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Gradient-Dependent Multiaxial High-Cycle Fatigue Criterion

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ABSTRACT The purpose of the present paper is precisely the formulation of a multiaxial fatigue criterion that takes into account the stress gradient effect. The work starts with a careful analysis of experimental results of uniaxial cyclic normal stresses (such as bending) and of cyclic shear stresses (such as torsion). This examination shows that the fatigue endurance is mainly affected by the gradient of normal stresses. Variations of the gradient of shear stress in torsion tests, do not alter significantly the corresponding endurance limit. Based on these conclusions a gradient dependent fatigue criterion of the –critical plane type– is formulated. The critical plane is the plane of maximum shear stress amplitude. The criterion is expressed as a function of the shear stress amplitude, of the normal stress acting on the critical plane and of the gradient of the normal stress. The proposed formula is then applied in fully reversed in-phase bending and torsion loading. It is theoretically demonstrated that the proposed gradient-dependent criterion leads, for this kind of loading, to the well-known ellipse arc formula of Gough and Pollard. A noticeable result is that within the proposed framework the discrepancies apparently existing between experimental results with gradient-free stress conditions and of results incorporating non-zero stress gradients are eliminated.

1 Introduction and Brief Overview of Previous Work

For a uniaxial normal stress load, it is well known that the presence of a stress gradient has a net beneficial effect on the corresponding endurance limit. For example, the endurance limit in fully reversed bending of many metals, is significantly higher than the corresponding limit in fully reversed tension–compression. The higher values of bending endurance limits have to be attributed to the beneficial influence of the gradient of the normal stress on the fatigue strength of the metal. Although some attempts for modelling the stress gradient effect under uniaxial normal stress cyclic loads have already been presented, similar attempts for multiaxial stress conditions have not yet been carried out in a systematic way.

The multiaxial high-cycle fatigue criterion is a rather old research topic. The systematic study of the fatigue behaviour of metals under multiaxial stress conditions was initiated by Gough and Pollard (1) who proposed as early as 1935 the ellipse quadrant empirical formula for mild metals subjected to in-phase bending-twisting. The work of Gough and Pollard has been completed with

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the participation of Clenshaw in 1951 (2), with the proposal of the ellipse arc formula for hard metals:

$$\left(\frac{\tau_a}{t}\right)^2 + \left(\frac{f}{t} - 1\right)\left(\frac{\sigma_a}{f}\right)^2 + \left(2 - \frac{f}{t}\right)\frac{\sigma_a}{f} \leqslant 1 \quad (1)$$

where σ_a and τ_a are respectively the amplitude of the normal stress due to bending and of the shear stress due to torsion. The constants f and t are the endurance (fatigue) limits under fully reversed bending and fully reversed torsion respectively. The above-mentioned ellipse quadrant formula for mild metals is a particular case of equation (1) corresponding to $f/t = 2$. After the pioneering work of Gough and Pollard many researchers addressed the question of the multiaxial fatigue criterion. Several proposals have been accumulated over the years. Some well-known old formulas are the Sines, Crossland and Findley criteria. More recent proposals have been formulated by Dang Van and McDiarmid. A critical examination of these proposals can be found in (3). However, none of these approaches have addressed the question of the influence of the stress gradient on the fatigue endurance of metals.

The study of the stress gradient related problem has been developed in a somewhat independent way mainly within the framework of uniaxial stress systems (bending) or in relation to the fatigue resistance of notched specimens. The notch effect will not be examined specifically in this work. Early research of Moore and Morkovin (4) on rotating–cantilever bending of plain cylindrical specimens of the same length but of different radii showed a substantial increase of the fatigue limit with the decrease of the radius (that is with the increase of the normal stress gradient). These authors proposed an explanation of the size effect based on the assumption that a specimen which fails under reversed bending behaves as if a fatigue crack started slightly below the surface of the specimen, where the stress is slightly lower than that at the surface. The depth of this critical surface layer is supposed to be a material constant. Rotating–cantilever bending tests have been conducted also by Cazaud (5) in France, in the early sixties. Cazaud reached similar conclusions as Moore and Morkovin. Constant moment rotating bending tests on specimens of various radii and lengths have been carried out by Pogoretskii and Karpenko in the middle of the seventies (6). They found a decrease of the fatigue limit with the increase of the radius but also with the increase of the length of the specimens. A comprehensive analysis of the gradient and size effects based on a statistical theory of fatigue failure has been carried out by Pavan (7). More recently, an empirical gradient-dependent model for uniaxial normal stress states has been proposed by Brand and Sutterlin (8).

An attempt at modelling the gradient effect under multiaxial fatigue loading has been made by Flavenot and Skalli (9). This work is merely an extension of the concept of the critical surface layer depth advanced for uniaxial stress states by Moore and Morkovin. The more recent work on gradient-dependent multiaxial fatigue criterion that the authors are aware of is a paper by Munday and Mitchell (**10**). These authors used the Sines criterion to analyse fully reversed biaxial fatigue data obtained by Sawert (**11**) in the early fifties. In the plane of the amplitudes of the two existing principal stresses σ_{1a} and σ_{2a} the Sines criterion is an ellipse. Monday and Mitchell showed that among Sawert's results the gradient-free data fall on the Sines ellipse, whereas the data with a non-zero stress gradient fall outside the Sines ellipse, thus proving the beneficial effect of the stress gradient. However these authors did not formulate any fatigue formula incorporating the stress gradient.

