5th International Conference on Biaxial/Multiaxial Fatigue and Fracture
Cracow'97, Poland

Andrzej BUCZYNSKI* and Grzegorz GLINKA**

Elastic-Plastic Stress-Strain Analysis of Notches under Non-Proportional Loading

* Institute of Heavy Machinery Engineering, Warsaw University of Technology
ul. Narbutta 85, 02-524 Warsaw, Poland
** Department of Mechanical Engineering, University of Waterloo, Waterloo ,
Ontario N2L 3G1, Canada

Keywords: notches, nonproportional loading, inelastic strains

ABSTRACT: An analytical method for calculating notch tip stresses and strains in elastic-plastic bodies subjected to non-proportional loading sequences is discussed in the paper. The method is based on axis invariant incremental relationships between the elastic and elastic-plastic strain energy densities at the notch tip, and material stress-strain behavior simulated according to the Mroz-Garud cyclic plasticity model. Two formulations are described involving the strain energy density and the complimentary strain density which appear to give the lower and the upper bound estimations for the elastic-plastic strains and stresses at the notch tip. Each formulation consists of a set of incremental algebraic equations that can easily be solved for elastic-plastic stress and strain increments, knowing the increments of the hypothetical elastic notch tip stress history and the material stress-strain curve. The validation of the proposed model against numerical data obtained for non-proportional loading is also presented. The method is particularly suitable for fatigue life analysis of notched bodies subjected to multiaxial cyclic loading paths.

Notation

$\Delta\varepsilon_{22}$	-	normal strain range in the critical plane
$\Delta\varepsilon_{ij}^{p}$	-	plastic strain increments
δ_{ij}	-	Kronecker delta, $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$
$\Delta\varepsilon_{kj}^{e}$	-	elastic strain increments
$\Delta\varepsilon_{kj}^{E}$	-	elastic-plastic strain increments according to ESED method
$\Delta\varepsilon_{kj}^{N}$	-	elastic-plastic strain increments according to Neuber's rule
$\Delta\varepsilon_{eq}^{pE}$	-	equivalent plastic strain increment according to ESED method
$\Delta\varepsilon_{eq}^{pN}$	-	equivalent plastic strain increment according to Neuber's rule
$\Delta\sigma_{ik}^{e}$	-	pseudo-elastic stress components
$\Delta\sigma_{ik}^{E}$	-	actual elastic-plastic stress components according to ESED method
$\Delta\sigma_{ik}^{N}$	-	actual elastic-plastic stress components according to Neuber's rule

$\Delta\sigma_{eq}^{E}$ - equivalent stress increment according to ESED method
$\Delta\sigma_{eq}^{N}$ - equivalent stress increment according to Neuber's rule
ΔS_{ij}^{e} - symmetric tensor of elastic strain energy density increment
ΔS_{ij}^{E} - symmetric tensor of elastic-plastic strain energy density increment
ΔW_{ij}^{e} - tensor of elastic strain energy density increment
ΔW_{ij}^{E} - tensor of elastic-plastic strain energy density increment
$\Delta\Omega_{ij}^{e}$ - tensor of elastic total strain energy density increment
$\Delta\Omega_{ij}^{N}$ - tensor of elastic-plastic total strain energy density increment
ΔT_{ij}^{e} - symmetric tensor of elastic total strain energy density increment
ΔT_{ij}^{N} - symmetric tensor of elastic-plastic total strain energy density increment
E - modulus of elasticity
ε_{kj}^{a} - actual elasti-plastic strains at the notch tip
ε_{eq}^{p} - equivalent plastic strain
ε_{eq}^{pE} - equivalent plastic strain determined from the ESED method
ε_{kj}^{E} - elasto-plastic notch-tip strains obtained from the ESED method
ε_{kj}^{e} - notch tip strain components obtained from linear elastic analysis
ε_{kj}^{N} - elasto-plastic notch-tip strains obtained from the Neuber method
ε_{kj}^{p} - plastic components of the notch-tip strain tensor
ε_{n} - nominal strain
ESED - equivalent strain energy density
H - plastic modulus
K' - cyclic strength coefficient
ν - Poisson's ratio
n' - cyclic strain hardening exponent
P - axial load
T - torque
R - radius of a cylindrical specimen
S_{n} - equivalent nominal stress
S_{eq} - equivalent nominal stress
S_{ij} - deviatoric stress components
σ_{eq} - equivalent stress
σ_{ik}^{a} - actual stress tensor components in the notch tip
σ_{ik}^{e} - notch tip stress tensor components obtained from linear elastic analysis
σ_{ik}^{E} - notch tip stress tensor components obtained from the ESED model
σ_{ik}^{N} - notch tip stress tensor components obtained from the Neuber solution
σ_{n} - nominal stress
σ_{n}^{P} - nominal (average) stress in the net cross section due to axial load P
σ_{o} - parameter of the material stress-strain curve (yield limit)
T - torque
t - wall thickness
τ_{n} - nominal shear stress in the net cross section

Introduction

Fatigue analyses of machine and structure components require determination of elastic-plastic stresses and strains at critical locations, such as notches, where the stress concentration occurs. In most cases the stress state in the notch tip region is multiaxial. However, if one of the stress components is the dominant one an uniaxial stress or plane strain state is often assumed. Such an approximation might be satisfactory in many practical applications but there are cases where all the stress and strain components have to be accounted for. This is particularly true when several loads are applied simultaneously and the stress components at the notch root change non-proportionally. For example, axles and shafts may experience combined out of phase torsion and bending loads.

The main focus of this paper is presentation and critical review of a method for calculating multiaxial elasto-plastic stresses and strains in notched bodies subjected to non-proportional loading histories.

Loading Histories

Fatigue cracks most often initiate at the notch tip where the highest stress concentration occurs. Therefore, most fatigue analyses are focused on the determination of fatigue life of the material volume which is under the effect of the notch tip stress-strain history. The notch tip stresses and strains are subsequently dependent on the notch geometry, material properties and the loading history applied to the notched body. If the various cyclic stress components are in phase and change proportionally with each other (Fig.1), the loading is called *proportional.* When the applied load causes the directions of the principal stresses and the ratio of the principal stress magnitudes to change after each load increment, the loading is termed *non-proportional.*

If plastic yielding takes place at the notch tip then almost always the stress path in the notch tip region is non-proportional regardless whether the remote loading is proportional or not. However, the remote proportional loading does not make the notch tip stress tensor to rotate and therefore it makes the stress analysis easier in spite of the fact that some non-proportionality of the notch tip stress history may occur.

