5th International Conference on Biaxial/Multiaxial Fatigue and Fracture
Cracow'97, Poland

B.H. LEE* and S.B. LEE*

A Stochastic Modeling of Multiaxial Fatigue Life and Damage Accumulation of SM45C Steel

* Department of Mechanical Engineering, Korea Advanced Institute of Science and Technology, 373-1, Kusong-dong, Yusung-gu, Taejon, S.KOREA

Keywords : Multiaxial Fatigue, Critical Plane, Stochastic Modeling, Markov Chain Model, Damage Vector

ABSTRACT *For the prediction of fatigue life and modeling of reliability and damage accumulation behavior of SM45C steel under variable multiaxial loading, a stochastic model which employing Markov chain model and damage vector at critical orientation was developed. In the case of SM45C steel maximum tensile stress plane was found to be the critical plane and positive tensile stress amplitude was taken to be the equivalent stress in the multiaxial loading. Under the assumption that the same stochastic model can be applied to the multiaxial loading cases with same equivalent stress level, 20 uniaxial fatigue tests were conducted at each three stress levels 828, 803, and 731 MPa and fatigue tests of variable multiaxial loading cases which are composed of multiaxial loading blocks whose equivalent stresses are 828, 803, and 703MPa were conducted. Markov chain model and damage vector at specific orientations of interest was applied for the simulation of life and damage accumulation behavior. And the crack initiation orientation was also predicted by this model.*

Notation

σ_a	axial stress amplitude
τ_a	shear stress amplitude
σ_m	axial stress mean
τ_m	shear stress mean
ϕ	phase shift of tension and torsion
σ_n	normal stress amplitude on a plane
σ_{eq}	equivalent stress
N_f	cycles to failure
W_f	duty cycles to failure

σ^2	variance of life
μ	mean value of life
DC	duty cycle
$\boldsymbol{P}$	b×b Transition Matrix
b	number of damage states
p_i	probability of remaining in state i in a DC
q_i	probability of going to state i+1 from state i in a DC
$\mathbf{p_o}$	initial damage vector of material
π_i	components of initial damage vector
$\mathbf{p_x}$	damage vector of material after x DC was applied
$p_x(i)$	probability of being in state i after x DC was applied
$\mathbf{p}_x^\theta$	damage vector at the plane of orientation θ after x DC was applied
$p_x^\theta(i)$	probability of being in state i at the plane of orientation θ after x DC was applied

Introduction

Generally engineering components such as automobile transmission, suspension, nuclear plant, pressure vessels are subjected to variable multiaxial loading, which may be non-proportional and of variable amplitude. And there is a strong need for developing reliable methods to predict fatigue life under multiaxial loading to achieve desired reliability of the system. A number of investigations were conducted for fatigue life prediction and reliability assessment under constant multiaxial loading by means of equivalent stress and strain, critical plane method, energy method and so on. Lee(1-2) proposed a modified ellipse quadrant type out-of-phase multiaxial fatigue life criterion and good agreement with experimental results. You and Lee(3) reviewed multiaxial fatigue assessments of metal. However in the case of variable multiaxial loading it is very difficult to make general and reliable method of fatigue life prediction, damage accumulation and reliability assessment due to its extreme complexity. Only a few methods have been proposed in the literature and up to now. Furthermore there is even no agreement on whether to look for damage in the volume or on a critical plane.

Bannantine and Socie(4-6) proposed a variable multiaxial fatigue life prediction method employing local strain and strain history and damage occurring on the critical plane which experiences maximum damage during multiaxial loading. Wang and Brown(7-8) proposed a life prediction method employing critical plane concept and a new multiaxial cycle counting method. Stefanov(10-11) proposed curvilinear method for random multiaxial fatigue. Lagoda and Macha (12) calculated damage on the most dangerous plane decided by the method of maximum variance of the stress on the plane. Most of the works are focused at the most important question "How to quantify damage cycle by cycle with changing principal axes?" In reliability assessment, one major problem of fatigue life prediction is the randomness of fatigue life data. Fatigue life is not deterministic variable but a random variable and in fact one can only state about the probability of failure not the life of material or components. There are generally two methods of dealing with this problem. First one can employ a statistical method assuming the distribution of fatigue life. Normal, lognormal and Weibull distributions are most frequently employed distribution types of fatigue life. But one should know the statistical properties of the population and it needs a number of experiments and the application is restricted to the same condition. Second one can employ a stochastic modeling and it requires relatively less data and one can simulate many situations including life prediction and reliability assessment.