The purpose of the present work is precisely to propose a gradient-dependent fatigue criterion and evaluate its predictive capability for non-zero gradient multiaxial stress systems. First, uniaxial experimental results (bending and torsion) will be carefully examined to get some insight into the gradient effect. Finally, it is useful to notice that often the gradient problem has been referred to in the literature as a size effect problem. In this work an effort will be made to distinguish between gradient effect and pure size effect.

2 Uniaxial Non-Zero Gradient Fatigue Tests

2.1 *Fully reversed bending*

Many tests indicate that the fully reversed tension–compression fatigue limit is the lowest that can be obtained from a fully reversed normal stress uniaxial test. Fatigue limits obtained under fully reversed bending conditions are always higher than the tension–compression limits. The higher values of bending fatigue limits have to be attributed to the beneficial influence of the normal stress gradient on the fatigue resistance of the metal.

Fully reversed bending conditions can be distinguished in plain bending and rotating bending. For identical cylindrical specimens of the same material the fatigue limit in plain bending is usually higher than the limit obtained in rotating bending (**5**). This difference could be explained on the basis of a statistical approach as in rotating bending all the external surface of the specimen undergoes the same (maximum) stress range, whereas in plain bending only the upper and lower parts of the external surface of the specimen are submitted to the maximum stress range. However, this problem will not be addressed here. Therefore, for the purpose of the present work, which is a first approach to the gradient effect under multiaxial stress conditions, differences between plain bending and rotating bending will be disregarded.

A more important distinction to be taken into account is the distinction between constant moment tests (such as four-point bending) and varying moment experiments (cantilever bending). In the four-point bending test the critical section is any section in the interval $L \leqslant x \leqslant L + l$, where the bending moment is the same at any time and equal to $M = FL$, Fig. 1a. In this interval the bending stress σ_{xx} does not vary in the axial direction (x-axis). It varies only along the depth of the specimen (y-axis). The gradient of σ_{xx} is a vector the

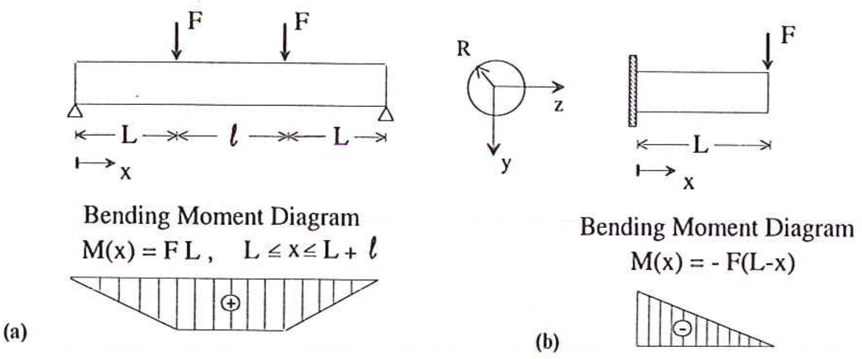

Fig 1 Cantilever and constant moment bending tests.

components of which are the partial derivatives of σ_{xx} with respect to x, y and z. Here the gradient has only one non-zero component, the derivative with respect to y. The bending stress and its gradient are then

$$\sigma_{xx} = \frac{FL}{I}\,y, \left[\frac{\partial\sigma_{xx}}{\partial x} = 0 \quad \frac{\partial\sigma_{xx}}{\partial y} = \frac{FL}{I} \quad \frac{\partial\sigma_{xx}}{\partial z} = 0\right],$$

$$L \leqslant x \leqslant L + l, \; -R \leqslant y \leqslant R \tag{2}$$

In the cantilever test the bending moment is a linear function of x, $M = F(L - x)$, Fig. 1b. Therefore, the normal stress due to bending varies along the depth and the length of the specimen. Its gradient has two non-zero components, the derivatives with respect to x and y. The bending stress and its gradient are given by the formulas

$$\sigma_{xx} = \frac{-F(L-x)}{I}\,y, \left[\frac{\partial\sigma_{xx}}{\partial x} = \frac{F}{I}\,y \quad \frac{\partial\sigma_{xx}}{\partial y} = \frac{-F(L-x)}{I} \quad \frac{\partial\sigma_{xx}}{\partial z} = 0\right],$$

$$0 \leqslant x \leqslant L, \; -R \leqslant y \leqslant R \tag{3}$$

At the most strained point of the cantilever ($x = 0$, $y = R$), equations (3), provide the gradient of the stress component

$$\frac{\partial\sigma_{xx}}{\partial x} = \frac{-\sigma_{xx}}{L} \quad \frac{\partial\sigma_{xx}}{\partial y} = \frac{\sigma_{xx}}{R} \quad \frac{\partial\sigma_{xx}}{\partial z} = 0 \tag{4}$$

where $\sigma_{xx} = -(FL/I)R$ is the normal stress at $x = 0$, $y = R$. Hence, in view of equations (4), the fatigue limit in the case of cantilever bending will depend on both the radius and the length of the specimen. For constant moment tests a fatigue crack can potentially appear at $L \leqslant x \leqslant L + l$, $y = R$. The gradient of the normal stress from equations (2) is

