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Calculation of Notch Strains for Nonproportional Cyclic Loading using a Structural Yield Surface

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ABSTRACT An approximate analytical method is developed to calculate notch-root stresses and strains in a notched bar of elastoplastic, isotropic material subjected to nonproportional multiaxial nominal loading. The notched structure is treated as an equivalent anisotropic material/structural element and the theory of plasticity of anisotropic metals is used to define a yield surface in nominal stress space that incorporates both the isotropic material properties and the anisotropic geometry factors of the notch. Notch-root plastic strain increments and workhardening effects are then related to this yield surface using standard methods of plasticity. The method is compared with strain gauge measurements and finite-element analyses of circumferentially notched shafts subjected to nonproportional tension–torsion nominal loading paths with zero mean nominal load. The strain calculations agree well both qualitatively and quantitatively with the experimental results and finite-element analyses, and are suitable for strain–life fatigue calculations.

Notation

E	elastic modulus
F, G, H, L, M and N	coefficients of anisotropy
K_{xz}	shear stress concentration factor
K_z	uniaxial stress concentration factor
K_z'	transverse stress concentration factor
$\bar{S}$	equivalent nominal stress
S_{ij}	nominal stress tensor
X, Y, Z, R, S and T	nominal yield strengths
$d\bar{\varepsilon}^p$	equivalent notch root plastic strain increment
$d\varepsilon_{ij}^p$	notch root plastic strain increment tensor
$d\lambda$	plasticity constant
h	hardening parameter
ε_{ij}^e	notch root elastic strain tensor
ν	Poisson's ratio
σ_{ij}	notch root tensor

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1 Introduction

Recent advances in the strain–life theory of fatigue have allowed the estimation of the fatigue life of a component from strains measured at critical locations for proportional and nonproportional multiaxial loading. However, to estimate the fatigue life of a component during the design process, strains at critical locations must be calculated from loads the component must carry and the geometry of the component.

For uniaxial loading of notched bodies, approximate formulas to calculate the notch root behaviour have been proposed (**3**, **4**, **14**, **15**). The behaviour of multiaxially loaded notched bodies has been of more recent interest. For proportional loading, approximate formulas have also been proposed (**7**, **12**). And lastly, nonproportional loading have been considered (**1**, **2**, **8**).

In this paper, the yield surface approach to calculate notch strains (**1**) will be restated, and compared with experimental results and nonlinear finite-element analyses of additional loading paths.

2 Strain Behaviour of Smooth and Notched Bars

For a smooth bar of isotropic material subjected to uniaxial tension in the elastoplastic range of the material, the strains at the surface of the bar in the transverse and normal directions are equal. That is, for the smooth bar, there are no preferred directions of material flow. For multiaxial states of stress, the strains are determined by treating a yield criterion (**11**) as a potential function in a suitable flow rule. Material workhardening characteristics are often related to the yield surface by translating or deforming the yield surface in stress space.

Analogous to the strain behaviour developed during uniaxial tension of a smooth bar, the strain behaviour developed during uniaxial tension of a notched bar will motivate the mathematical description of an assumed *structural* yield surface of the notched bar. In the elastic range of the notched bar, the notch root strains are related to the applied loading through the use of a *nominal* stress, S_z, an elastic stress concentration factor in tension, K_z, and a transverse stress concentration factor K'_z. The transverse stress concentration factor is a measure of the elastic notch constraint, and is a function of the geometry of the notch. The stress concentration factors can be determined by experiment, finite-element analysis, or in some cases, by theory of elasticity. By substitution of the non-zero components of local stress into Hooke's law, the elastic strains are related to the nominal stress by

$$\varepsilon_x^e = \frac{1}{E}[K'_z - \nu K_z]S_z \qquad \varepsilon_y^e = \frac{1}{E}[-\nu(K'_z + K_z)]S_z$$

$$\varepsilon_z^e = \frac{1}{E}[K_z - \nu K'_z]S_z \tag{1}$$

at the traction-free surface of the notch root.

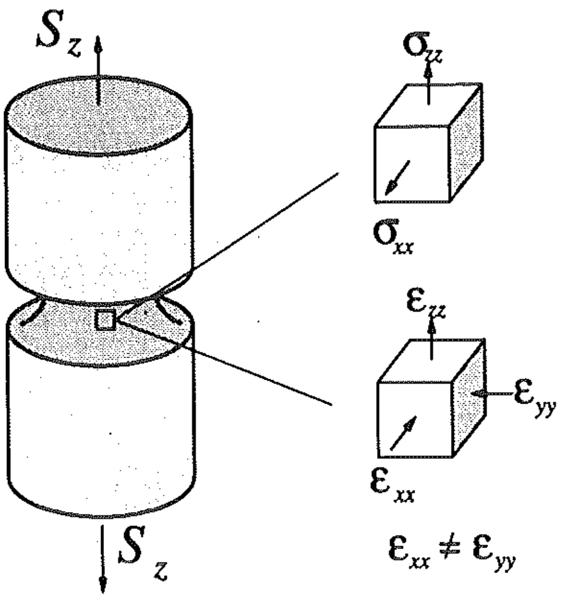


Fig 1 Notched bar stress and strain due to uniaxial load.

