLI (*)

Expected Principal Stress Directions under Multiaxial Random Loading through Weight Function Method

(*) Dept. Civil Engineering, University of Parma, Viale delle Scienze, Parma, Italy
(**) Dept. Mechanics and Machine Design, Technical University of Opole, Opole, Poland

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ABSTRACT: According to the critical plane approach, the expected fracture plane needs to be determined in order to calculate the fatigue life of a body under multiaxial random loading. As experimentally observed by many authors, the position of such plane strongly depends on the directions of the principal stresses or strains. In the present paper, the expected principal stress directions are obtained by averaging the instantaneous values of the three Euler angles through some suitable weight functions which can take into account the main factors influencing the fatigue fracture behaviour. A numerical simulation is presented to illustrate the proposed procedure.

Notation

A	matrix of the principal direction cosines
a_i, b_i, c_i, d_i	coefficients of the i-th one-dimensional recursive digital filter
c	constant coefficient, with $0 < c \leq 1$
f	generic frequency
$f_{\max 11} \cdots f_{\max 66}$	maximum frequencies
$G(f)$	power spectral density functions
l_n, m_n, n_n	principal direction cosines (eigenvectors of the stress tensor)
$m_\sigma = -1/m$	coefficient depending on the slope m of the S-N curve
N	number of time instants being considered
$N_{t_k}^{-1}$	reciprocal of the fatigue life

$N_{fc} = N_f / (c)^{m_\sigma}$	constant for the S-N curve
PDF	joint probability density function
PSDF	power spectral density function
$R_x(\tau)$	correlation matrix
$S_{ij}(f)$	two-sided PSDF
$t_1, t_2, \ldots, t_k, \ldots, t_N$	time instants
$X(t) = [X_1(t), \ldots, X_6(t)]$	six-dimensional vectorial process
$\hat{x} = [\hat{x}_1, \ldots, \hat{x}_6]$	mean value vector
λ, μ, ν, ρ	Euler-Rodriguez parameters
$\mu_x(\tau)$	covariance matrix
$\sigma_{xx}(t), \sigma_{yy}(t), \sigma_{zz}(t)$	normal stresses
$\sigma_{xy}(t), \sigma_{xz}(t), \sigma_{yz}(t)$	shear stresses.
σ_n	principal stresses (eigenvalues of the stress tensor)
σ_{af}	fatigue limit stress
ϕ, θ, ψ	Euler angles
$W(t_k)$	weight function

Introduction

Numerous models of fatigue crack initiation and propagation under multiaxial loadings do not consider changes of the principal stress and strain directions. Such models, introduced by Irwin [1], Brown and Miller [2] and Socie [3], concern only some simple cases of loadings. This is a reason why they may be applied for some particular cases under cyclic loading, while they are not applicable to the case of multiaxial random loading, where various damage mechanisms are mixed.

One group of multiaxial fatigue criteria is based on the critical plane approach [4]. In order to calculate the fatigue life, we must know the critical or expected fracture plane. From the review of many test results obtained under multiaxial stress state, caused by cyclic in- and out-of-phase loadings, it appears that the fatigue fracture plane position strongly depends on the directions of the maximum principal stress or strain and the maximum shear stress or strain [5,6]. However, changes of the principal axes positions are often ignored, and directions of stresses or strains with the maximum amplitudes or ranges are groundlessly assumed as directions of principal stresses or strains.

Under multiaxial random loading, predominant damage mechanisms and factors influencing the fatigue fracture plane position can be expressed through averaging, i.e. evaluation of some statistical parameters. The expected or critical fatigue fracture plane position can be determined through the weight function method, which consists in averaging the instantaneous values of the parameters determining the position of the principal stress or strain axes through some suitable weight functions [6,7]. The 3 × 3 matrix of direction cosines can be obtained, and its elements are used to determine the expected position of the fatigue fracture plane. However, no averaging procedure gives, in a general case, an orthogonal matrix because only 3 out of 9 direction cosines are independent. It is difficult to say which 3 out of 9 parameters should be averaged [6,7,8]. In the present paper the authors are going to avoid the controversial problem of selecting 3 independent parameters by averaging the 3 Euler angles. A numerical simulation is presented as an example of the proposed pocedure.