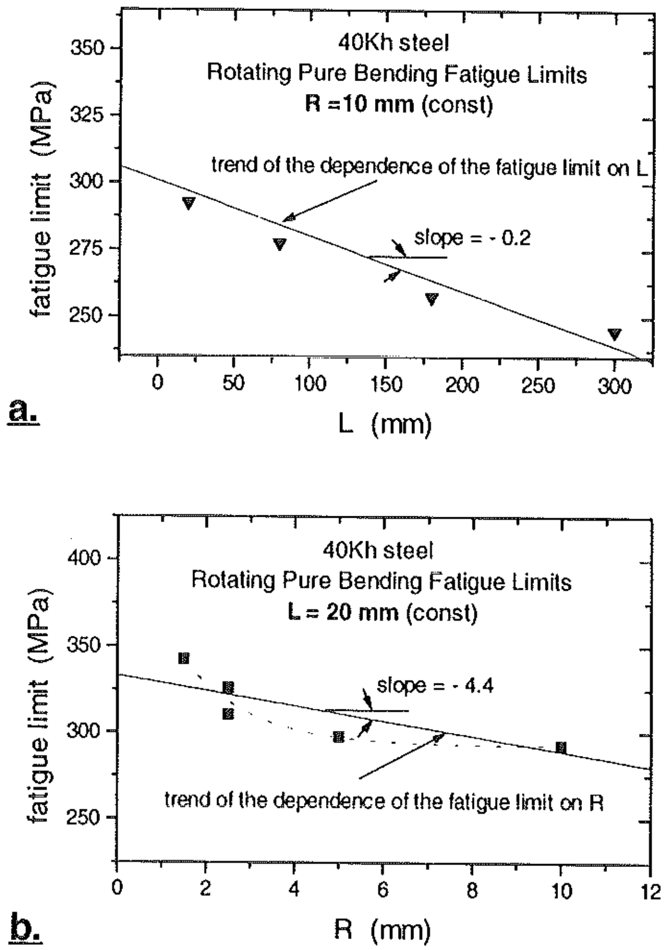

Fig 2 Pogorotskii and Karpenko constant moment bending fatigue limit data.

$$\frac{\partial \sigma_{xx}}{\partial x} = 0 \qquad \frac{\partial \sigma_{xx}}{\partial y} = \frac{\sigma_{xx}}{R} \qquad \frac{\partial \sigma_{xx}}{\partial z} = 0 \tag{5}$$

where $\sigma_{xx} = (FL/I)R$. Therefore the fatigue limit will *a priori* be independent of the length of the specimen.

Constant moment rotating bending experiments performed by Pogoretskii and Karpenko (6) and by Pavan (7) on cylindrical specimens of the same radius and different lengths indicate however a dependence of the fatigue limit on the length of the specimen. This dependence cannot be explained within the framework of a gradient-dependent theory and must be attributed to a pure size effect. To model the pure size effect a statistical theory of fatigue (non) failure might be used. Varying moment experiments (cantilever bending) also indicate a dependence of the fatigue limit on the length of the specimens. This, in view of equations (4), may be captured, at least partly, through the gradient of the stress components. Figure 2 shows constant moment rotating bending results adapted from the work of Pogoretskii and Karpenko (6).

In Fig. 2a the pairs (fatigue limit, L) have been plotted for specimens of the same radius. This figure shows the pure size effect on the fatigue limit and not any gradient effect, because for constant moment bending tests the gradient of the normal stress depends only on R, see equation (5). In Fig. 2b the pairs (fatigue limit, R) have been reported for constant length specimens. This figure represents the gradient effect on the fatigue limit. A rough estimate of the influence of the radius R and the length L on the fatigue limit f, can be gained by inspecting the general (linear) trends of the graphs f vs R and f vs L.

It can be seen from these figures that the general trend of the dependence of the fatigue limit on R has a slope equal to -4.4 (Fig. 2b), where the trend of the dependence on L has a slope of only -0.2 (Fig. 2a). This means that the influence of the stress gradient on the fatigue limit seems to be an order of magnitude higher than the influence of the pure size effect. According to tension–compression experiments performed by Phillips and Heywood (**12**), there is no influence of the specimen size on the fatigue limit. Similar conclusions can be drawn from the work of Massonet (**13**) who observed reductions of the order of 5% in tension–compression fatigue limits and of more than 20% in rotating bending fatigue limits for cylindrical specimens with increasing radii. To summarize, the size effect on constant-moment bending fatigue limits is an order of magnitude weaker than the stress gradient effect. The influence of the size seems to be even weaker on the tension–compression fatigue limit.

Only fully reversed uniaxial normal stress systems have been examined in this section. It is reminded that a tensile mean normal stress reduces the endurance limit, whereas a compressive mean stress leads to a net increase. Experiments show that the dependence of the fatigue limit on a mean normal stress can be accurately described by a linear relationship (**14**).