Equations (1) state that for an applied nominal load in the axial direction, strains in the directions transverse to the axial direction are not equal (Fig. 1), as in the case of the smooth bar. Similar behaviour occurs in the plastic range. The geometry of the notch, unyielded material around the notch, and the local multiaxial stress state constrain the deformation in the notch plane. For volume-conserving metals, this necessitates a larger plastic strain normal to the notch surface than in the transverse direction of the notched shaft.

3 Equations for a Notched Round Bar

In terms of the nominal stress, the notch root strains exhibit preferred directions of plastic flow, or anisotropy, when subjected to uniaxial nominal stress. Therefore, the notched structure may be treated as an equivalent element of anisotropic material that has a yield surface described by the developed theory of plasticity of anisotropic materials, stated in terms of nominal stress.

This structural yield criterion must not only incorporate the deformation behaviour induced by the geometry of the notch, it must also incorporate restrictions to the deformation behaviour induced by the material. Thus, for a metal component, the mathematical implications of the observed zero dilatation of plastic strain must also be incorporated into the yield criterion. In the following section, one particular anisotropic yield criterion will be chosen to represent the yield surface; however, other yield criteria could also be used.

3.1 *Equations for plastic strain increments*

Hill's theory of plasticity of anisotropic materials (5, 6) was developed to model metals that exhibited preferred directions of volume conserving plastic flow due to internal microstructure. As the strains at the notch root of a bar also exhibit preferred directions of plastic flow, Hill's theory will be applied to this application as well.

Hill generalized Mises' yield criterion by introducing coefficients in the Mises yield criterion. Hill's yield criterion is

$$2f(S_{ij}) \equiv F(S_y - S_z)^2 + G(S_z - S_x)^2 + H(S_x - S_y)^2 + 2LS_{yz}^2 + 2MS_{zx}^2 + 2NS_{xy}^2 = 1 \tag{2}$$

where the coefficients are determined from the current values of the directional yield strengths X, Y, Z, R, S, and T associated with the directions of the stress components S_x, S_y, S_z, S_{yz}, S_{xz}, and S_{xy}. These relations are

$$\begin{aligned} 2F &= \frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2} \qquad & 2L &= \frac{1}{R^2} \\ 2G &= \frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{Y^2} \qquad & 2M &= \frac{1}{S^2} \\ 2H &= \frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2} \qquad & 2N &= \frac{1}{T^2} \end{aligned} \tag{3}$$

An assumption made during the development of the yield criterion was that elastic normal and shear strains are uncoupled. The directions where this occurs are known as the principal directions of anisotropy. For the notched round bar, these directions coincide with the specimen axis, but can be found for other components made of isotropic material by a tensor rotation.

The yield criterion can be used as a plastic potential and the normality flow rule

$$d\varepsilon_{ij}^p = d\lambda \frac{\partial f}{\partial S_{ij}} \tag{4}$$

can be used in the principal directions of anisotropy to develop equations analogous to the Prandtl–Reuss equations of plasticity for isotropic materials. The equations for the notch root element of a bar subjected to nominal tension and torsion are

$$\begin{aligned} d\varepsilon_x^p &= d\lambda[-GS_z] \qquad & d\varepsilon_y^p &= d\lambda[-FS_z] \\ d\varepsilon_z^p &= d\lambda[(G + F)S_z] \qquad & d\varepsilon_{xz}^p &= d\lambda MS_{xz} \end{aligned} \tag{5}$$

Hill (6) then assumed an equivalent stress, derived an expression for equivalent strain, and determined an expression for the plasticity constant, $d\lambda$. These expressions are summarized as follows

$$\bar{S} = \left(\frac{3}{2}\right)^{1/2} \frac{h}{(F_0 + G_0 + H_0)^{1/2}} \tag{6}$$

$$\begin{aligned} h = [&F_0(S_y - S_z)^2 + G_0(S_z - S_x)^2 + H_0(S_x - S_y)^2 \\ &+ 2L_0 S_{yz}^2 + 2M_0 S_{zx}^2 + 2N_0 S_{xy}^2]^{1/2} \end{aligned} \tag{7}$$

$$\begin{aligned} d\bar{\varepsilon}^p = \left(\frac{2}{3}\right)^{1/2} &(F_0 + G_0 + H_0)^{1/2} \left[F_0 \left(\frac{G_0 d\varepsilon_y^p - H_0 d\varepsilon_z^p}{F_0 G_0 + G_0 H_0 + H_0 F_0}\right)\right. \\ &+ G_0 \left(\frac{H_0 d\varepsilon_z^p - F_0 d\varepsilon_z^p}{F_0 G_0 + G_0 H_0 + H_0 F_0}\right)^2 \\ &+ \left. H_0 \left(\frac{F_0 d\varepsilon_x^p - G_0 d\varepsilon_y^p}{F_0 G_0 + G_0 H_0 + H_0 F_0}\right)^2 + \frac{2d\varepsilon_{yz}^{p2}}{L_0} + \frac{2d\varepsilon_{zx}^{p2}}{M_0} + \frac{2d\varepsilon_{xy}^{p2}}{N_0}\right]^{1/2} \end{aligned} \tag{8}$$