In this paper a stochastic modeling of variable multiaxial fatigue was developed based on a critical plane concept and uniaxial fatigue data. Two kinds of variable multiaxial fatigue tests were conducted using round specimens of SM45C structural steel and compared with the results of the stochastic fatigue damage accumulation model.

Stochastic Model of Variable Multiaxial Fatigue

Since Weibull(13) introduced a cumulative distribution function(CDF) of extreme value type into failure studies, extensive use has been made of CDF in life and failure studies because the constructing CDF by stochastic modeling is simple and useful. Bogdanoff(14-17) regarded fatigue phenomenon as a Markov process and employed Markov chain model and damage vector for the stochastic modeling of fatigue. Tanaka et al(18) applied this model to uniaxial block loading cases. A Markov process is a stochastic process that satisfies the following equation (1).

$$\text{Prob}\,[S_{n+1} = j \mid S_n = i, S_{n-1} = k, \cdots, S_o = z] = \text{Prob}\,[S_{n+1} = j \mid S_n = i] \tag{1}$$

S_{n+1}, the probability of being in state j at n+1 is only affected by the value of S_n and the history of state value has no effect and the fatigue damage accumulation can be regarded as a Markov process and Markov chain model is employed for the stochastic modeling of the fatigue damage accumulation as in Fig. 1.

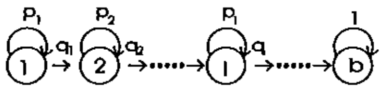


Fig.1 Markov chain model for fatigue damage accumulation

DC :a representative period of operation in the life of a component in which damage can be accumulate. For example, 100cycles can be define as one duty cycle

i : damage state number

b : total number of damage states including b-1 transient states and one absorbing state, final failure state

p_i : probability of remaining in state i in a DC if damage was state i at the start of the DC

q_i : probability of going to state i+1 in a DC if damage was state i at the start of the DC

Transition matrix ***P*** is defined as equation (2).

$$\boldsymbol{P} = \begin{bmatrix} p_1 & q_1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & p_2 & q_2 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & p_3 & q_3 & & & \cdot \\ \cdot & \cdot & & & & & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & p_{b-1} & q_{b-1} \\ 0 & 0 & 0 & \cdot & \cdot & 0 & 1 \end{bmatrix} \tag{2}$$

where p_i and q_i must satisfy equation (3) and the initial damage vector $\mathbf{p}_o$ can be defined as equation (4).

$$p_i > 0, \quad p_i + q_i = 1 \tag{3}$$

$$\mathbf{p}_o = \{\pi_1, \pi_2, \ldots, \pi_b\}, \ \pi_i = \text{probability of being in state i at DC=0} \qquad \sum_{i=1}^{b} \pi_i = 1 \tag{4}$$

Generally π_1 has the value 1 and the other components are all zero for virgin materials. The probability of being in damage state i after x DCs have been applied is given by damage vector $\mathbf{p}_x$ as shown in equation (5).

$\mathbf{p}_x=\{ p_x(1), p_x(2), \ldots, p_x(b)\}$, $p_x(i)$ = probability of being in state i after x DCs

$$\sum_{i=1}^{b} p_x(i) = 1 \tag{5}$$

The damage vector $\mathbf{p}_x$ is calculated by equation (6).

$$\mathbf{p}_x = \mathbf{p}_{x-1} P = \mathbf{p}_o P^x \tag{6}$$

If the transition matrix P is different from duty cycle to duty cycle, then the damage vector is calculated by equation (7)

$$\mathbf{p}_x = \mathbf{p}_o \prod_{1}^{x} P_i \tag{7}$$

In order to estimate the value of b, p_i, and q_i we can use expected value and variance of duty cycles to failure W_f with the condition that $\pi_1 = 1$.