2.2 *Fully reversed and asymmetrical torsion tests*

The study of the fatigue endurance of metals under shear stress loading is carried out with the help of cyclic torsion tests. These experiments can be fully reversed and/or asymmetrical torsion tests. An asymmetrical torsion test is a test where a static torque is superimposed on a varying torque. The gradient of the shear stress is inherent in torsion tests. Therefore, to study the influence of the shear stress gradient on the fatigue endurance we have not at our disposal two distinct groups of results; one gradient free and the other with non-zero stress gradient, as in uniaxial normal stress loading (such as tension–compression and bending).

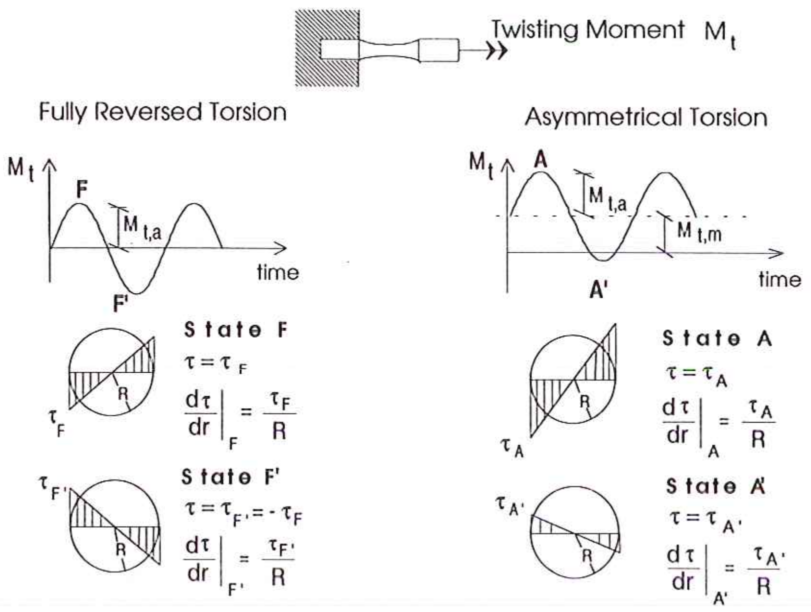

Fig 3 Stress states in fully reversed and asymmetrical cyclic torsion experiments.

To draw some conclusions on the shear stress gradient effect, results from fully reversed and asymmetrical torsion tests will be analysed. First of all it is important to point out that the fatigue limit in torsion does not depend on the mean value of torsion. This means that the limiting amplitude of the shear stress, corresponding to the non-fracture by fatigue of two specimens, one subjected to fully reversed torsion and the other to asymmetrical torsion, is the same. The uniqueness of the fatigue limit in torsion is a well-established experimental fact (**14**).

In Fig. 3 an analysis of the shear stress states is presented for two sinusoidal torsion loadings, one fully reversed and the other with a mean value. Let us assume that the two loadings are of a magnitude corresponding to the fatigue limit level. The extreme states in the case of fully reversed torsion will be denoted as states F and F', whereas the corresponding states for asymmetrical torsion will be denoted as A and A' (see Fig. 3). Let us examine the shear stress distribution for the homologous states F and A. Clearly the shear stresses τ_F and τ_A are different due to the mean torsion present in the asymmetrical loading case. Consequently the gradient of the shear stress is also different in the two cases (Fig. 3). Identical observations can be made for the states F' and A'. Therefore one has

$$\left\{\tau_F \neq \tau_A, \left.\frac{d\tau}{dr}\right|_F \neq \left.\frac{d\tau}{dr}\right|_A\right\} \quad \text{and} \quad \left\{\tau_{F'} \neq \tau_{A'}, \left.\frac{d\tau}{dr}\right|_{F'} \neq \left.\frac{d\tau}{dr}\right|_{A'}\right\} \tag{6}$$

However, the amplitude of the shear stress is the same in both loading cases as it was pointed out before and equal to the (unique) fatigue limit in torsion.

$$\frac{\tau_F - \tau_{F'}}{2} = \frac{\tau_A - \tau_{A'}}{2} = t \tag{7}$$

The uniqueness of the fatigue limit in torsion implies two things: its independence of a mean torsion but also *its independence with respect to the gradient of the shear stress.* The two cyclic torsion loads examined above led to the same (unique) torsion fatigue limit, although the mean shear stress states and the shear stress gradients were different. Based on this conclusion it is natural to assume that under a general multiaxial cyclic loading the limiting stress state corresponding to non-fracture by fatigue is unaffected by the presence of a static shear stress state and also by the gradient of the shear stresses.

3 Multiaxial Gradient-Dependent Fatigue Criterion

3.1 *Formulation of the criterion*

A critical plane fatigue formula will be used as a basis for the development of a gradient-dependent fatigue criterion. The critical plane, denoted as P, is the plane on which the shear stress amplitude becomes maximum. The fatigue formula is a linear combination of this shear stress amplitude, denoted as $\mathcal{T}_a$, and of the maximum value $\mathcal{N}_{max}$, that the normal stress $\mathcal{N}$ acting on the critical plane P reaches during a loading cycle, i.e.