$$d\lambda = \bar{S}\, d\bar{\varepsilon}^p \tag{9}$$

The equivalent quantities may be related to the uniaxial nominal stress–notch-root plastic strain curve by

$$\bar{S} = \left(\frac{3}{2}\right)^{1/2} \left(\frac{F_0 + G_0}{F_0 + G_0 + H_0}\right)^{1/2} S_z \tag{10}$$

$$d\bar{\varepsilon}^p = \left(\frac{2}{3}\right)^{1/2} \left(\frac{F_0 + G_0 + H_0}{F_0 + G_0}\right)^{1/2} d\varepsilon_z^p \tag{11}$$

The uniaxial behaviour of the notch can be determined from experiment, finite-element analysis, or a suitable approximate formula for uniaxial loading, such as those previously discussed.

3.2 *Material workhardening*

A model of material/structural workhardening is based on the assumption used in the analysis of anisotropic materials (6). Hill's assumption was that if there existed a pronounced preferred orientation in the material, then this orientation will remain in the same relative magnitude as the material workhardens. In this case, the current values of the directional yield strengths and, consequently the coefficients of anisotropy remain in the same proportion, and are related by the hardening parameter, h, equation (7). In other words, $X = hX_0$, $Y = hY_0$, ..., where the subscript 0 denotes the initial value.

For a multiple yield surface plasticity model, this assumption amounts to requiring that all surfaces are mathematically similar. For the notched bar, this should be approximately the case, until the notch is grossly distorted and the

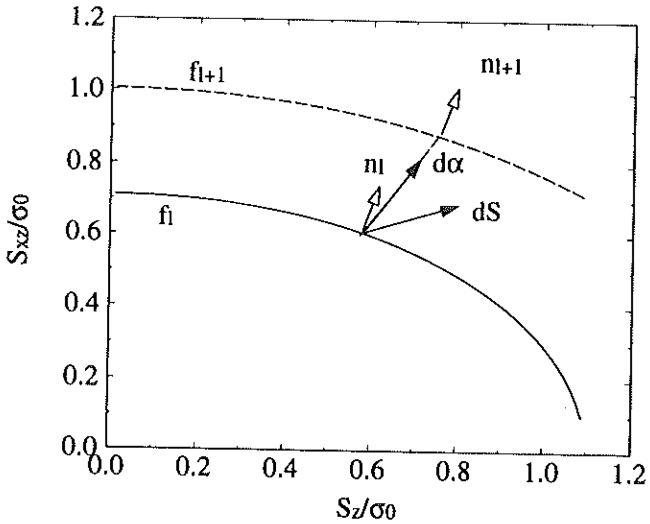


Fig 2 Mróz model in normalized nominal stress space.

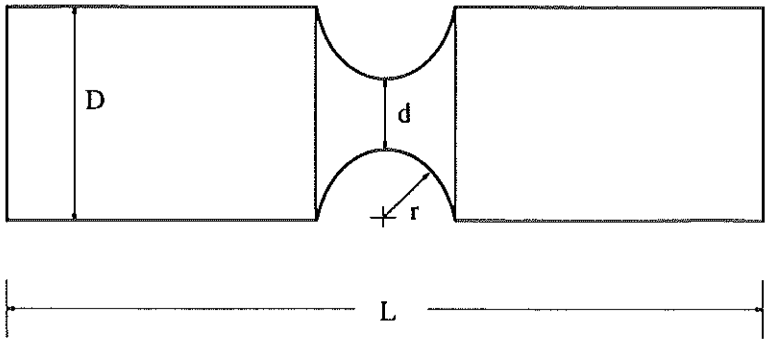
bar approaches net section plasticity. For the calculations presented later, a multiple yield surface model using the Mróz hardening rule (13) in nominal stress space (Fig. 2) was selected.

3.3 *Determination of coefficients of anisotropy*

The initial coefficients of anisotropy may be determined from the value of nominal stress that causes yielding at the notch-root, or from the ratios of strains during simple tension. The assumption that the yield stresses will remain in proportion leads to plastic strain increments remaining in the same proportion. For smooth bars, the Hill yield criterion reduces to Mises', and equation (5) reduces to the Prandt–Reuss equations. For mildly notched bars, the initial yield strengths can be calculated from the stress concentration factors at the notch root and the material yield strength. For sharply notched bars, where the notch root approaches a state of plane strain, the fully developed ratio of the plastic strain components deviates from the initial ratio as calculated from the stress concentration factors. For these cases, the coefficients can be determined from either an elastoplastic tensile finite-element analysis, or by assuming that the plastic flow in the hoop direction of the notched bar is zero.