$$\begin{aligned} E[W_f] &= b-1+\sum_{i=1}^{b-1}\frac{p_i}{q_i} \\ Var[W_f] &= \sum_{i=1}^{b-1}\frac{p_i}{q_i}\left(1+\frac{p_i}{q_i}\right) \end{aligned} \tag{8}$$

In this model damage vector has no concern about orientation or the plane on which damage is accumulated because it is developed for uniaxial fatigue life. However in the case of variable multiaxial loading whose critical plane keeps changing during the loading and consideration of orientation of damage should be considered. In order to consider the orientation of damage accumulation we define damage vector at the plane of orientation θ after x DC was applied as equation(4) with the assumption that the same stochastic model of uniaxial fatigue data can be applied to the multiaxial loading cases with same equivalent stress level. Then $\mathbf{p}_x^\theta$ can be calculated for every orientation of interest similarly to $\mathbf{p}_x$ as following,

$$\mathbf{p}_x^\theta=\{ p_x^\theta(1), p_x^\theta(2), \ldots, p_x^\theta(b)\} \tag{9}$$

where, $p_x^\theta\ (i)$ = probability of damage state of orientation θ being in i after x DCs

$\mathbf{p}_x^\theta$ can be calculated from equation (10), and the orientation that gives the largest $p_x^\theta\ (b)$ is the most plausible failure plane, and x that gives $p_x^{\theta_{critical}}\ (b)$ the value 0.5 is the expected value of W_f, where $\theta_{critical}$ is the orientation of most plausible failure plane.

$$\mathbf{p}_x^\theta = \mathbf{p}_o^\theta \prod_1^x P_i \qquad (10)$$

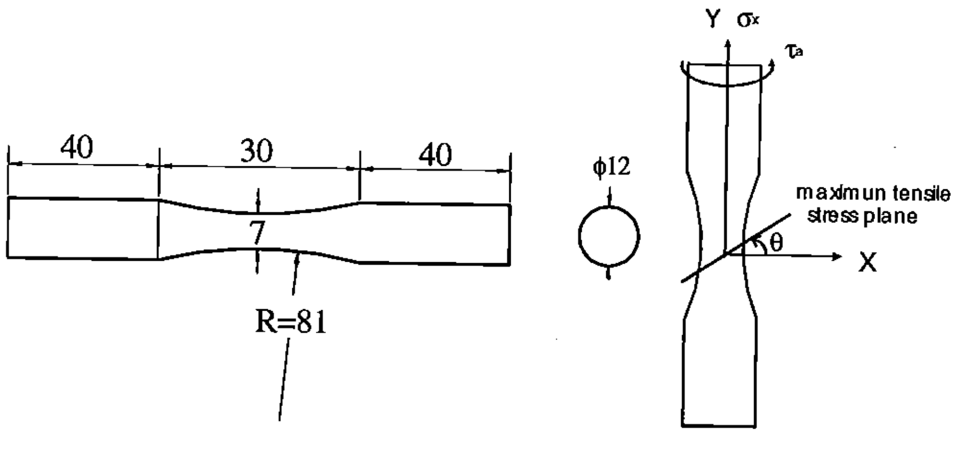
where $\mathbf{p}_o^\theta$, the initial damage vector at orientation θ can be defined similarly to $\mathbf{p}_o$

Experiment and Results

Uniaxial experiments

Uniaxial fatigue experiments were conducted using hour glass type specimens made of SM45C structural steel. The specimens' radii of curvature are 40.5 mm and the specimens were polished to number 2000 paper. Fig. 2. shows specimen geometry and coordinate system of the specimen.

Uniaxial fatigue tests were conducted to get S-N curve of uniaxial fatigue data for seven stress levels. Fig. 3. shows uniaxial S-N curve of stress amplitude versus mean life time. The stochastic modeling of variable multiaxial loading was developed under the assumption that the same stochastic model can be applied to the multiaxial loading cases with same equivalent stress level, so equivalent parameter should be decided considering cracking behavior of the material and it was found that maximum tensile plane is critical plane and positive tensile stress amplitude on that plane was taken to be equivalent stress. From the uniaxial S-N curve 828MPa, 803MPa, and 731 MPa were chosen to be test stress levels.