Random stress state

A multiaxial random stress state can be expressed by a six-dimensional vectorial process:

$$X(t) = [X_1(t), \ldots, X_6(t)] \qquad (1)$$

where $X_i(t), i = 1, \ldots, 6$, are unidimensional stochastic processes representing the components of the stress tensor in the following order

$\sigma_{xx}(t), \sigma_{yy}(t), \sigma_{zz}(t)$ normal stresses

$\sigma_{xy}(t), \sigma_{xz}(t), \sigma_{yz}(t)$ shear stresses.

As an example, Figure 1 shows a fragment of the single time history of the random stress $\sigma_{xx}(t) = X_1(t)$ generated by means of a numerical simulation, as is illustrated at the end of the present Section.

According to the correlation theory, the stationary and ergodic vectorial process $X(t)$ is usually described through its mean value, $\hat{x} = [\hat{x}_1, \ldots, \hat{x}_6]$, and its correlation matrix, $R_x(\tau)$, or covariance matrix, $\mu_x(\tau)$, with $\tau = t_k - t_h$, for $h, k = 1,2,\ldots,N$ and $h < k$, where N is the number of time instants being considered.

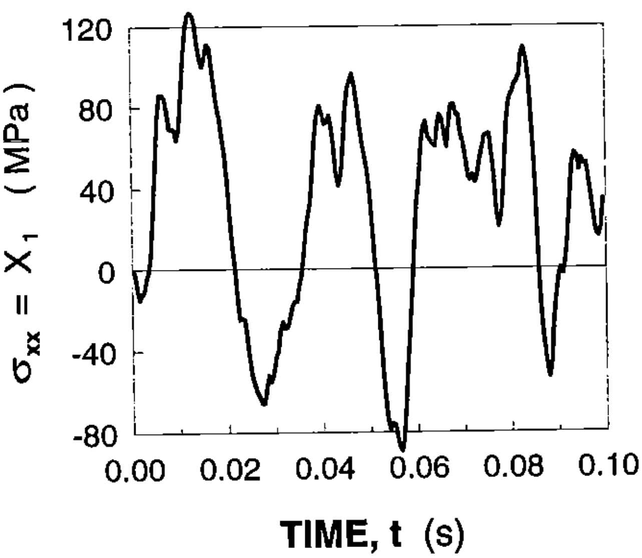

Fig. 1 Time history of the random stress $\sigma_{xx}(t)$ generated by a numerical simulation.

By assuming the above process to be Gaussian, the joint probability density function (PDF) is given by [9]

$$f_{x_1,\ldots,x_6}(x_1,\ldots,x_6,\tau) = \frac{1}{\sqrt{(2\pi)^6 \cdot |\mu_x(\tau)|}} \exp[-0.5 \cdot \sigma_x \cdot \mu_x^{-1}(\tau) \cdot \sigma_x^T] \tag{2}$$

where

$$\mu_x(\tau) = \begin{bmatrix} \mu_{x11}(\tau) & \ldots & \mu_{x16}(\tau) \\ \ldots & \ldots & \ldots \\ \mu_{x61}(\tau) & \ldots & \mu_{x66}(\tau) \end{bmatrix}$$

$$\sigma_x = [x_1 - \hat{x}_1, \ldots, x_6 - \hat{x}_6]$$

$\mu_x(\tau)$ = positive definite covariance matrix of random variables $X_1, \ldots, X_6$

$|\mu_x(\tau)|$ = determinant of the matrix $\mu_x(\tau)$

$\mu_x^{-1}(\tau)$ = inverse of covariance matrix $\mu_x(\tau)$

σ_x = row vector of variables $x_1, \ldots, x_6$ and mean values $\hat{x}_1, \ldots, \hat{x}_6$

σ_x^T = column vector (σ_x transposed).