3.4 *Elastic strains*

If the shaft is in the elastic range, the initial stress concentration factors may be used to find the elastic strain increments from



D = 50.8 mm d = 25.4 mm

L = 254 mm r = 12.7 mm

Fig 3 Fully notched round shaft geometry and dimensions.

$$d\varepsilon_x^e = \frac{1}{E}[K_z' - \nu K_z]\,dS_z \qquad d\varepsilon_y^e = \frac{1}{E}[-\nu(K_z' + K_z)]\,dS_z \tag{12}$$

$$d\varepsilon_z^e = \frac{1}{E}[K_z - \nu K_z']\,dS_z \qquad d\varepsilon_{xz}^e = \frac{1+\nu}{E}K_{xz}dS_{xz}.$$

However, if the material is not in the elastic range, the initial elastic stress concentration factors no longer hold. In this case, the elastic strain increments can be determined by considering the local notch stresses and the boundary conditions of the notch.

Local stress increments can be determined from the previously computed plastic strain increments and from the condition of the traction-free surface at the notch-root (or known pressure in the case of the inside wall of a pressure vessel). These stress increments can be substituted into Hooke's law to determine the local elastic strain increments. Total strains are then obtained by the addition of the elastic and plastic components of strain.

4 Experimental Set-up

To test the proposed method of calculating notch strains, solid shafts made of 1070 steel were machined with a circumferential notch to the dimensions indicated in Fig. 3. A strain gauge rosette was placed at the notch-root to record the surface strain state during testing. Proportional and nonproportional cyclic tension–torsion tests were conducted at several values of load, and on several specimens. The elastic stress concentration factors for the notched shaft were experimentally found to be $K_z = 1.45$, $K_{xz} = 1.15$, and $K_z' = 0.30$.

5 Finite-element Analysis

A converged finite-element mesh of the notched shaft was constructed by Köttgen (**10**). A three-dimensional slice of the shaft was modelled with boundary conditions of the faces of the slice being constrained to move the same in the radial and hoop directions. An Abaqus user material implementation of the Mróz work-hardening plasticity model (**9**) was used for the analyses of the nonproportional loading paths. The elastic stress concentration factors for the notched shaft from the finite-element analysis were found to be $K_z = 1.41$, $K_{xz} = 1.15$, and $K_z' = 0.26$.

6 Comparison of Methods

To demonstrate the applicability of a yield surface representation of the material behaviour, the notched shaft was first subjected to a nonproportional box-shaped tension–torsion loading path, Fig. 4(a). As demonstrated at low levels of load (**1**), such that the material at the notch-root had not yielded, the axial strain and shear strain response was uncoupled. This uncoupled behaviour appears as portions of the strain response parallel to the axes of axial or shear strain, in an axial strain *versus* shear strain plot. At increasing values of load, the material at the notch-root yields, and the shear and axial strains become coupled. The regions of strain coupling can be seen as the nonaxis parallel portions of the strain response in Fig. 4(b), for which the direction of travel around the loading path was counter-clockwise.

Such material behaviour is qualitatively expected from a yield surface material model. The coupled portions of the strain response correspond with the influence of the incremental plasticity constant, $d\lambda$, that increases as the equivalent stress increases when approaching each corner. At the corner points, the loading direction is abruptly changed to unload into the initial yield surface, which is modelled as the elastic region of the material. When the initial yield surface is completely traversed, the incremental plasticity constant again increases until the next corner is reached, and so on.

Both the simplified model and the finite-element model exhibit this equivalent strain behaviour, as demonstrated in Fig. 4(c–d). The finite-element model uses isotropic plasticity theory in terms of real stresses at each integration point in the model, using the material's uniaxial stress–strain response as the basis for the equivalent stress–strain curve. The simplified model, however, uses anisotropic plasticity theory to directly relate the applied nominal stresses to notch strain, using the structural uniaxial stress–notch strain response as the basis of the equivalent nominal stress–notch strain curve. Because the strain response is directly calculated from the nominal load, the calculation time required is significantly less for the simplified model than the finite-element model.