$$\mathcal{T}_a + \alpha\mathcal{N}_{max} \leqslant \gamma \tag{8}$$

where α and γ are material parameters. To fix ideas let us consider a body cyclically loaded in a proportional manner with the external loads varying according to a sinusoidal law. The stress state around the most strained point of the body, denoted as O, can then be written as

$$\sigma_{ij} = \hat{\sigma}_{ij}\sin(\omega t) + \bar{\sigma}_{ij}, \quad i, j = x, y, z \tag{9}$$

where $\hat{\sigma}_{ij}$ is the amplitude of the stress and $\bar{\sigma}_{ij}$ its mean value. Let us denote by n_i, $i = x, y, z$, the unit vector normal to the critical plane P. For the loading described by equation (9), the shear stress amplitude $\mathcal{T}_a$ is then given by

$$\mathcal{T}_a = \sqrt{\hat{\sigma}_{kl}n_l\hat{\sigma}_{km}n_m - (n_i\hat{\sigma}_{ij}n_j)^2} \tag{10}$$

and the maximum value $\mathcal{N}_{max}$ of the normal stress is given by

$$\mathcal{N}_{max} = n_i(\hat{\sigma}_{ij} + \bar{\sigma}_{ij})n_j \tag{11}$$

The summation convention over repeated indices has been assumed. The criterion described by equation (8) reproduces correctly the independence of the limiting stress state from a superimposed static shear as well as its linear dependence with respect to a static normal stress state. This criterion will now be modified to include the experimentally observed beneficial effect of the stress gradient. If the stress state given by equation (9) is not homogeneous, then the shear stress amplitude $\mathcal{T}_a$ and the normal stress $\mathcal{N}_{max}$ will vary with little variations of the coordinates x, y and z, around the point O of the body under consideration. However the variation of $\mathcal{T}_a$ (its gradient) should not have any influence on the limiting stress state since as it was seen in the previous section the torsion fatigue limit is unique. It remains to examine the gradient of $\mathcal{N}_{max}$. The normal stress $\mathcal{N}_{max}$, as defined by equation (11) is a scalar. Therefore its gradient will be a vector, denoted as $\mathcal{G}$, with the components

$$\mathcal{G} = \left[\frac{\partial \mathcal{N}_{max}}{\partial x}, \frac{\partial \mathcal{N}_{max}}{\partial y}, \frac{\partial \mathcal{N}_{max}}{\partial z}\right] \tag{12}$$

The norm of this vector, denoted by $\mathcal{G}$ and given by the following relation,

$$\mathcal{G} = \sqrt{\left(\frac{\partial \mathcal{N}_{max}}{\partial x}\right)^2 + \left(\frac{\partial \mathcal{N}_{max}}{\partial y}\right)^2 + \left(\frac{\partial \mathcal{N}_{max}}{\partial z}\right)^2} \tag{13}$$

will be used as an indicator of the influence of the normal stress gradient. It is natural to assume at this point that the influence of the gradient $\mathcal{G}$ is associated with a non-zero value of $\mathcal{N}_{max}$. A possible way to satisfy this assumption is by introducing in the new criterion the product of $\mathcal{G}$ by $\mathcal{N}_{max}$ so that the influence of $\mathcal{G}$ vanishes with vanishing normal stress. In view of these, the following multiaxial gradient-dependent fatigue formula is proposed.

$$\mathcal{T}_a + \alpha \mathcal{N}_{max} - \beta\sqrt{\mathcal{G}\langle \mathcal{N}_{max}\rangle} \leqslant \gamma \tag{14}$$

where β is a material parameter and the MacCauley bracket $\langle\cdot\rangle$ is defined as

$$\begin{aligned} \langle \mathcal{N}_{max}\rangle &= \mathcal{N}_{max} \qquad \text{if} \qquad \mathcal{N}_{max} > 0 \\ \langle \mathcal{N}_{max}\rangle &= 0 \qquad \text{if} \qquad \mathcal{N}_{max} \leqslant 0 \end{aligned} \tag{15}$$

The above choice means that the gradient effect is neglected not only in the case of vanishing normal stress ($\mathcal{N}_{max} = 0$), but also in the case of a fully compressive normal stress cycle ($\mathcal{N}_{max} < 0$). This choice can be considered as a reasonable precaution at this stage of the model development, as in the authors best knowledge, experimental data where $\mathcal{N}_{max}$ is compressive do not exist.

3.2 *Identification of the material parameters*

The parameters α and γ of the proposed criterion given by equation (14) can be related easily to the fully reversed tension–compression fatigue limit denoted by s and to the torsion fatigue limit denoted by t. In fact in the case of torsion one has $\mathcal{T}_a = t$, $\mathcal{N}_{max} = 0$, $\mathcal{G} = 0$, and application of equation (14) provides γ as

$$\gamma = t \tag{16}$$

For fully reversed tension–compression, $\mathcal{T}_a = s/2$, $\mathcal{N}_{max} = s/2$, $\mathcal{G} = 0$ and α is equal to:

$$\alpha = \frac{2t}{s} - 1 \tag{17}$$

A fully reversed constant moment bending test can be used to obtain β. Let us denote by f the corresponding fatigue limit. Then $\mathcal{T}_a = \sigma_{xx}/2 = f/2$ and $\mathcal{N}_{max} = \sigma_{xx}/2 = f/2$. With the help of equations (5) and (13) the components of the gradient of $\mathcal{N}_{max}$ and its norm $\mathcal{G}$ are respectively

$$\mathcal{G} = \left[\frac{\partial \mathcal{N}_{max}}{\partial x} = 0 \ \frac{\partial N_{max}}{\partial y} = \frac{\partial(\sigma_{xx}/2)}{\partial y} = \frac{f}{2R} \ \frac{\partial \mathcal{N}_{max}}{\partial z} = 0\right], \quad \mathcal{G} = \frac{f}{2R} \tag{18}$$

Applying the criterion, equation (14), substituting the parameters γ and α from equations (16–17) and solving for β we obtain

$$\beta = 2\sqrt{R}\left(\frac{t}{s} - \frac{t}{f}\right) \tag{19}$$

In the following section a comparison of the predictions of the proposed criterion described by equation (14) with experimental results will be provided.