By carrying out spectral analysis [10,11], we are mainly concentrated on power distribution, i.e. distribution of mean square values of amplitudes of particular harmonic components appearing in the random process, and on frequency band width. Probabilistic relations between random components of the vectorial process are also important. These properties are expressed with power spectral density functions (PSDF) $G(f)$, which give a rectangular 6 x 6 matrix for the random tensor

$$G(f)=\begin{bmatrix} G_{11}(f) & \dots & G_{61}(f) \\ \dots & \dots & \dots \\ G_{61}(f) & \dots & G_{66}(f) \end{bmatrix} \tag{3}$$

One-sided PSDFs $G_{ij}(f)$, $i, j = 1, \dots, 6$, of stress state components are determined for frequency $f \geq 0$ and they are equal to double values of two-sided PSDFs $S_{ij}(f)$:

$$G_{ij}(f)=\begin{cases} 2S_{ij}(f) & \text{for } 0 \leq f < \infty \\ 0 & \text{for } f < 0 \end{cases} \tag{4}$$

where

$G_{ii}(f), S_{ii}(f)$ = autospectral density function of stresses $X_i(t)$

$G_{ij}(f), S_{ij}(f)$ = cross-spectral density function between stresses $X_i(t)$ and $X_j(t)$.

The cross PSDFs are complex functions

$$G_{ij}(f) = \mathrm{Re}\left[G_{ij}(f)\right] + \mathrm{Im}\left[G_{ij}(f)\right] \tag{5}$$

where

$\mathrm{Re}\left[G_{ij}(f)\right]$ = coincident spectral density function, a real part of the complex function $G_{ij}(f)$

$\mathrm{Im}\left[G_{ij}(f)\right]$ = quadrature spectral density function, an imaginary part of the complex function $G_{ij}(f)$.

In the following, six ergodic and stationary random components of the stress tensor are generated by means of a numerical simulation. The generic tensor element $x_i(t)$, with i = 1, ..., 6 and $t = t_1, t_2, \dots, t_k, \dots, t_N$, is obtained by filtering a sequence of random numbers $y_i(t)$ through the following one-dimensional recursive digital filter:

$$x_i(t_k) = a_i\, y_i(t_k) + b_i\, y_i(t_{k-1}) + c_i\, x_i(t_{k-1}) + d_i\, x_i(t_{k-2}) \tag{6}$$

The above coefficients a_i, b_i, c_i, d_i are determined in order to produce a sequence X_i with normal probability distribution $\mathcal{N}(0,\mu)$ (that is, expected value and variance are equal to zero and μ, respectively) and with desired one-sided autospectral density function G_{ii} (element of the principal diagonal in matrix (3)) [12] given by:

$$G_{ii}(\omega) = \frac{4\mu}{\omega_{ii}^{*}\left[1+\left(\frac{\omega}{\omega_{ii}^{*}}\right)^{2}\right]^{2}} \tag{7}$$

where $G_{ii}(f) = 2\,\pi\, G_{ii}(\omega)$ and $\omega = 2\,\pi f$, f = generic frequency. The parameter ω_{ii}^{*} is connected with the maximum frequency value, $f_{\max,ii}$, taken into account when numerically generating the sequence $x_i(t)$. More precisely, we calculate ω_{ii}^{*} by assuming

$$G_{ii}(\omega_{\max,ii}) = 0.05\; G_{ii}(0), \tag{8}$$

where $\omega_{\max,ii} = 2\,\pi f_{\max,ii}$.

Principal stress directions through Euler angles

As is well known, the principal stresses, σ_n, n = 1, 2, 3, correspond to the eigenvalues of the stress tensor, whereas the eigenvectors represent the principal direction cosines l_n, m_n, n_n, n = 1, 2, 3 (Fig.2). Assume to arrange the eigenvalues in the following order: $\sigma_1 \geq \sigma_2 \geq \sigma_3$; that is, the directions of maximum and minimum principal stresses are called 1-axis and 3-axis, respectively. The matrix A of the principal direction cosines

$$A = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \tag{9}$$

consists of nine elements, but only three of them are independent because of six orthogonality conditions.

The orthogonal coordinate system P123 with origin at the generic point P, for which the stress tensor is known, and axes coincident with the principal directions (Fig.2) can also be described through the Euler angles ϕ, θ, ψ which represent three counterclockwise rotations around Z-axis, Y'-axis and 3-axis, sequentially. Analogously to the case of the direction cosines, we only need three parameters to define the principal stress directions.
If the above Euler angles are given, the principal direction cosines can be written as follows:

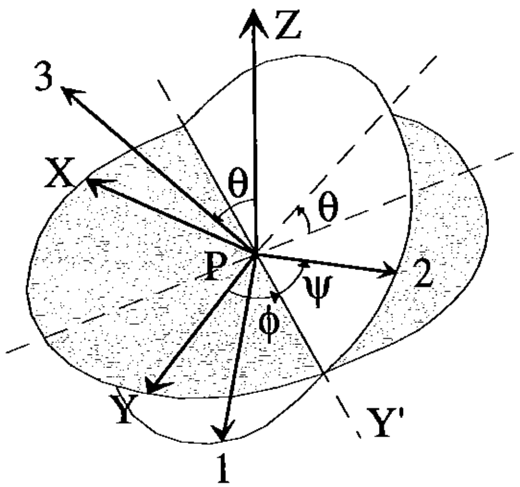

Fig. 2 Principal stress directions 123 described through the Euler angles ϕ, θ, ψ.

$$A = \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix} \qquad (10)$$

where s and c correspond to sin and cos, respectively, while the subscripts represent the arguments of such trigonometric functions.
The procedure to obtain the Euler angles from the components of matrix A (see eqn (9)) is quite simple, even if some calculation steps are needed. First of all, the following quantities must be computed [13]:

$$\chi = arc\ cos\frac{1}{2}(l_1 + m_2 + n_3 - 1)$$
$$u_1 = \frac{m_3 - n_2}{2\ sin\,\chi}, \qquad u_2 = \frac{n_1 - l_3}{2\ sin\,\chi}, \qquad u_3 = \frac{l_2 - m_1}{2\ sin\,\chi} \tag{11}$$

Then the Euler - Rodriguez parameters can be deduced:

$$\lambda = u_1\ sin\frac{\chi}{2} \qquad \mu = u_2\ sin\frac{\chi}{2}$$
$$\nu = u_3\ sin\frac{\chi}{2} \qquad \rho = cos\frac{\chi}{2} \tag{12}$$

Note that the parameter χ is not univocally identified from the first expression in eqs (11), since $\chi \pm 2\,\pi\,k$ and $-\chi \pm 2\,\pi\,k$ (with k = natural number) present the same value of the function arc cos. Nevertheless, the representation (12) is not ambiguous, since only one set of Euler - Rodriguez parameters can be obtained from all the different results determined for χ.

Finally, the values of the Euler angles are derived from the following expressions:

$$\varphi = arc\ tg(\nu/\rho) - arc\ tg(\lambda/\mu)$$
$$\psi = arc\ tg(\nu/\rho) + arc\ tg(\lambda/\mu) \tag{13}$$
$$\theta = arc\ sin(m_3 / sin\ \varphi)$$

Averaging process

Since every component $\sigma_{ij}(t)$ of the stress tensor is a random function of time t, we can determine the above Euler angles $\phi(t)$, $\theta(t)$ and $\psi(t)$ at each time instant t, with $t = t_1, t_2, \ldots, t_k, \ldots, t_N$. By assuming that the expected position $(\hat{\phi}, \hat{\theta}, \hat{\psi})$ of the principal axes is influenced by their generic position ($\phi(t)$, $\theta(t)$, $\psi(t)$) in the same way for any value of t, independently of the stress values, the mean principal stress directions $\hat{1}, \hat{2}$ and $\hat{3}$ can be obtained from simple arithmetic averages:

$$\hat{\phi} = \frac{1}{N}\sum_{t_1}^{t_N}\phi(t_k), \quad \hat{\theta} = \frac{1}{N}\sum_{t_1}^{t_N}\theta(t_k), \quad \hat{\psi} = \frac{1}{N}\sum_{t_1}^{t_N}\psi(t_k) \tag{14}$$

Since the expected fatigue fracture plane position may be assumed to depend on the mean directions of the principal stress axes $\hat{1},\hat{2}$ and $\hat{3}$ [5,6,8], it seems logic from a physical point of view to carry out the averaging of Euler angles by employing suitable weight functions, $W(t_k)$, to take into account the main factors influencing the fatigue fracture behaviour:

$$\hat{\phi} = \frac{1}{W}\sum_{t_1}^{t_N}\phi(t_k)W(t_k) \qquad \hat{\theta} = \frac{1}{W}\sum_{t_1}^{t_N}\theta(t_k)W(t_k)$$
$$\hat{\psi} = \frac{1}{W}\sum_{t_1}^{t_N}\psi(t_k)W(t_k) \qquad W = \sum_{t_1}^{t_N} W(t_k) \tag{15}$$

If the above weight function is expressed as follows:

$$W_1(t_k) = 1, \quad \text{for each} \quad t_k \in [t_1, t_2, \ldots, t_N], \tag{16}$$

the sum, W, of the weights is equal to the number, N, of realizations of the stress tensor. Consequently, the weighted mean values of the Euler angles obtained from eqs (15) for $W_1(t_k)$ coincide with the arithmetic averages, given by eqs (14).