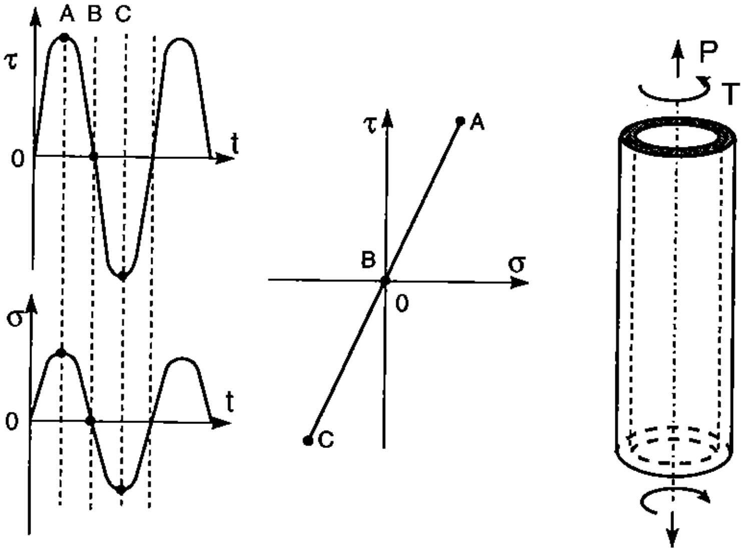


Fig. 1. An example of proportional cyclic torsion-tension loading applied to a hollow cylinder

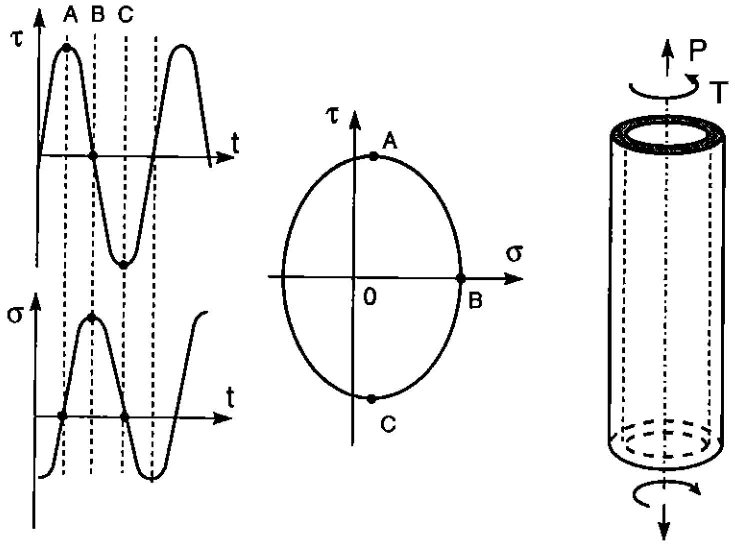


Fig. 2. An example of non-proportional cyclic torsion-tension loading applied to a hollow cylinder

The non-proportional loading/stress paths are usually defined by successive increments of load/stress parameters and therefore all calculations have to be carried out incrementally, particularly when plastic yielding takes place at the point of stress concentration. For this reason the entire analysis discussed below has been limited to incremental formulations involving principles of theory elasticity and plasticity.

The Stress State at the Notch Tip

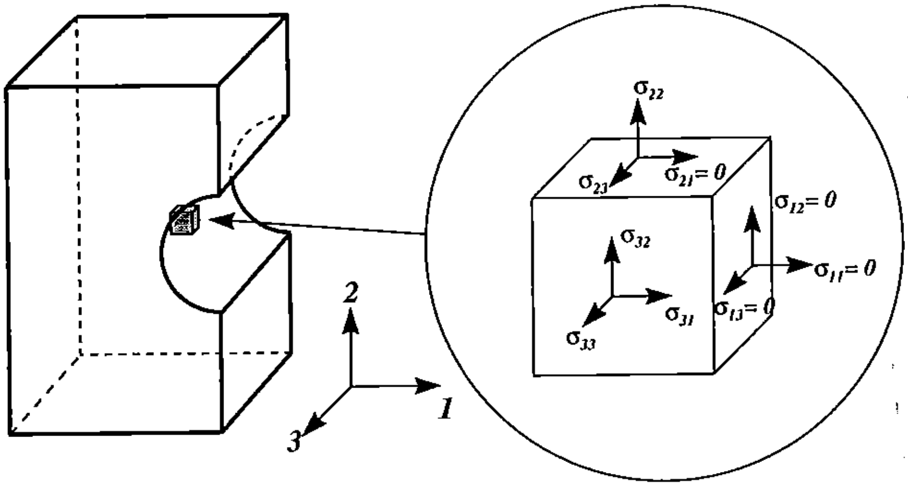
If the dimensions and external loads applied to a body are such that the plane stress state dominates in the body, then the stress state at the notch tip is uni-axial (Fig. 3a) providing that the surface at the notch tip is stress free. If the notched body is in the state of plane strain (Fig. 3b), there are only two principal non-zero stress and two non-zero strain components at the notch tip.

For the case of general multiaxial loading applied to a notched body, the state of stress near the notch tip is tri-axial. At the notch tip, the stress state is biaxial because of the stress free surface (Fig. 3c). Since equilibrium of the element at the notch tip must be maintained, $\sigma_{23} = \sigma_{32}$ and $\varepsilon_{23} = \varepsilon_{32}$,there are in general three non-zero stress components and four non-zero strain components. Therefore, there are seven unknowns and a set of seven independent equations is required for the determination of all stress and strain components at the notch tip. The material constitutive relationships provide four equations, leaving three additional relationships to be established.