4 Applications

4.1 *Fully reversed bending*

The purpose of this paragraph is to relate the fatigue limits in fully reversed bending and fully reversed tension–compression. For the constant moment case the calculations have already been done before, during the evaluation of the parameter β. Applying the criterion, substituting γ and α and solving for f we obtain

$$f = \frac{s}{1 - \kappa/\sqrt{R}} \tag{20}$$

where the constant $\kappa = \beta(s/2t)$ has been introduced for convenience. It is interesting to apply the proposed criterion in the case of cantilever fully reversed bending. The corresponding endurance limit is denoted by f'. Then, $\mathcal{T}_a = \sigma_{xx}/2 = f'/2$, $\mathcal{N}_{max} = \sigma_{xx}/2 = f'/2$. With the help of equations (4) the components of the gradient $\mathcal{G}$ of $\mathcal{N}_{max}$ are

$$\frac{\partial \mathcal{N}_{max}}{\partial x} = \frac{\partial(\sigma_{xx}/2)}{\partial x} = \frac{-f'}{2L}, \ \frac{\partial \mathcal{N}_{max}}{\partial y} = \frac{\partial(\sigma_{xx}/2)}{\partial y} = \frac{f'}{2R}, \ \frac{\partial \mathcal{N}_{max}}{\partial z} = \frac{\partial(\sigma_{xx}/2)}{\partial z} = 0 \tag{21}$$

The norm $\mathcal{G}$ is equal to

$$\mathcal{G} = \sqrt{\left(\frac{-f'}{2L}\right) + \left(\frac{f'}{2R}\right)^2} \Rightarrow \mathcal{G} = \frac{f'}{2}\sqrt{\frac{1}{L^2} + \frac{1}{R^2}} \tag{22}$$

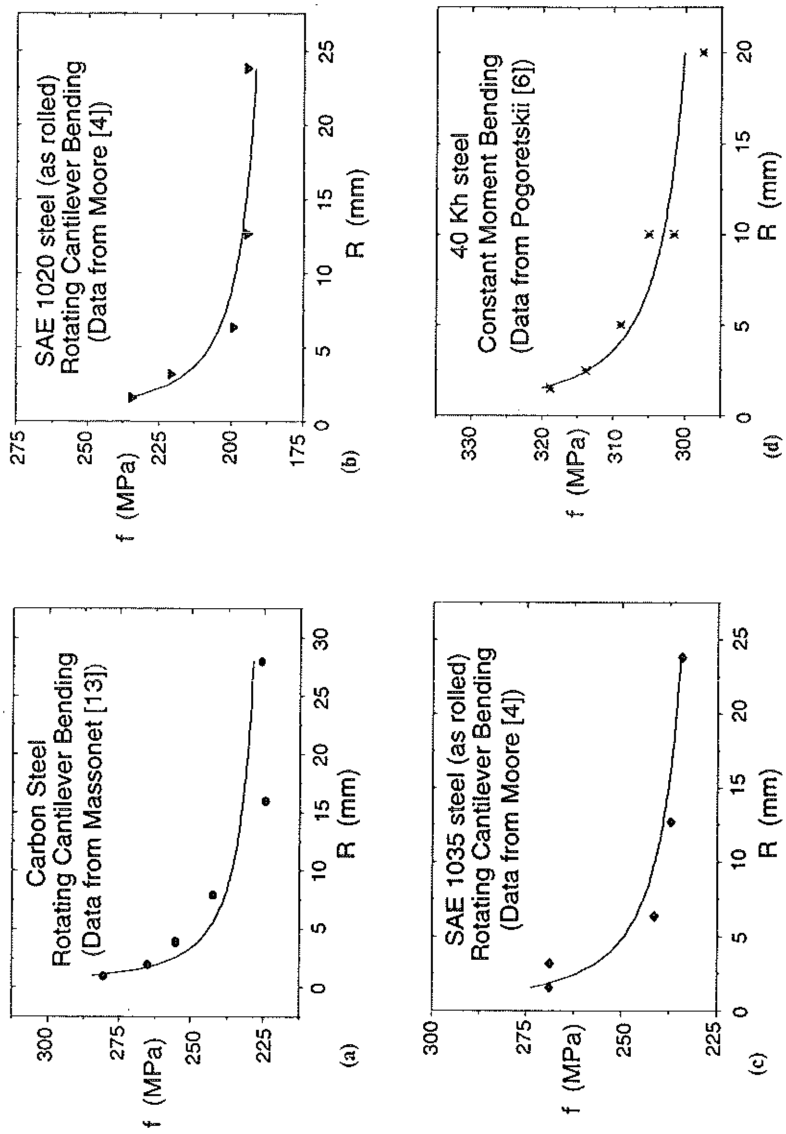

Fig 4 Bending fatigue limits of cylindrical specimens.