The data required for the calculation of the surface strains of notched bars are given in Table 1. The minimum requirements for mildly notched shafts are the elastic stress concentration factors, smooth specimen material properties for

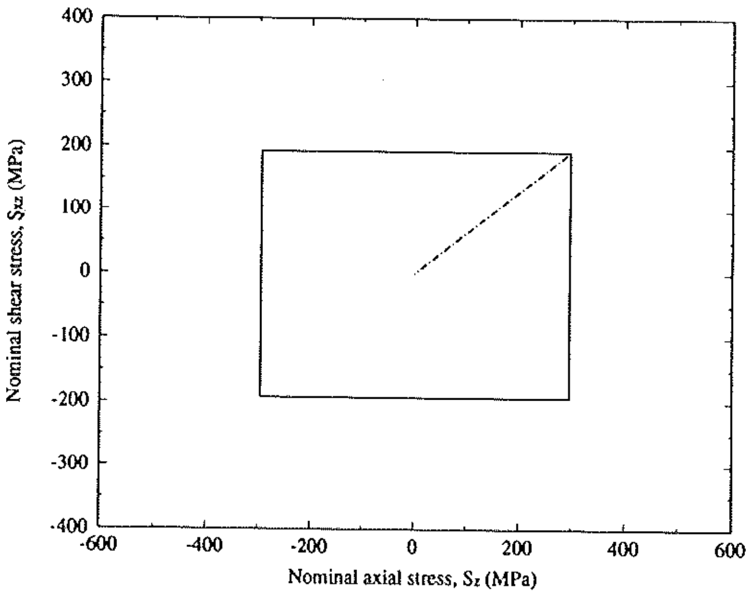


Fig 4(a) Nominal stress path for maximum nominal stresses of S_z = 296 MPa and S_{xz} = 193 MPa.

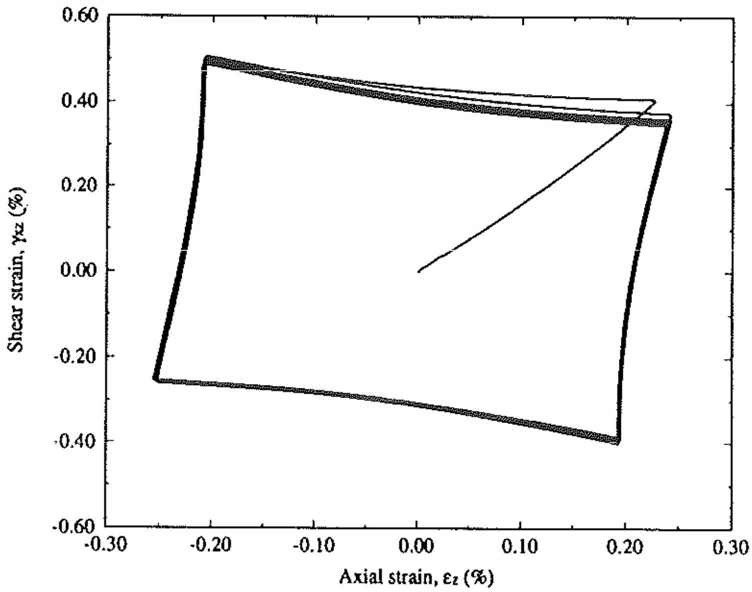


Fig 4(b) Measured strain response for maximum nominal stresses of S_z = 296 MPa and S_{xz} = 193 MPa.

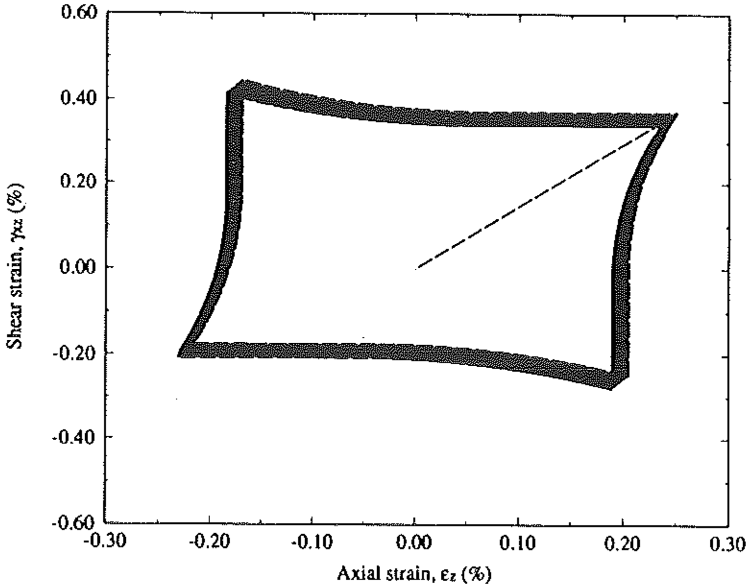


Fig 4(c) Calculation using the simplified method for maximum nominal stresses of S_z = 296 MPa and S_{xz} = 193 MPa.