Material Constitutive Model

The material constitutive model most frequently used in incremental plasticity is the Prandtl-Reuss flow rule associated with the von Mises plastic yielding criterion. For an isotropic body, the Prandtl-Reuss relationship can be expressed as:

$$\Delta \varepsilon_{ij} = \frac{1+\nu}{E} \Delta \sigma_{ij} - \frac{\nu}{E} \Delta \sigma_{kk} \delta_{ij} + \frac{3}{2} \frac{\Delta \varepsilon_{eq}^{p}}{\sigma_{eq}} S_{ij} \tag{1}$$



(a) Plane Stress

$$\sigma_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix}$$

(b) Plane Strain

$$\sigma_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} \qquad \varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) General

$$\sigma_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} \\ 0 & \sigma_{32} & \sigma_{33} \end{bmatrix} \qquad \varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & \varepsilon_{23} \\ 0 & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

Fig. 3. Stress state at a notch tip (notation): (a) body in plane stress, (b) body in plane strain, (c) general bi-axial stress state.

The multiaxial incremental stress-strain relation (1) is obtained from the uniaxial stress-strain curve by relating the equivalent plastic strain increment to the equivalent stress increment such that:

$$\Delta \varepsilon_{eq}^{p} = \frac{d\, f(\sigma_{eq})}{d\, \sigma_{eq}} \Delta \sigma_{eq} \tag{2}$$

The function $f(\sigma_{eq})$ is identical to that one obtained under uniaxial loading.

The Load-Notch Tip Stress-Strain Relations

The load in the case of notched bodies is usually represented by the nominal or reference stress being proportional to the remote applied load. In the case of notched bodies in plane stress or plane strain state the relationship between the load and the elastoplastic notch tip strains and stresses is most often approximated by the Neuber rule [1] or the Equivalent Strain Energy Density (ESED) equation [2]. It was later shown [3, 4] that both methods can be extended for multiaxial proportional and non-proportional modes of loading. However, the multiaxial Neuber and ESED [3, 4] models are not the only methods for determination of multiaxial elastoplastic strain and stress states at the notch tip. Similar methods were also proposed by Hoffman and Seeger [5] and Barkey et al. [6]. All of the approximate methods consists, in general, of two parts namely the constitutive equations and the relationships linking the fictitious linear elastic stress-strain state ($\sigma_{ij}^e, \varepsilon_{ij}^e$) at the notch tip with the elastic-plastic stress-strain response as shown in Fig. 4.

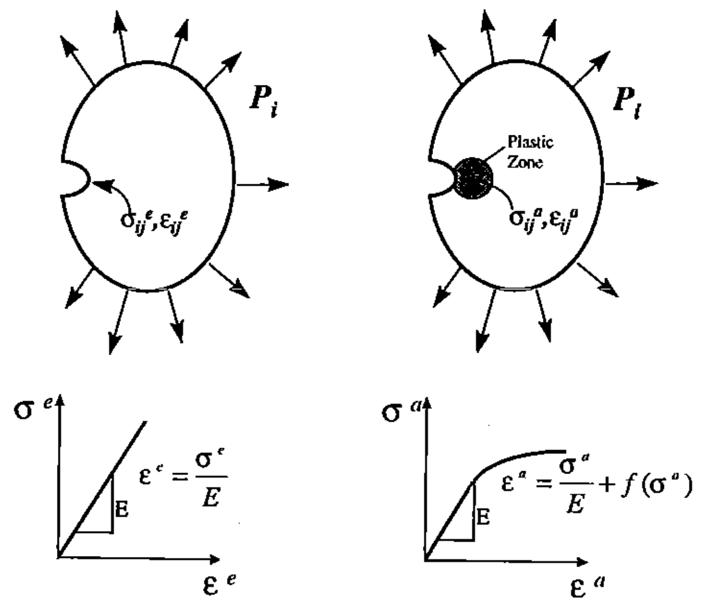


Fig. 4. The linear elastic and elasto-plastic strain and stress states in geometrically identical bodies

The incremental Neuber rule (3) and the ESED equation (4) can be found in reference [4].

- incremental Neuber's rule

$$\sigma_{ij}^{e}\Delta\varepsilon_{ij}^{e}+\varepsilon_{ij}^{e}\Delta\sigma_{ij}^{e}=\sigma_{ij}^{N}\Delta\varepsilon_{ij}^{N}+\varepsilon_{ij}^{N}\Delta\sigma_{ij}^{N}. \qquad (3)$$

- incremental ESED equation

$$\sigma_{ij}^{e}\Delta\varepsilon_{ij}^{e}=\sigma_{ij}^{E}\Delta\varepsilon_{ij}^{E}. \qquad (4)$$

The overall energy equivalence in the form of eq.(3) or (4) relating the pseudo-elastic and the actual elastic plastic notch tip strains and stresses has been accepted in general but the additional conditions necessary for the complete formulation of the problem are being the subject of controversy. Hoffman and Seeger [5] assumed that the ratio of the actual principal strains at the notch tip is to be equal to the ratio of fictitious elastic principal strain components while Barkey et al. [6] suggested to use the ratio of principal stresses. The data presented by Moftakhar [7] indicates that the accuracy of the stress or strain ratio based notch tip stress-strain analysis depends on the constraint at the notch tip. Unfortunately, it is very difficult to define objective criteria enabling appropriate choice of those additional conditions. On the other hand, the accuracy of the additional energy equations presented by Singh et.al. [4] seems to be less dependent on the geometry and constraint conditions at the notch tip and therefore the analyst is not forced to make any arbitrary decisions while using them. However, they have a theoretical drawback indicated by Chu [8] because they do not have tensor properties and thus the estimated elastic-plastic notch tip strains and stresses depend on the system of coordinates. The dependence is not very strong and with suitably chosen system of reference it could be sufficiently accurate for engineering applications. Nevertheless, it is possible to formulate an axis invariant system of equations similar to those discussed in references [4, 8].