Consider the following weight function:

$$W_2(t_k) = \begin{cases} 0 & \text{if } \sigma_1(t_k) < c\,\sigma_{af} \\ \left[\sigma_1(t_k)/(c\,\sigma_{af})\right]^{m_\sigma} & \text{if } \sigma_1(t_k) \geq c\,\sigma_{af} \end{cases} \qquad 0 < c \leq 1 \tag{17}$$

It only includes into the averaging process, described by eqs (15), those positions of principal axes for which the maximum principal stress σ_1 is greater than or equal to the product of the constant coefficient c, with $0 < c \leq 1$, and the fatigue limit stress, σ_{af}, deduced from the S-N curve plotted for loading ratio, R, equal to -1 (continuous thick line in Fig.3). The weight of such positions, which is defined in eqs (17), exponentially depends on the coefficient $m_\sigma = -\frac{1}{m}$, where m is the slope of the S-N curve being considered.

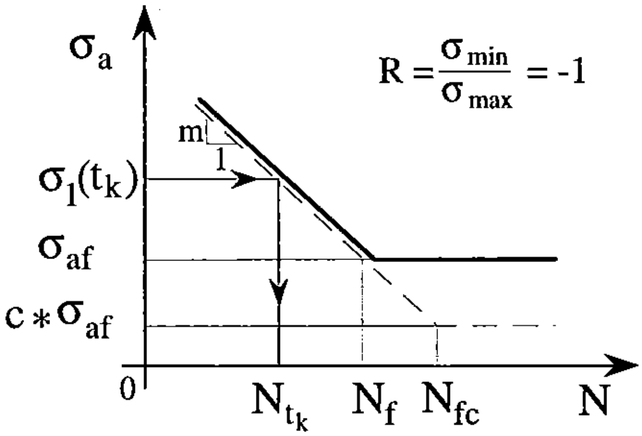

Fig. 3 The Wöhler curve.

The above weight functions are plotted in Fig.4. In particular, Figure 4(b) shows the constant weight W_1, whereas Figures 4(c) and 4(d) represent the function W_2 for $c = 1$ and $c < 1$, respectively. Finally, the function $\phi * W_i$, with $i = 1,2$, to be averaged to determine the expected Euler angle $\hat{\phi}$ according to eqs (15) is displayed in Fig.4(e).