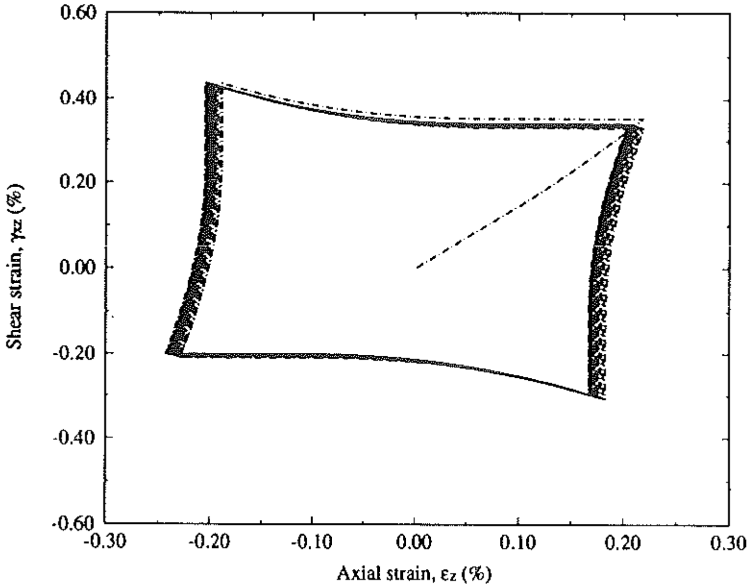


Fig 4(d) Calculation using the finite-element method for maximum nominal stresses of S_z = 296 MPa and S_{xz} = 193 MPa.

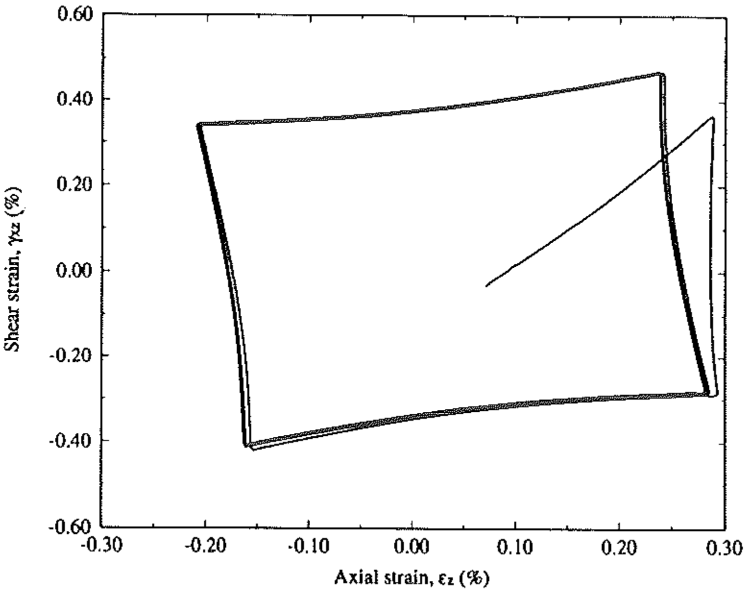


Fig 4(e) Strain response for clockwise loading path for maximum nominal stresses of $S_z = 296$ MPa and $S_{xz} = 193$ MPa.

use with an appropriate uniaxial nominal stress–notch strain formula, and an assumed hardening rule.

Table 1 Data required for use of the simplified method.

Elastic geometry factors:	K_z, K_{xz}, K'_z
plastic strain ratio, $\varepsilon^p_x/\varepsilon^p_y$	one for smooth bars, zero for highly constrained notches; can be determined from K_t for mild notches
Material properties:	k', n', E, ν
Uniaxial structure load–strain curve	Glinka's formula, or FEA
Hardening rule for multiaxial loading	Mróz hardening rule

If the direction of travel around the loading path of Fig. 4(a) is reversed, so that it is clockwise, the same basic features will be observed in the models. This is indeed the case experimentally, as Fig. 4(e) shows for the strain response of such a loading path.

Another common nonproportional loading path is one in which the frequency of the applied loads is unequal. For example, Fig. 5(a), is a loading path in which there are five cycles of tension for one cycle of torsion. For the same values of maximum nominal load, the experimental strain response is given in Fig. 5(b) for one hundred repeats of the loading path. The presence of plasticity

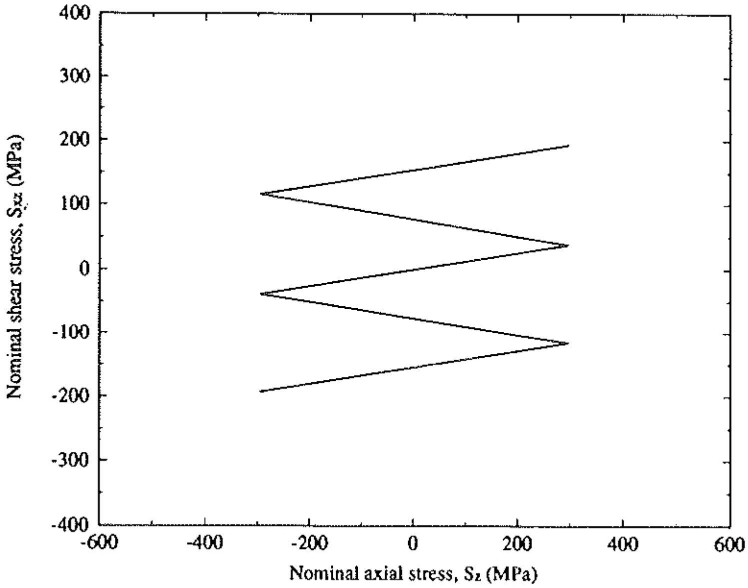


Fig 5(a) Nominal stress path for maximum nominal stresses of S_z = 296 MPa and S_{xz} = 193 MPa.