Axis Invariant Multiaxial Equivalent Strain Energy Density Equations

The strain energy density tensor, ΔW_{ij}^{e}, resulting from the pseudo-elastic stress and strain increments at the notch tip (Fig. 4) can determined as:

$$\Delta W_{ij}^{e} = \sigma_{ik}^{e} \cdot \Delta\varepsilon_{kj}^{e} \tag{5}$$

The strain energy density tensor, ΔW_{ij}^{E}, for the elastic-plastic body resulting from the elastic-plastic stress and strain increments at the notch tip (Fig. 4a) can also be determined as the inner product of the current stress tensor and the strain increment tensor.

$$\Delta W_{ij}^{E} = \sigma_{ik}^{E} \cdot \Delta\varepsilon_{kj}^{E} \tag{6}$$

In the case of the notch tip stress free surface (Fig. 3c) tensors (5) and (6) can be presented in the matrix form with all the elements in the first column and the first row being equal to zero.

$$\Delta W_{ij}^{e} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_{22}^{e}\Delta\varepsilon_{22}^{e} + \sigma_{23}^{e}\Delta\varepsilon_{32}^{e} & \sigma_{22}^{e}\Delta\varepsilon_{23}^{e} + \sigma_{23}^{e}\Delta\varepsilon_{33}^{e} \\ 0 & \sigma_{32}^{e}\Delta\varepsilon_{22}^{e} + \sigma_{33}^{e}\Delta\varepsilon_{32}^{e} & \sigma_{32}^{e}\Delta\varepsilon_{23}^{e} + \sigma_{33}^{e}\Delta\varepsilon_{33}^{e} \end{bmatrix} \tag{7}$$

and

$$\Delta W_{ij}^{E} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_{22}^{E}\Delta\varepsilon_{22}^{E} + \sigma_{23}^{E}\Delta\varepsilon_{32}^{E} & \sigma_{22}^{E}\Delta\varepsilon_{23}^{E} + \sigma_{23}^{E}\Delta\varepsilon_{33}^{E} \\ 0 & \sigma_{32}^{E}\Delta\varepsilon_{22}^{E} + \sigma_{33}^{E}\Delta\varepsilon_{32}^{E} & \sigma_{32}^{E}\Delta\varepsilon_{23}^{E} + \sigma_{33}^{E}\Delta\varepsilon_{33}^{E} \end{bmatrix} \tag{8}$$

The sum of diagonal terms in both tensors represents the increment of strain energy density. Both tensors (7) and (8) are generally non-symmetric. However, they can be easily converted into symmetric tensors, ΔS_{ij}^{e} and ΔS_{ij}^{E}, by setting the off-diagonal terms equal to the mean of the off-diagonal terms in tensors ΔW_{ij}^{e} and ΔW_{ij}^{E} respectively without changing any of the diagonal terms.

$$\Delta S_{ij}^{e} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_{22}^{e}\Delta\varepsilon_{22}^{e} + \sigma_{23}^{e}\Delta\varepsilon_{32}^{e} & \dfrac{(\sigma_{22}^{e} + \sigma_{33}^{e})\Delta\varepsilon_{23}^{e} + \sigma_{23}^{e}(\Delta\varepsilon_{22}^{e} + \Delta\varepsilon_{33}^{e})}{2} \\ 0 & \dfrac{(\sigma_{22}^{e} + \sigma_{33}^{e})\Delta\varepsilon_{23}^{e} + \sigma_{23}^{e}(\Delta\varepsilon_{22}^{e} + \Delta\varepsilon_{33}^{e})}{2} & \sigma_{32}^{e}\Delta\varepsilon_{23}^{e} + \sigma_{33}^{e}\Delta\varepsilon_{33}^{e} \end{bmatrix} \tag{9}$$

and

$$\Delta S_{ij}^{E} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_{22}^{E}\Delta\varepsilon_{22}^{E} + \sigma_{23}^{E}\Delta\varepsilon_{32}^{E} & \dfrac{\left(\sigma_{22}^{E} + \sigma_{33}^{E}\right)\Delta\varepsilon_{23}^{E} + \sigma_{23}^{E}\left(\Delta\varepsilon_{22}^{E} + \Delta\varepsilon_{33}^{E}\right)}{2} \\ 0 & \dfrac{\left(\sigma_{22}^{E} + \sigma_{33}^{E}\right)\Delta\varepsilon_{23}^{E} + \sigma_{23}^{E}\left(\Delta\varepsilon_{22}^{E} + \Delta\varepsilon_{33}^{E}\right)}{2} & \sigma_{32}^{E}\Delta\varepsilon_{23}^{E} + \sigma_{33}^{E}\Delta\varepsilon_{33}^{E} \end{bmatrix} \quad (10)$$

Analogously to the hypothesis proposed in references [2, 3, 4] it is assumed that the strain energy increments at the notch tip in the geometrically identical elastic and the elastic-plastic bodies (Fig. 4) are equal. Such a hypothesis can be expressed by the equality of tensors (9) and (10).

$$\Delta S_{ij}^{e} = \Delta S_{ij}^{E} \quad (11)$$

The hypothesis written in the form of equation (11) results in three independent equations relating the pseudo-elastic strain and stress components and the actual elastic-plastic response at the notch tip in the elastic-plastic body (Fig. 4).

$$\sigma_{22}^{e}\Delta\varepsilon_{22}^{e} + \sigma_{23}^{e}\Delta\varepsilon_{23}^{e} = \sigma_{22}^{E}\Delta\varepsilon_{22}^{E} + \sigma_{23}^{E}\Delta\varepsilon_{23}^{E} \quad (12)$$

$$\sigma_{33}^{e}\Delta\varepsilon_{33}^{e} + \sigma_{23}^{e}\Delta\varepsilon_{23}^{e} = \sigma_{33}^{E}\Delta\varepsilon_{33}^{E} + \sigma_{23}^{E}\Delta\varepsilon_{23}^{E} \quad (13)$$

$$\left(\sigma_{22}^{e} + \sigma_{22}^{e}\right)\Delta\varepsilon_{23}^{e} + \sigma_{23}^{e}\left(\Delta\varepsilon_{22}^{e} + \Delta\varepsilon_{33}^{e}\right) = \left(\sigma_{22}^{E} + \sigma_{22}^{E}\right)\Delta\varepsilon_{23}^{E} + \sigma_{23}^{E}\left(\Delta\varepsilon_{22}^{E} + \Delta\varepsilon_{33}^{E}\right) \quad (14)$$

Equations (12-14) can be supplemented with four stress-strain relationships derived from the general constitutive equation (1).