(a) (b)

Fig.2 (a) Specimen geometry (b) coordinate system, Dimensions in mm.

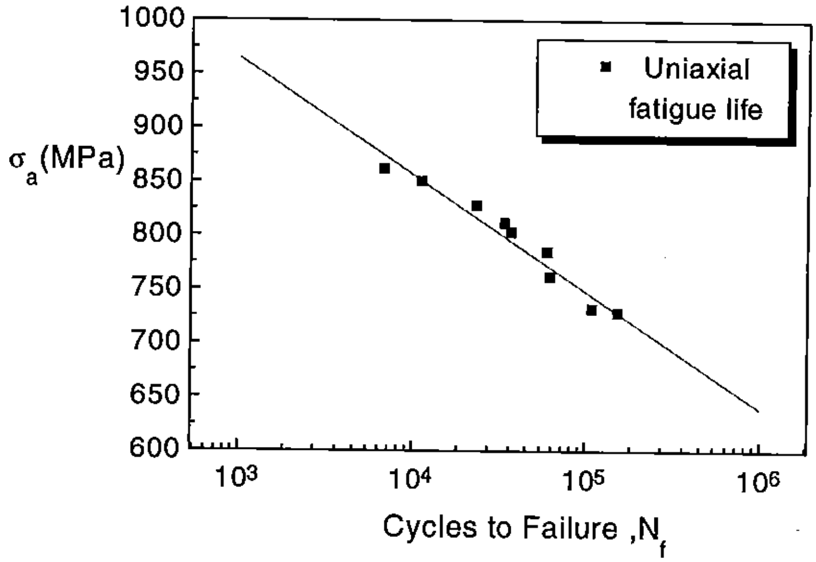


Fig. 3 Uniaxial S-N curve of SM45C steel

At each level of equivalent stresses uniaxial fatigue tests were repeated 20 times to construct the transition matrix P at each level. The transition matrix of each equivalent stress level denotes as P_a, P_b, and P_c. The statistical properties of fatigue life of three equivalent stress levels are given in Table 1 for DC=50cycles. According to the statistical property results of Table 1, P_a, P_b, and P_c were constructed to be the same size as shown in Table 2.

Table 1. Estimates of statistical properties of uniaxial fatigue life

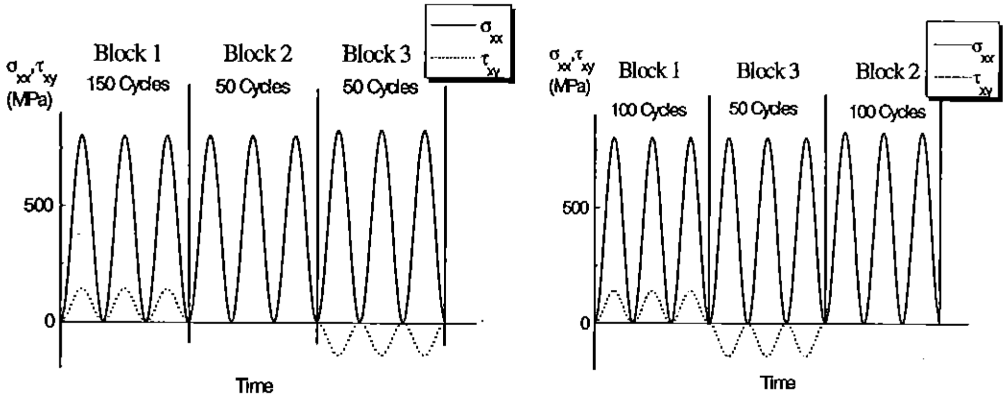
stress level	duty cycle mean, μ	duty cycle variance, σ^2	standard error
A (828 MPa)	464.28	5671.99	20.89
B (803 MPa)	738.81	11587.30	29.86
C (731 MPa)	2178.80	135674	47.10

Table 2. Transition matrices

stress level	b	p_i	q_i	
P_a	45	0.96057	0.03943	i =1,2,...,5
		0.88443	0.11557	i =6,...,44
P_b	45	0.94044	0.05956	i =1,2,...,44
P_c	45	0.99183	0.00817	i =1,2,...,5
		0.97511	0.02489	i =6,...,44

Variable multiaxial fatigue tests

Multiaxial fatigue tests were performed under two kinds of variable multiaxial loading cases composed of three multiaxial load blocks, namely loading case A and loading case B. Fig. 4 shows loading case A and case B, and stress state of each block and critical plane orientation θ are given at Fig. 5 and Table 3.