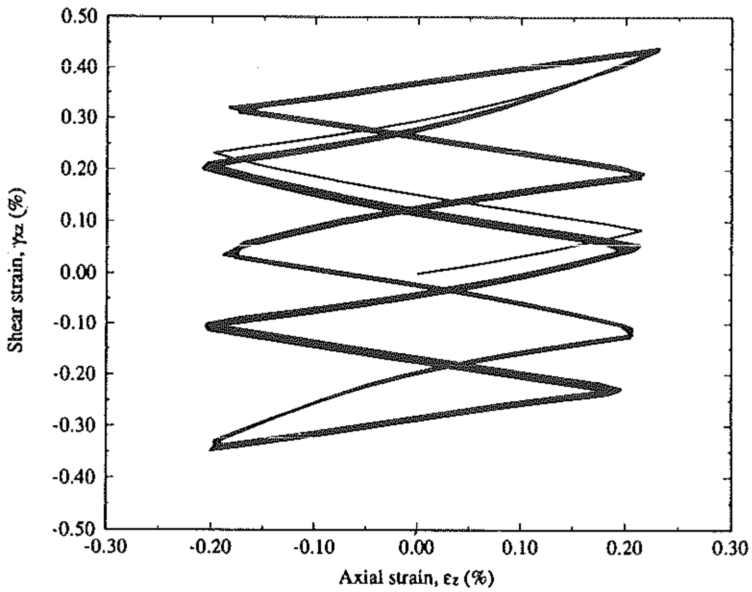


Fig 5(b) Measured strain response for maximum nominal stresses of S_z = 296 MPa and S_{xz} = 193 MPa.

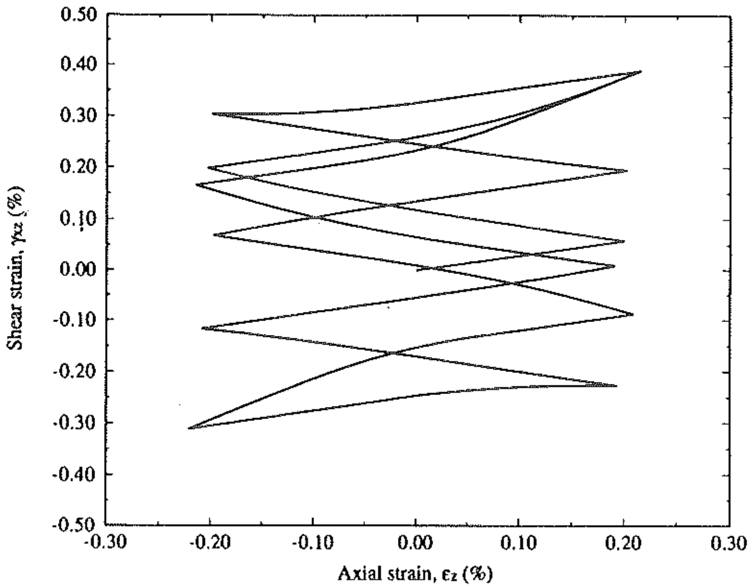


Fig 5(c) Calculation using the simplified method for maximum nominal stresses of S_z = 296 MPa and S_{xz} = 193 MPa.

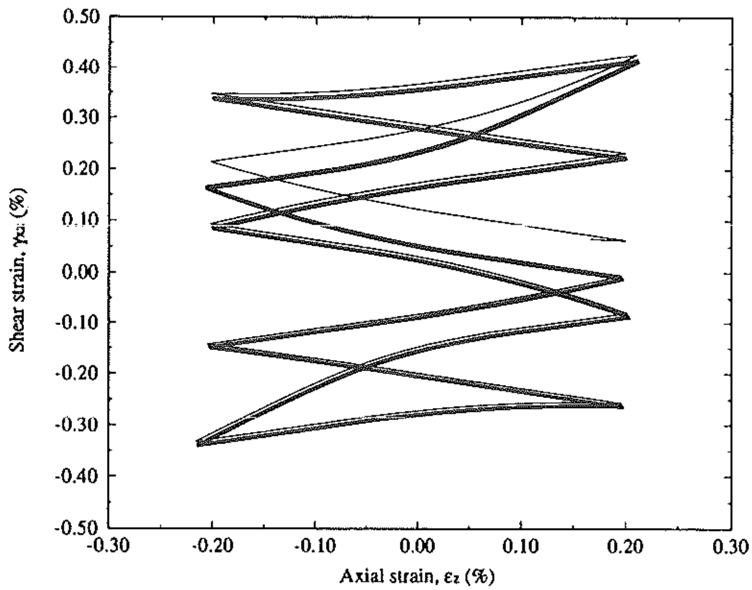


Fig 5(d) Calculation using the finite-element method for maximum nominal stresses of S_z = 296 MPa and S_{xz} = 193 MPa.