$$\Delta\varepsilon_{11}^{E} = -\frac{\nu}{E}\left(\Delta\sigma_{22}^{E} + \Delta\sigma_{33}^{E}\right) - \frac{1}{2}\left(\sigma_{22}^{E} + \sigma_{33}^{E}\right)\frac{\Delta\varepsilon_{eq}^{pE}}{\sigma_{eq}^{E}} \quad (15)$$

$$\Delta\varepsilon_{22}^{E} = \frac{1}{E}(\Delta\sigma_{22}^{E} - \nu\Delta\sigma_{33}^{E}) + \frac{1}{2}(2\sigma_{22}^{E} - \sigma_{33}^{E})\frac{\Delta\varepsilon_{eq}^{pE}}{\sigma_{eq}^{E}} \quad (16)$$

$$\Delta\varepsilon_{33}^{E} = \frac{1}{E}(\Delta\sigma_{33}^{E} - \nu\Delta\sigma_{22}^{E}) + \frac{1}{2}(2\sigma_{33}^{E} - \sigma_{22}^{E})\frac{\Delta\varepsilon_{eq}^{pE}}{\sigma_{eq}^{E}} \quad (18)$$

$$\Delta\varepsilon_{23}^{E} = \frac{1+\nu}{E}\Delta\sigma_{23}^{E} + \frac{3}{2}\frac{\Delta\varepsilon_{eq}^{pE}}{\sigma_{eq}^{E}}\sigma_{23}^{E} \quad (19)$$

where: $\left(\sigma_{eq}^{E}\right)^2 = \left(\sigma_{22}^{E}\right)^2 + \left(\sigma_{33}^{E}\right)^2 - \sigma_{22}^{E}\sigma_{33}^{E} + 3\left(\sigma_{23}^{E}\right)^2$

$$\Delta\sigma_{eq}^{E} = \frac{\left(\sigma_{22}^{E} - \sigma_{33}^{E}\right)\left(\Delta\sigma_{22}^{E} - \Delta\sigma_{33}^{E}\right) + 3\sigma_{23}^{E}\Delta\sigma_{23}^{E}}{\sigma_{eq}^{E}}$$

$$\Delta\varepsilon_{eq}^{pE} = \frac{df\left(\sigma_{eq}^{E}\right)}{d\sigma_{eq}^{E}}\Delta\sigma_{eq}^{E}$$

Equations (12-19) form a complete set of equations enabling all the elastic-plastic strains, ($\Delta\varepsilon_{11}^{E}$, $\Delta\varepsilon_{22}^{E}$, $\Delta\varepsilon_{33}^{E}$, $\Delta\varepsilon_{23}^{E}$), and stress increments (σ_{22}^{E}, σ_{33}^{E}, σ_{23}^{E}) to be calculated based on the pseudo-elastic stress history at the notch tip. A graphical representation of the incremental ESED method is shown in Fig. 5a, where the strain energy densities are represented by the vertical bars of the trapezoidal shape whose areas, according to eqns. (12-14), must be equal.

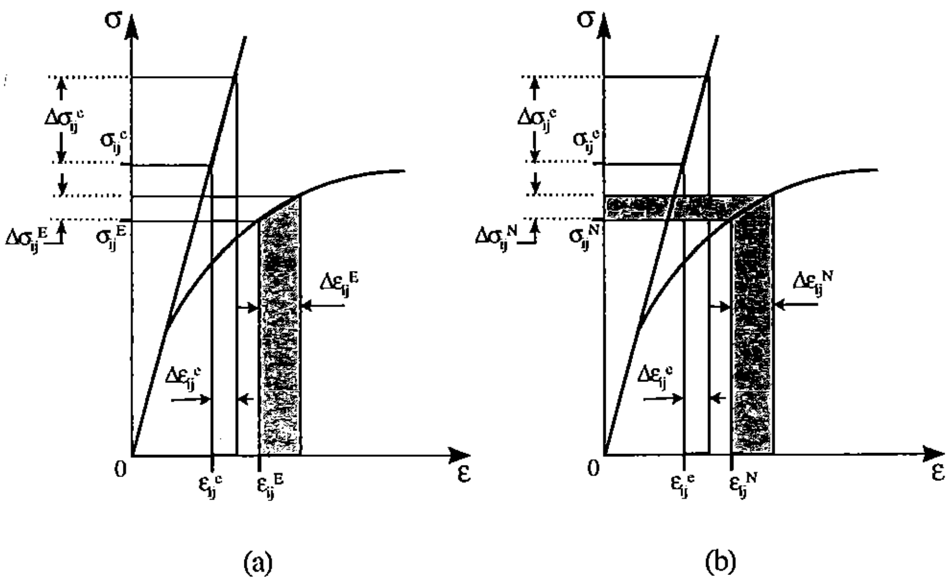


Fig. 5. Graphical representation of : a) Incremental ESED method, b) Incremental Neubers's rule

Axis Invariant Total Strain Energy Density Equations

A set of axis invariant equations similar to equations (12-14) can also be written in terms of the total strain energy density, i.e. the sum of the strain energy density and the complementary strain energy density. The tensor representation of the increments of the total strain energy density at the notch tip of linear elastic body (Fig. 4) can be written as:

$$\Delta\Omega_{ij}^{e} = \sigma_{ik}^{e} \cdot \Delta\varepsilon_{kj}^{e} + \Delta\sigma_{ik}^{e} \cdot \varepsilon_{kj}^{e} \tag{20}$$

Analogously, the tensor representation of the total strain energy density increments at the notch tip of geometrically identical elastic-plastic body (Fig. 4) can be written as well.

$$\Delta\Omega_{ij}^{N} = \sigma_{ik}^{N} \cdot \Delta\varepsilon_{kj}^{N} + \Delta\sigma_{ik}^{N} \cdot \varepsilon_{kj}^{N} \tag{21}$$

Both tensors (20) and (22) are generally non-symmetric but they can be converted into symmetric tensors, T_{ij}^{e} and T_{ij}^{N}, by setting the off-diagonal terms equal to the mean of the off-diagonal terms in tensors $\Delta\Omega_{ij}^{e}$ and $\Delta\Omega_{ij}^{N}$ respectively without changing any of the diagonal terms. Similarly to the strain energy density tensors the sums of the diagonal terms of tensors T_{ij}^{e} and T_{ij}^{N} represent the increment of the total strain energy density, ie. the strain energy density plus the complementary strain energy density.