(a) (b)

Fig. 4 Load combination of loading case (a) case A (b) case B

Table 3. Stress state of the blocks

Block	σ_a	σ_m	τ_a	τ_m	ϕ	θ
1	803	401.5	141.6	70.8	0°	10°
2	828	414	0	0	0°	0°
3	803	401.5	141.6	-70.8	180°	-10°

When blocks are repeatedly applied to the material, critical plane i.e. maximum tensile plane keeps changing and different amount of damage is accumulated on each critical plane by the blocks. For example, when block 1 is applied $\sigma_{eq}=\sigma_n$ of $\theta = 10°$ plane is 828MPa, $\theta = 0°$ plane is 803MPa, and $\theta = 10°$ plane is 731MPa. Now we define the transition matrix applied to a plane for 1 repetition of loading cases as follows. The results are shown in Table 4.

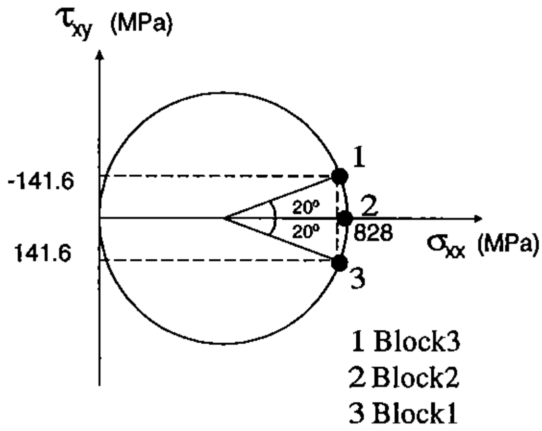


Fig. 5 Stress states of blocks

P_{α}^{θ} : Transition matrix of the plane of orientation θ under loading case α

Table 4. Transition matrices of the plane of interest

Loading case	$P_{\alpha}^{10^\circ}$	$P_{\alpha}^{0^\circ}$	$P_{\alpha}^{-10^\circ}$
A	$P_{A}^{10^\circ} = P_a P_a P_a P_b P_c$	$P_{A}^{0^\circ} = P_b P_b P_b P_a P_b$	$P_{A}^{-10^\circ} = P_c P_c P_c P_b P_a$
B	$P_{B}^{10^\circ} = P_a P_a P_c P_b P_b$	$P_{B}^{0^\circ} = P_b P_b P_b P_b P_a$	$P_{B}^{-10^\circ} = P_c P_c P_b P_a P_a$

Damage accumulation of each plane is calculated as equation (11) for 1 repetition of loading case. As 1 DC is 50 cycles, 1 repetition of loading case is equal to 5 DC.

$$\mathbf{p}_{5x}^{\theta} = \mathbf{p}_{o}^{\theta}\left(P_{\alpha}^{\theta}\right)^{x} \qquad (11)$$

CDFs (Cumulative density function) of the plane of interest under loading case A and case B are plotted in Figs. 6 and 7. Under loading case A 10° plane is most dangerous and under loading case B all three planes are almost equally dangerous.