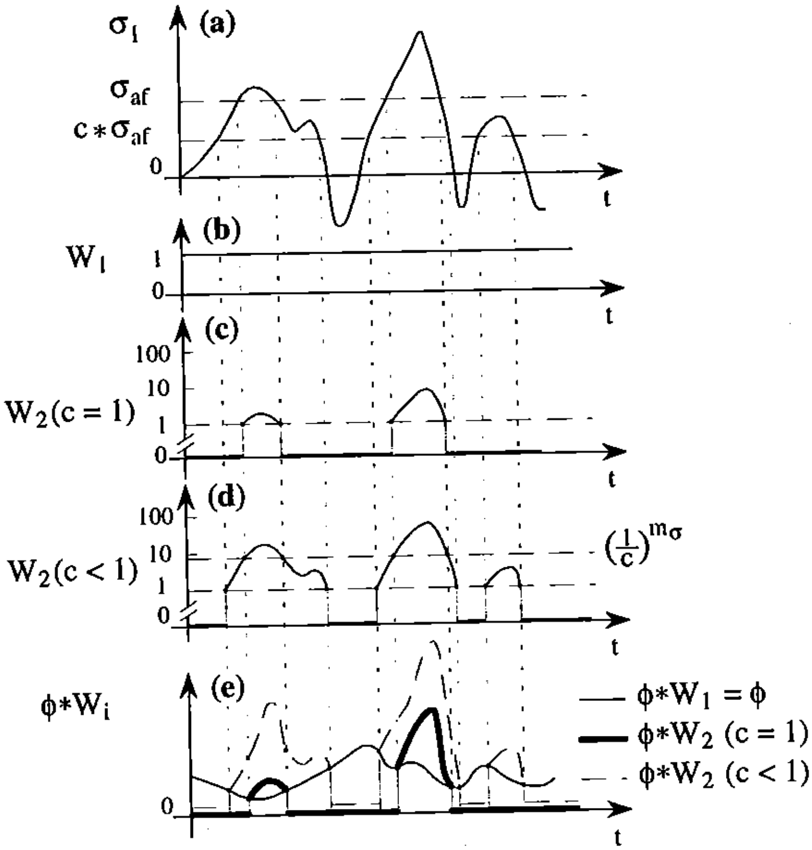

Fig. 4 Modifications of the time history of angle $\phi(t)$ by means of weight functions $W_1(t)$ and $W_2(t)$.

Numerical simulation

The components of a triaxial random stress state (1) have been numerically generated with six independent sequences of numbers, $x_1(t_k),\ldots,x_6(t_k)$, (k = 1, ... , N = 131072). Each of the digital time series has zero mean value $\hat{x}_i = 0$, $(i = 1,\ldots,6)$, normal probability distribution with variances μ_{x11} = 7021.198 MPa^2, μ_{x22} = 6291.326 MPa^2, μ_{x33} = 5359.036 MPa^2, μ_{x44} = 4465.860 MPa^2, μ_{x55} = 4092.515 MPa^2, μ_{x66} = 3540.544 MPa^2 and low-band PSDF (7) with $f_{\max 11}$ = 80.00 Hz,

$f_{max 22} = 90.00\ Hz$, $f_{max 33} = 65.00\ Hz$, $f_{max 44} = 70.00\ Hz$, $f_{max 55} = 100.00\ Hz$, $f_{max 66} = 85.00\ Hz$. The constants of the digital filters (6), determined as discussed in the second Section, are equal to:

i	a_i	b_i	c_i	d_i
1	0.01829	0.06841	1.74764	-0.76357
2	0.02146	0.08036	1.71847	-0.73825
3	0.01374	0.05135	1.79241	-0.80318
4	0.01523	0.05692	1.77736	-0.78975
5	0.02471	0.09254	1.68970	-0.71377
6	0.01986	0.07431	1.73297	-0.75080

The remaining related data are: sampling time $\Delta t = 0.5\ ms$, sampling frequency $1/\Delta t = 2\ kHz$, $\omega_{11}^* = 269.756\ s^{-1}$, $\omega_{22}^* = 303.475\ s^{-1}$, $\omega_{33}^* = 219.176\ s^{-1}$, $\omega_{44}^* = 236.037\ s^{-1}$, $\omega_{55}^* = 286.616\ s^{-1}$, $\omega_{66}^* = 337.195\ s^{-1}$. As an example, Figure 1 shows a part of the time history of stress $\sigma_{xx}(t)$ generated on the basis of the above digital filter data.

Then, such digital time series $x_i(t_k)$, (i=1,...,6) are treated as elements of a symmetric matrix, whose eigenvalues and eigenvectors are calculated through the Jacobi method. After arranging the eigenvalues and eigenvectors according to the relation $\sigma_1(t_k) \geq \sigma_2(t_k) \geq \sigma_3(t_k)$, the components of the eigenvectors form the matrix of direction cosines (9).

Euler angles calculation from the matrix $A(t_k)$ consists of two stages. In the first stage, the Euler angle ranges $0 \leq \phi(t_k), \psi(t_k) < 2\pi$ and $0 \leq \theta(t_k) < \pi$ are reduced to the new range $-\pi/2 \leq \phi(t_k), \theta(t_k), \psi(t_k) \leq \pi/2$, which is related to $l_1(t_k), m_2(t_k), n_3(t_k) \geq 0$, by multiplying particular columns of the matrix $A(t_k)$ by ± 1, in order to fulfill the condition $Det\, A(t_k) = 1$. The angles $\phi(t_k), \theta(t_k), \psi(t_k)$ are calculated according to equations (11) to (13).

In the second stage, the signs of the previous results are changed to average the values of the Euler angles in a correct way with respect to the physical meaning of such angles. The ranges of the Euler angles are reduced as follows: $0 \leq \phi(t_k),\ \theta(t_k) \leq \pi/2$ and $-\pi/2 \leq \psi(t_k) \leq \pi/2$.