It is postulated, similarly to the original Neuber's concept, that in the case of localized plastic yielding in the notch tip region the symmetric tensors are equal.

$$\Delta T_{ij}^{e} = \Delta T_{ij}^{N} \tag{22}$$

It can be shown that in the case of uni-axial stress state equation (22) reduces to the well known Neuber's rule [1,2] and to the model proposed by Moftakhar et.al. [3] for multiaxial proportional loading. For the problem with one surface free of stress, as it often occurs in notches (Fig. 3), the tensor equation (22) results in three independent equations relating the pseudo-elastic and the elastic-plastic strain and stress increments at the notch tip.

$$\sigma_{22}^{e}\Delta\varepsilon_{22}^{e} + \Delta\sigma_{22}^{e}\varepsilon_{22}^{e} + \sigma_{23}^{e}\Delta\varepsilon_{23}^{e} + \Delta\sigma_{23}^{e}\varepsilon_{23}^{e} = \sigma_{22}^{N}\Delta\varepsilon_{22}^{N} + \Delta\sigma_{22}^{N}\varepsilon_{22}^{N} + \sigma_{23}^{N}\Delta\varepsilon_{23}^{N} + \Delta\sigma_{23}^{N}\varepsilon_{23}^{N} \tag{23}$$

$$\sigma_{33}^{e}\Delta\varepsilon_{33}^{e} + \Delta\sigma_{33}^{e}\varepsilon_{33}^{e} + \sigma_{23}^{e}\Delta\varepsilon_{23}^{e} + \Delta\sigma_{23}^{e}\varepsilon_{23}^{e} = \sigma_{33}^{N}\Delta\varepsilon_{33}^{N} + \Delta\sigma_{33}^{N}\varepsilon_{33}^{N} + \sigma_{23}^{N}\Delta\varepsilon_{23}^{N} + \Delta\sigma_{23}^{N}\varepsilon_{23}^{N} \quad (24)$$

$$\begin{aligned}&\left(\sigma_{22}^{e} + \sigma_{33}^{e}\right)\Delta\varepsilon_{23}^{e} + \left(\Delta\sigma_{22}^{e} + \Delta\sigma_{33}^{e}\right)\varepsilon_{23}^{e} + \sigma_{23}^{e}\left(\Delta\varepsilon_{22}^{e} + \Delta\varepsilon_{33}^{e}\right) + \Delta\sigma_{23}^{e}\left(\varepsilon_{22}^{e} + \varepsilon_{33}^{e}\right) = \\ &\left(\sigma_{22}^{N} + \sigma_{33}^{N}\right)\Delta\varepsilon_{23}^{N} + \left(\Delta\sigma_{22}^{N} + \Delta\sigma_{33}^{N}\right)\varepsilon_{23}^{N} + \sigma_{23}^{N}\left(\Delta\varepsilon_{22}^{N} + \Delta\varepsilon_{33}^{N}\right) + \Delta\sigma_{23}^{N}\left(\varepsilon_{22}^{N} + \varepsilon_{33}^{N}\right)\end{aligned} \quad (25)$$

Equations (23-25) and four constitutive equations (26-29) form a set of seven equations necessary for complete formulation of the notch tip stress-strain problem.

$$\Delta\varepsilon_{11}^{N} = -\frac{\nu}{E}\left(\Delta\sigma_{22}^{N} + \Delta\sigma_{33}^{N}\right) - \frac{1}{2}\left(\sigma_{22}^{N} + \sigma_{33}^{N}\right)\frac{\Delta\varepsilon_{eq}^{pN}}{\sigma_{eq}^{N}} \quad (26)$$

$$\Delta\varepsilon_{22}^{N} = \frac{1}{E}(\Delta\sigma_{22}^{N} - \nu\Delta\sigma_{33}^{N}) + \frac{1}{2}(2\sigma_{22}^{N} - \sigma_{33}^{N})\frac{\Delta\varepsilon_{eq}^{pN}}{\sigma_{eq}^{N}} \quad (27)$$

$$\Delta\varepsilon_{33}^{N} = \frac{1}{E}(\Delta\sigma_{33}^{N} - \nu\Delta\sigma_{22}^{N}) + \frac{1}{2}(2\sigma_{33}^{N} - \sigma_{22}^{N})\frac{\Delta\varepsilon_{eq}^{pN}}{\sigma_{eq}^{N}} \quad (28)$$

$$\Delta\varepsilon_{23}^{N} = \frac{1+\nu}{E}\Delta\sigma_{23}^{N} + \frac{3}{2}\frac{\Delta\varepsilon_{eq}^{pN}}{\sigma_{eq}^{N}}\sigma_{23}^{N} \quad (29)$$

where: $\left(\sigma_{eq}^{N}\right)^2 = \left(\sigma_{22}^{N}\right)^2 + \left(\sigma_{33}^{N}\right)^2 - \sigma_{22}^{N}\sigma_{33}^{N} + 3\left(\sigma_{23}^{N}\right)^2$

$$\Delta\sigma_{eq}^{N} = \frac{\left(\sigma_{22}^{N} - \sigma_{33}^{N}\right)\left(\Delta\sigma_{22}^{N} - \Delta\sigma_{33}^{N}\right) + 3\sigma_{23}^{N}\Delta\sigma_{23}^{N}}{\sigma_{eq}^{N}}$$

$$\Delta\varepsilon_{eq}^{pN} = \frac{df\left(\sigma_{eq}^{N}\right)}{d\sigma_{eq}^{N}}\Delta\sigma_{eq}^{N}$$

In the case of uni-axial or plane strain state at the notch tip the set of seven equations reduces to only two equations as proposed originally by Neuber [1]. The equivalence of the increments of the total strain energy density is graphically shown in Fig. 5b, where the energies are represented by the horizontal and vertical rectangles whose areas are assumed to be equal.

Comparison of Calculated Elastic-Plastic Notch Tip Strains and Stresses with Finite Element Data

The accuracy of the proposed incremental Neuber rule and the incremental ESED method were assessed by comparing the calculated notch tip stress-strain histories to those obtained from the finite element method of stress analysis. The elastic-plastic results from the finite element analysis of reference [4] were obtained using the ABAQUS finite element package. The geometry of the notched element was that of the circumferentially notched bar shown in Fig. 6.