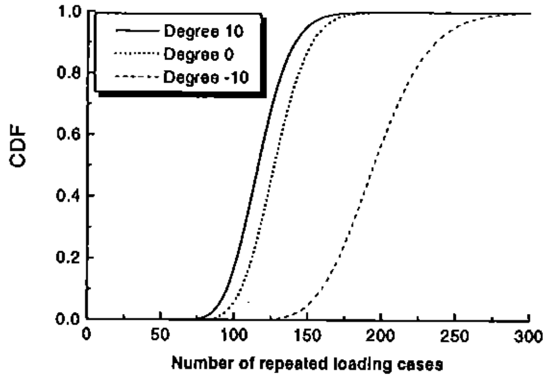


Fig. 6 CDFs of loading case A

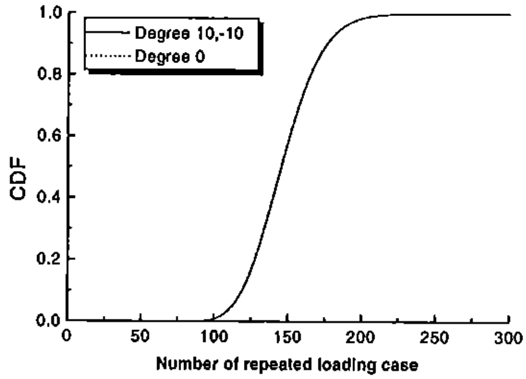


Fig. 7 CDFs of loading case B

The experimental results of variable multiaxial loading cases are give in Table 5. CDF from the model is plotted with experimental CDF in Figs. 8 and 9.

Table 5 Experimental results of variable multiaxial loading cases

	CDF	loading case A	loading case B
1	0.03571	57	99
2	0.10714	82	100
3	0.17857	86	108
4	0.25	89	114
5	0.32143	90	119
6	0.39286	100	124
7	0.46429	110	125
8	0.53571	112	126
9	0.60714	113	134
10	0.67857	114	137
11	0.75	121	142
12	0.82143	123	154
13	0.89286	152	155
14	0.96489	182	193

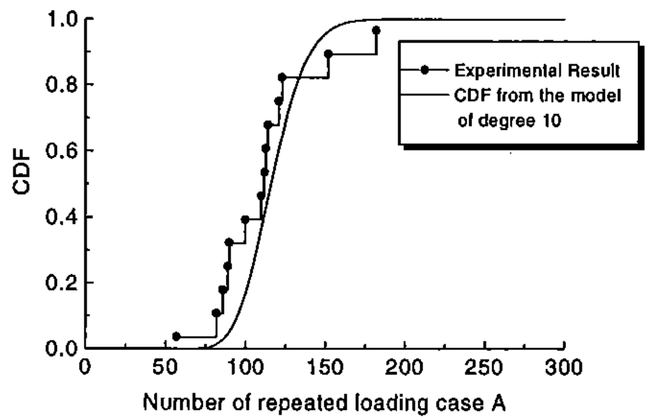


Fig. 8 CDF of loading case A

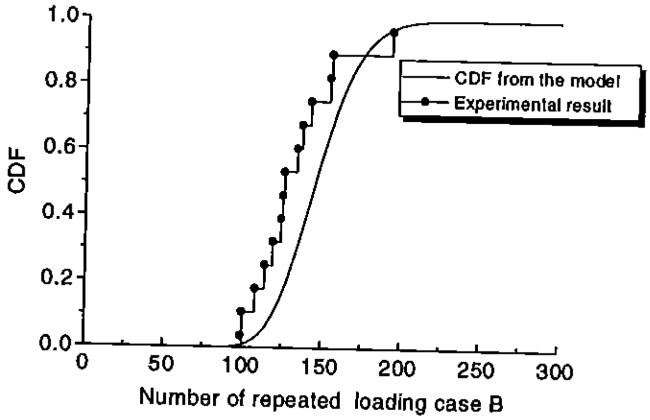


Fig. 9 CDF of loading case B

The CDFs calculated from the stochastic model showed relatively good accordance with experimental results as seen in previous figures and have a tendency to overestimate the life because it used positive tensile stress amplitude as the equivalent stress and shear stress was neglected, so it is expected that the model accuracy will increase if more proper equivalent parameter is selected. The expected number of repeated loading cases calculated from the model that gives 50% failure probability is 116 for loading case A and 145 for loading case B. Experimental results of number of repeated loading cases A and B of 50% failure probability are 111(4.3 % error) and 125.5 (12.6 % error). It is advantage of this model that one can conduct fatigue life prediction and reliability assessment simultaneously. Cracking orientation can also be predicted by selecting a plane that gives most small number of repeated loading cases of 50 % failure probability.