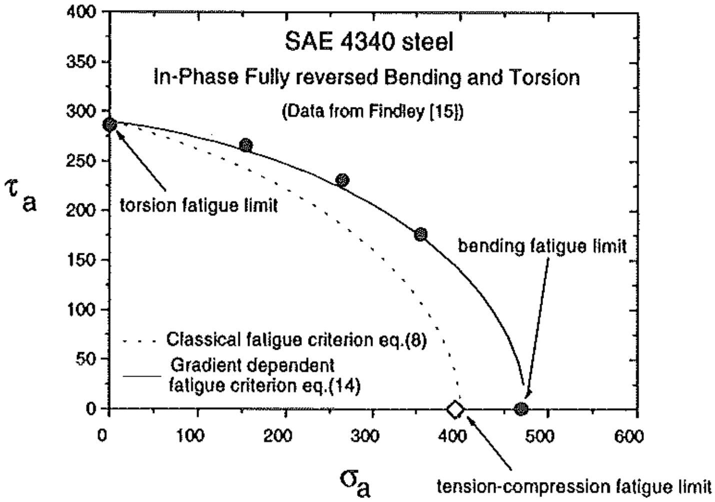

Fig 5 Fully reversed bending/twisting fatigue data for SAE 4340 steel.

Substituting the above values of $\mathscr{T}_a$, $\mathscr{N}_{max}$ and $\mathscr{G}$ into the proposed criterion, equation (14), introducing γ and α from equations (16–17) and solving for f' one obtains

$$f' = \frac{s}{1 - \dfrac{\kappa}{\sqrt{R}}\left(1 + \dfrac{R^2}{L^2}\right)^{1/4}} \tag{23}$$

where κ is the same as before. From equation (23) it is clear that the endurance limit in cantilever bending depends not only on the radius R of the circular section of the specimen but also on its length L. This dependence is due only to the gradient of the normal stress. It is not the demonstration of a size effect. For specimens with $R \ll L$, the ratio R^2/L^2 can be neglected. Under these circumstances the fatigue limits in fully reversed constant moment and cantilever bending of specimens of the same radius coincide and are related to the tension–compression limit by equation (20). Figure 4 shows some experimental results of rotating bending fatigue limits collected from the literature.

In the graphs above, the fatigue limits are plotted against the radius of the specimens. Figures 4a, b, c are from cantilever bending, where Fig. 4d is from constant moment tests. The solid curves in these graphs are the prediction by the new criterion. It is seen that the agreement with the experiments is satisfactory.

4.2 *Fully reversed combined bending and torsion fatigue tests*

Experimental results of fully reversed in-phase bending-twisting test of hard metals agree very well with the empirical ellipse quadrant criterion, given by

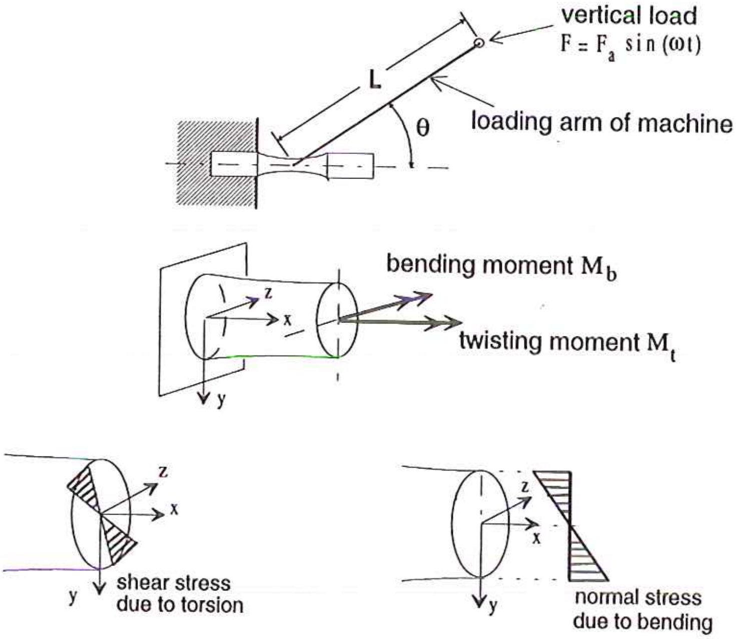

Fig 6 Stress state for combined bending and torsion experiments.

equation (1). From equation (1) it can be seen that the ellipse quadrant involves two material constants, the fatigue limit in fully reversed torsion t and the fatigue limit in fully reversed bending f. It is interesting to notice that if the classical 'critical plane type' criterion given by equation (8), is applied to combined tension–compression and torsion situations, an ellipse arc formula is obtained (**3**). This ellipse arc is identical to equation (1) with the exception that in the place of the bending fatigue limit f, the fully reversed tension–compression limit s appears

$$\left(\frac{\tau_a}{t}\right)^2 + \left(\frac{s}{t} - 1\right)\left(\frac{\sigma_a}{s}\right)^2 + \left(2 - \frac{s}{t}\right)\frac{\sigma_a}{s} \leqslant 1 \tag{24}$$

However, if one tries to predict the behaviour of the material in bending and torsion following the above tension–compression and torsion ellipse arc, he will find high discrepancies between predictions and experimental data. In Fig. 5 the results on bending-twisting obtained by Findley *et al.* (**15**) on SAE 4340 steel, have been reported.