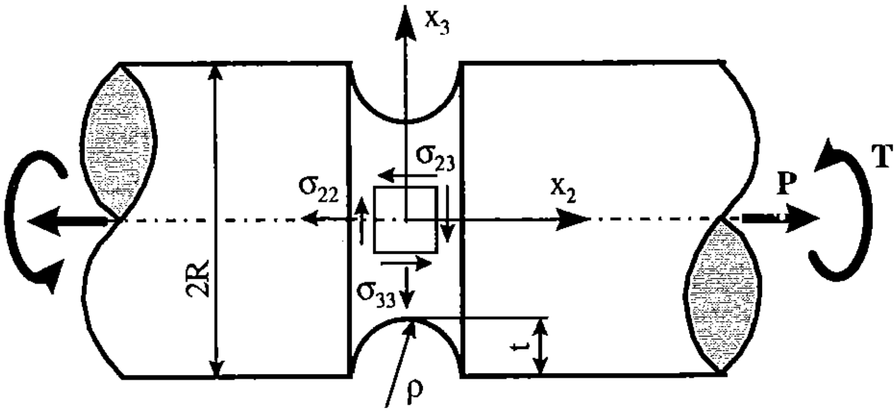


Fig. 6. Geometry and dimensions of the notched bar tested under non-proportional tension and torsion loading; ρ/t = 0.3, R/t = 7

The nominal torsional stresses, τ_n, and tensile stresses, σ_{nP}, were determined based on the net cross section according to eq. (30).

$$\sigma_{nP} = \frac{P}{\pi(R-t)^2} \quad \text{and} \quad \tau_n = \frac{2T}{\pi(R-t)^3} \tag{30}$$

The basic proportions of the cylindrical component were ρ/t = 0.3 and R/t = 7 resulting in the torsional and tensile stress concentration factor $K_T = 3.31$ and $K_P = 1.94$ respectively. The ratio of the notch tip hoop to axial stress under tensile loading was $\sigma_{33}^e / \sigma_{22}^e = 0.284$.

The material for the notched bar was SAE 1045 steel with a cyclic stress-strain curve approximated by the Ramberg-Osgood relation:

$$\varepsilon = \frac{\sigma}{E} + \left(\frac{\sigma}{K'}\right)^{\frac{1}{n'}} \tag{31}$$

The material properties were: E = 202 GPa, ν = 0.3, S_y = 202 MPa, n' = 0.208, and K' = 1258 MPa.

The loads applied to the bar were torsion and tension according to the path shown in Fig.7.

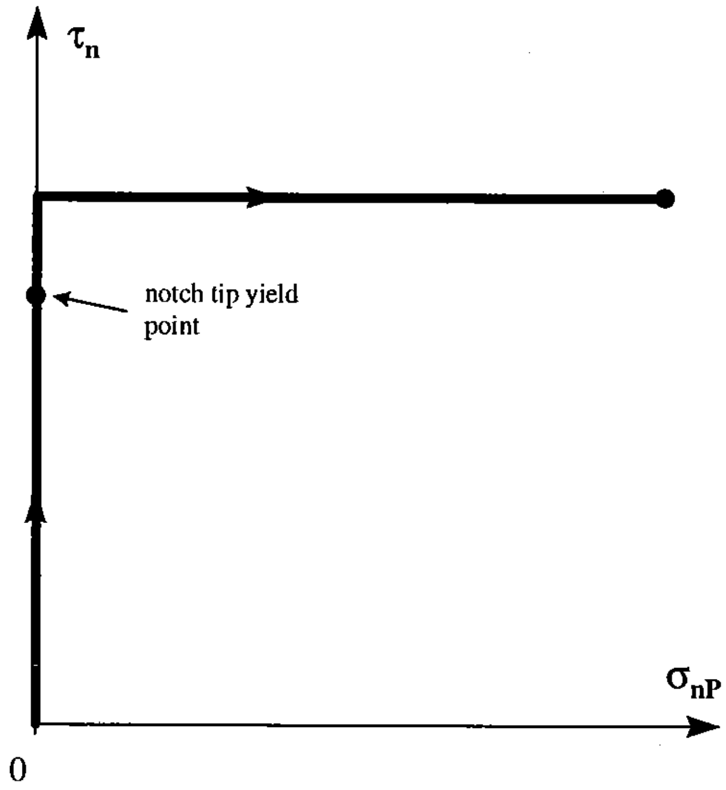


Fig. 7. The non-proportional torsion-tension load path

The maximum applied load levels were chosen to be 50% higher than it would be required to cause yielding at the notch tip if each load were applied separately. Specifically, the maxima were σ_{nP} = 103 Mpa, and τ_n = 90 MPa. The final ratio of the nominal stresses was σ_{nP}/τ_n = 1.133. The normalized nominal equivalent net cross sectional stress ratio defined as

$$\frac{S_{eq}}{S_y} = \frac{\sqrt{\sigma_{nP}^2 + 3\tau_n^2}}{S_y} \tag{32}$$

reached the value of 0.92 at the end of the loading path, a value that indicates almost general yielding of the net cross section.

The local pseudo-elastic stress histories at the notch tip induced by the load path shown in Fig. 7 were used as the input to calculate the elastic-plastic notch tip stres-strain response. They were inputted into eqns. (12-19) in the case of the ESED method and into eqns. (23-29) in the case of the total strain energy (Neuber) approach. The calculated strains and stresses were subsequently compared with the elastic-plastic finite element results. The strain components, ε_{22} and ε_{23}, and the stress components, σ_{22} and σ_{23}, that were calculated using the methods described above are shown in Figs. 8 and 9. Note, that the results from both models and the finite element analysis are identical in the elastic range. This is expected since the models converge to the elastic solution in the case of entirely elastic behavior.

Just beyond the onset of yielding at the notch tip, the strain results that were predicted using the proposed models and the finite element data begin gradually to diverge.

However, both methods give reasonably good estimation of the notch tip stress-strain behavior. It can be concluded that the incremental total strain energy density or the Neuber method predicts an upper bound, and the incremental ESED method, a lower bound approximation of the actual notch tip strains. The investigations up to date revealed that the actual notch tip strains are always within the band defined by the two methods and the average values of the two limits may be used as a good approximation of the actual stress-strain state at the notch tip.

In order to predict the notch tip stress-strain response of a notched component subjected to multiaxial cyclic loading, the incremental equations discussed above have to be linked with the cyclic plasticity model as described in reference [9].