Conclusions

(1) A stochastic model which employs Markov chain model was developed for life prediction and reliability assessment under variable multiaxial fatigue loading.

(2) The assumption that the same stochastic model can be applied to the multiaxial loading cases with same equivalent stress level was found to be acceptable.

(3) The transition matrices applied to construct multiaxial stochastic model are obtained from the transition matrices of uniaxial fatigue tests.

(4) The CDFs calculated from the stochastic model showed relatively good accordance with experimental results and it can be also used for reliability assessment and cracking orientation prediction.

References

(1) Lee, S. B.(1980) Evaluation of theories of multiaxial fatigue with discriminating specimen, *Ph. D. thesis*, **Stanford University.**

(2) Lee, S. B.(1985) A Criterion for fully reversed out-of-phase torsion and bending, *Multiaxial Fatigue*, **ASTM STP 853**, 553-568.

(3) You, B. R., Lee, S. B.(1996) A critical review on multiaxial fatigue assessments of metals, *Int. Journal of Fatigue*, **18**, 235-244.

(4) Bannantine, J. A., Socie, D. F.(1990) Multiaxial fatigue life estimation techniques, *Advances in Fatigue Lifetime Predictive Techniques*, **ASTM STP 1122**, 249-275.

(5) Bannantine, J. A., Socie, D. F.(1991) A variable amplitude multiaxial fatigue life prediction method, *Fatigue under biaxial and multiaxial loading*, **ESIS 10**, (K. Kussmaul, D. Mcdiarmid and D. Socie eds.) Mechanical Engineering Publications. London, 36-51.

(6) Bannantine, J. A., Socie, D. F.(1991) Variable amplitude multiaxial fatigue: The influence of stress state, **EIS 90**-Engineering Integrity Through Testing.

(7) Wang, C. H., Brown, M. W.(1996) Life prediction techniques for variable amplitude multiaxial fatigue-part1 : Theories, *Journal of Engineering Materials and Technology*, **118**, 367-370.

(8) Wang, C. H., Brown, M. W.(1996) Life prediction techniques for variable amplitude multiaxial fatigue-part2 : Comparison with experimental results, *Journal of Engineering Materials and Technology*, **118**, 371-374.

(9) Stefanov, S. H.(1993) A curvilinear integral method for multiaxial fatigue life computing under non-proportional, arbitrary or random stressing, *Int. Journal of Fatigue*, **15**, 467-472.

(10) Stefanov, S. H.(1995) The curvilinear integral method : computerization and testing 1 (under non-proportional reversed axial force and torque), *Int. Journal of Fatigue*, **17**, 567-575.

(11) Stefanov, S. H.(1995) The curvilinear integral method : computerization and testing 2 (under non-proportional pulsating axial force and internal pressure), *Int. Journal of Fatigue*, **18**, 41-48.

(12) Lagoda, T., Macha, E.(1994) Estimated and experimental fatigue lives of 30CrNiMo8 steel under in-and out-of phase combined bending and torsion with variable amplitudes, *Fatigue Fract. Engng. Mater. Struct.*, **17**, 1307-1318.

(13) Weibull, W. (1951) A statistical distribution function if wide application, *Journal of Applied Mechanics*, **18**, 293-297.

(14) Bogdanoff, J. L.(1978) A new cumulative damage model, Part 1, *Journal of Applied Mechanics*, **45**, 245-250.

(15) Bogdanoff, J. L., Kreiger, W.(1978) A new cumulative damage model, Part 2, *Journal of Applied Mechanics*, **45**, 251-257.

(16) Bogdanoff, J. L.(1978) A new cumulative damage model, Part 3, *Journal of Applied Mechanics*, **45**, 733-739

(17) Bogdanoff, J. L., Kozin, F.(1980) A new cumulative damage model, Part 4, *Journal of Applied Mechanics*, **47**, 40-44.

(18) Tanaka, S., Ichikawa, M., Akita, S.(1984) A probabilistic investigation of fatigue life and cumulative cycle ratio, *Engineering Fracture Mechanics*, **20**, 501-513.