In the same figure the ellipse arc, based on s and t equation (24), has also been plotted (dashed curve). All the experimental points fall considerably outside this ellipse arc. This demonstrates the beneficial effect of the gradient of the normal stress due to bending. Still, the same data agree very well with the Gough–Pollard ellipse arc, equation (1), based on the f and t fatigue limits (solid curve on Fig. 5).

It is our purpose in this section to prove that the proposed gradient-dependent fatigue criterion, equation (14), leads to the Gough–Pollard ellipse arc when applied in combined bending-twisting. On the other hand, for combined tension–compression and torsion, the proposed criterion reduces to the classical formula, equation (8), because of the absence of gradient of the normal stress (i.e. $\mathscr{G} = 0$). Therefore in that case the ellipse arc, equation (24), is recovered. Hence, the proposed approach links the gap between zero gradient and non-zero gradient stress states providing a 'smooth' passage from one state to the other. To prove this statement let us consider a combined bending and torsion experiment, (Fig. 6).

Specimens of toroidal shape are usually used for these tests. The minimal section of the specimen, where a fatigue crack can potentially appear, is subjected to pure bending and pure torsion. Considering a point lying on the y-axis, Fig. 6, the stress state is given by

$$\sigma_{xx} = \frac{M_b}{I} y, \quad \sigma_{xz} = \frac{M_t}{2I} y \tag{25}$$

The maximum straining occurs at $y = R$, where the amplitudes of the normal σ_{xx} and shear σ_{xz} stresses, denoted respectively as σ_a and τ_a, are

$$\sigma_a = \frac{M_b}{I} R, \quad \tau_a = \frac{M_t}{2I} R \tag{26}$$

The amplitude of the maximum shear stress $\mathscr{T}_a$ and the maximum normal stress $\mathscr{N}_{max}$ acting on the plane of $\mathscr{T}_a$ are given by

$$\mathscr{T}_a = \sqrt{\frac{\sigma_a^2}{4} + \tau_a^2}, \qquad \mathscr{N}_{max} = \frac{\sigma_a}{2} \tag{27}$$

Taking into account equation (25) the components of the gradient of $\mathscr{N}_{max}$ (the partial derivatives with respect to x, y and z) and its norm $\mathscr{G}$ are

$$\left[\frac{\partial \mathscr{N}_{max}}{\partial x} = 0, \frac{\partial \mathscr{N}_{max}}{\partial y} = \frac{\partial(\sigma_{xx}/2)}{\partial y} = \frac{M_b}{2I} = \frac{\sigma_a}{2R}, \frac{\partial N_{max}}{\partial z} = 0\right], \mathscr{G} = \frac{\sigma_a}{2R} \tag{28}$$

Applying the proposed criterion, equation (14), and substituting the parameters γ, α and β from equations (16–17) and (19) we obtain

$$\sqrt{\frac{\sigma_a^2}{4} + \tau_a^2} + \left(\frac{2t}{s} - 1\right)\frac{\sigma_a}{2} + 2\sqrt{R}\left(\frac{t}{s} - \frac{t}{f}\right)\frac{\sigma_a}{2\sqrt{R}} \leqslant t \tag{29}$$

It is recalled that β is identified from a fully reversed pure bending test. It is also assumed that this test has been performed on a specimen of the same radius

R as the specimens used for the combined bending–torsion tests. Obviously the radius R disappears from the above formula and after some elementary algebra equation (29) leads to the Gough–Pollard ellipse arc, equation (1). The proof is completed.

5 Conclusions

A gradient-dependent multiaxial fatigue criterion has been developed in the present work. A critical plane classical fatigue criterion has been used as a working hypothesis for the derivation of the gradient dependent criterion. This criterion has been adequately modified to include the stress gradient. The proposed formula has been obtained following a phenomenological approach. The guidelines for the development of the new criterion have been provided by the critical examination and comparison of uniaxial zero and non-zero gradient tests (bending, tension–compression, and torsion). Application of the proposed criterion in the case of fully reversed bending of cylindrical specimens of various radii, showed good agreement between theory and experiments. A remarkable result concerns the combined fully reversed bending and torsion tests. It is demonstrated that the proposed gradient-dependent criterion leads to the well-known ellipse arc formula of Gough and Pollard that depends on the fatigue limits in fully reversed bending and fully reversed torsion. Furthermore, the proposed formula for combined tension–compression and torsion loading leads to an ellipse arc that differs from the Gough–Pollard formula regarding only the bending fatigue limit which is replaced by the limit in tension–compression. Hence, the presented approach constitutes a unified framework for zero and non-zero gradient stress states, providing a 'smooth' passage from one state to the other. Finally it is of interest to note that by incorporating the stress gradient into the fatigue criterion, a so-called weak nonlocality has been introduced. In that respect this resembles the (weak) nonlocal gradient plasticity theories, (16).

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