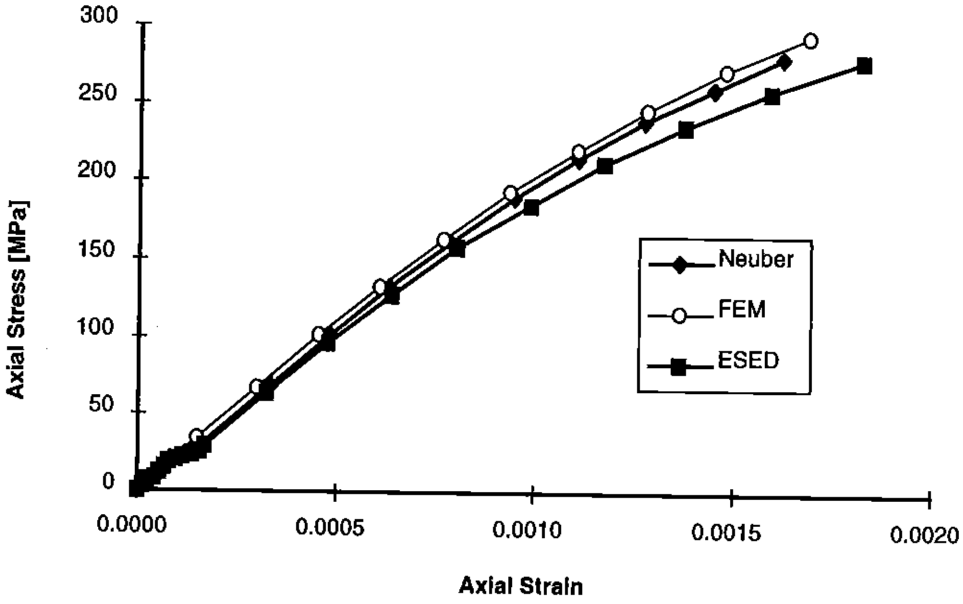


Fig. 8. The axial stress and strain, σ_{22} and ε_{22}, as a function of the torsion-tension load history

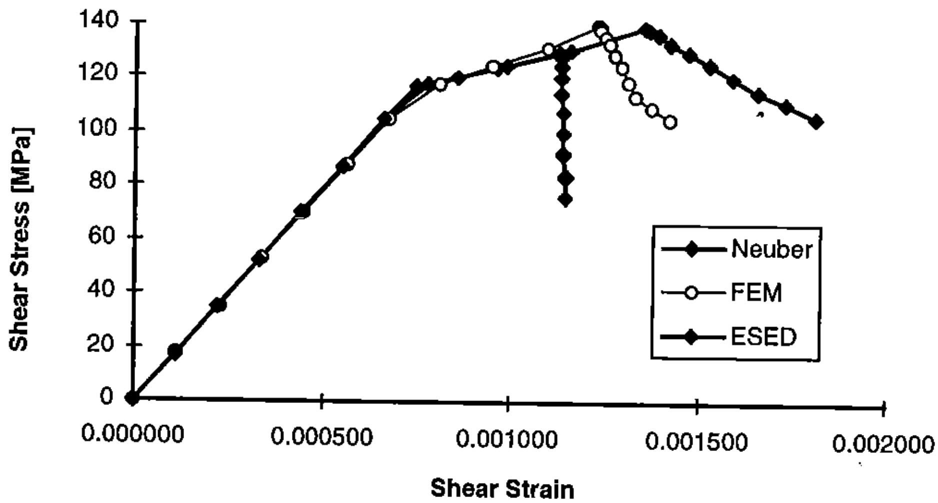


Fig. 9. The shear stress and strain, σ_{23} and ε_{23}, as a function of the torsion-tension load history

Conclusions

Two methods for calculating elastic-plastic notch tip strains and stresses induced by multiaxial loading paths have been proposed. The methods have been formulated using both the total strain energy density and the strain energy density relationships. It has been found that the generalized Neuber's rule, which represents the equality of the total strain energy density at the notch tip, gives an upper bound estimate for the elasto-plastic notch tip strains. The generalized equations of the equivalent strain energy density (ESED) yield a lower bound solution for the notch tip strains and stresses. The method has been verified by comparison with finite element data obtained for non-proportional loading path and non-linear stress-strain material model.

The calculated notch tip strains and stresses can be subsequently used for estimating fatigue damage and life prediction for multiaxial cyclic loading histories.

Literature

1. Neuber, H. (1961) Theory of Stress Concentration Shear Strained Prismatic Bodies with Arbitrary Non- Linear Stress-Strain Law, ASME Journal of Applied Mechanics **28**, 544-550.
2. Molski, K. and Glinka, G. (1981) A Method of Elastic-Plastic Stress and Strain Calculation at a Notch Root, Material Science and Engineering **50**, 93-100.
3. Moftakhar, A., Buczynski, A. and Glinka, G. (1995) Calculation of Elasto-Plastic Strains and Stresses in Notches under Multiaxial Loading, International Journal of Fracture **70**, 357-373.
4. Singh, M.N.K., Glinka, G. and Dubey, R.N., (1996) Elastic-Plastic Stress-Strain Calculation in Notched Bodies Subjected to Non-proportional Loading, International Journal of Fracture, to be published.
5. Seeger, T. and Hoffman, M. (1986) The Use of Hencky's Equations for the Estimation of Multiaxial Elastic-Plastic Notch Stresses and Strains, Report No. FB-3/1986, Technische Hochschule Darmstadt.
6. Barkey, M.E., Socie, D.F. and Hsia, K.J. (1994) A Yield Surface Approach to the Estimation of Notch Strains for Proportional and Non-proportional Cyclic Loading, ASME Journal of Engineering Materials and Technology **116**, 173-180.

7. Moftakhar, A. A., (1994), Calculation of Time-Independent and Time-Dependent Strains and Stresses in Notches, Doctoral Dissertation, University of Waterloo, Department of Mechanical Engineering, Waterloo, Ontario, Canada
8. Chu, C. -C., (1995), Incremental Multiaxial Neuber Correction for Fatigue Analysis, International Congress and Exposition, Detroit, 1995, SAE Technical Paper No. 950705
9. Singh, N.K., Buczynski, A. and Glinka, G., (1995) Notch Stress-Strain Analysis and Life Prediction in Multiaxial Fatigue, Proceedings of the Int. Conference on 'Fatigue Design 95', eds. G. Marquis et.al., Helsinki, Sept. 5-8, 1995