Figure 5 shows a part of the time history of the angle $\phi(t_k)$ (note that each dot represents a single time instant), whereas Figure 6 contains the probability density distributions of the Euler angles corresponding to the generated triaxial random stress state. It can be seen that PDFs of the Euler angles are asymmetric and strongly differ from the normal distribution.

After averaging the Euler angles according to relationships (14) to (17) we can determine the mean directions $(\hat{1}\hat{2}\hat{3})$ of the principal stress axes. In the case of the numerically generated triaxial random stress state, the following mean values of the Euler angles and the corresponding matrix of the mean direction cosines are obtained by assuming the weight function $W_1(t_k) = 1$:

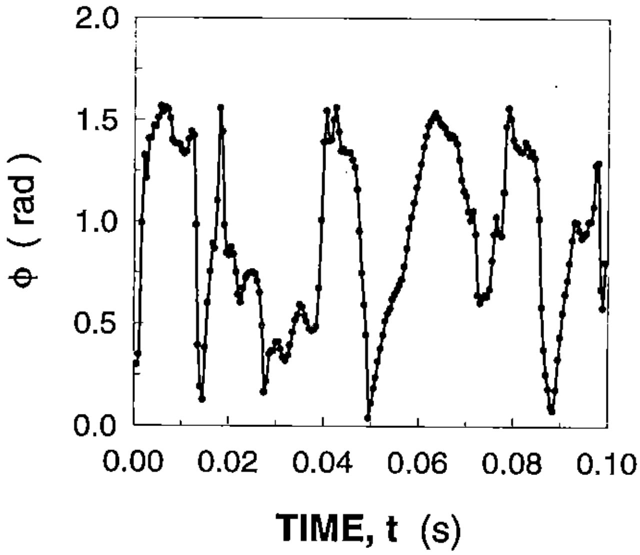

Fig.5 Time history part for the angle $\phi(t)$ obtained from the generated triaxial random state of stress.

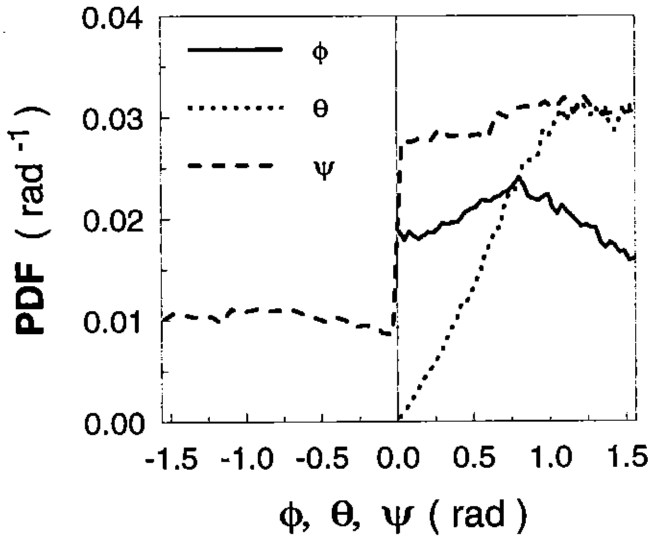

Fig.6 Probability density function (PDF) of the Euler angles corresponding to the numerically generated triaxial random stress state.

$\hat{\phi} = 0.77687$ rad $\hat{\theta} = 1.01483$ rad $\hat{\psi} = 0.80389$ rad

$$\hat{A} = \begin{bmatrix} -0.24364 & 0.75747 & 0.60571 \\ -0.77022 & 0.22842 & -0.59547 \\ -0.58941 & -0.61161 & 0.52776 \end{bmatrix}$$

Conclusion

Three independent parameters need to define the principal stress directions. Since the selection of 3 out of 9 direction cosines is a controversial problem, the three Euler angles are chosen here to describe the principal stress directions.

Under multiaxial random loading, the stress tensor and its eigenvectors (i.e., the principal direction cosines) change at each time instant. A procedure to univocally calculate the Euler angles from the matrix of the principal direction cosines for a given time instant is proposed in the present paper.

Then the instantaneous values of the Euler angles can be averaged by employing some suitable weight functions to take into account the main factors influencing the fatigue fracture process. Therefore the expected principal stress directions under multiaxial random stress state are determined and, finally, the expected fatigue fracture plane can be obtained from such principal directions.

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