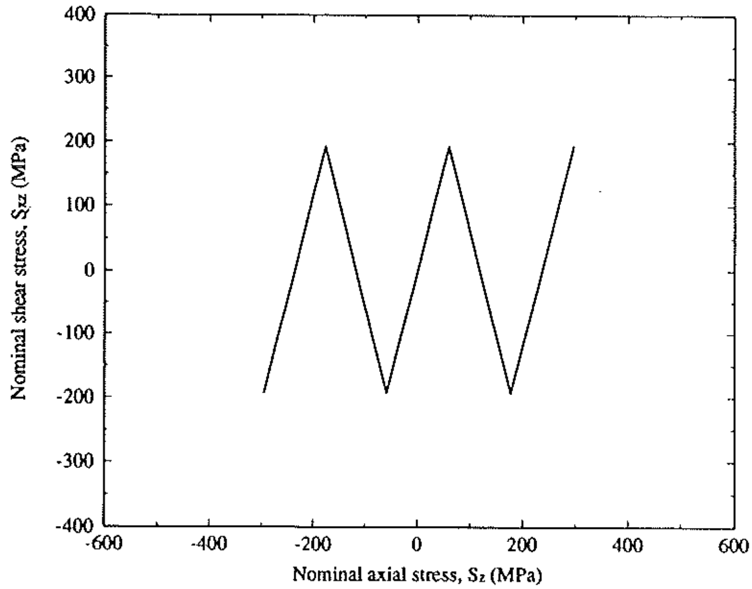


Fig 6(a) Nominal stress path for maximum nominal stresses of $S_z = 296$ MPa and $S_{xz} = 193$ MPa.

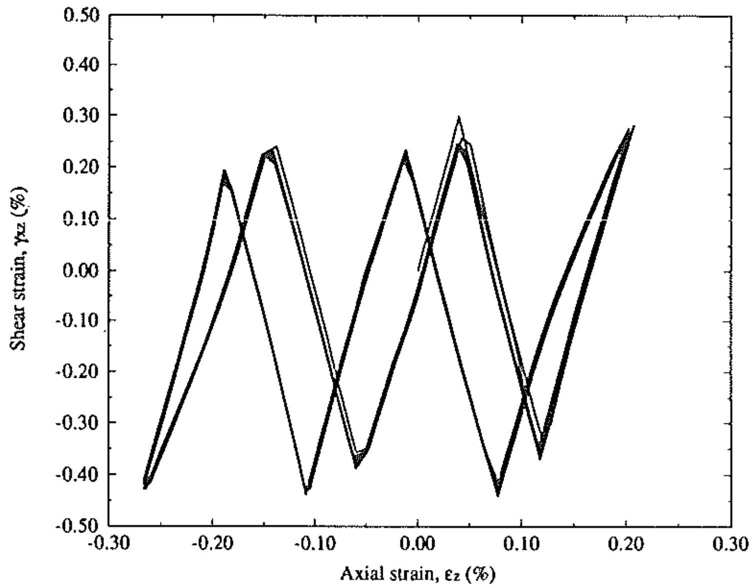


Fig 6(b) Measured strain response for maximum nominal stresses of $S_z = 296$ MPa and $S_{xz} = 193$ MPa.

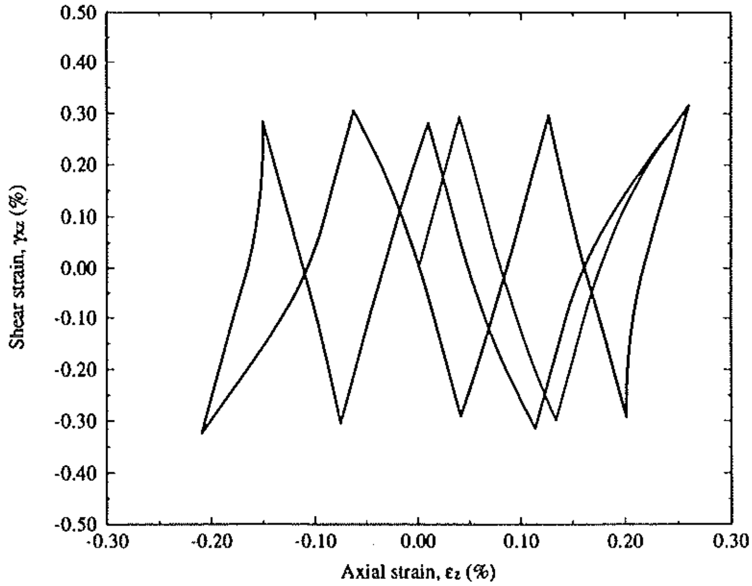


Fig 6(c) Calculation using the simplified method for maximum nominal stresses of S_z = 296 MPa and S_{xz} = 193 MPa.

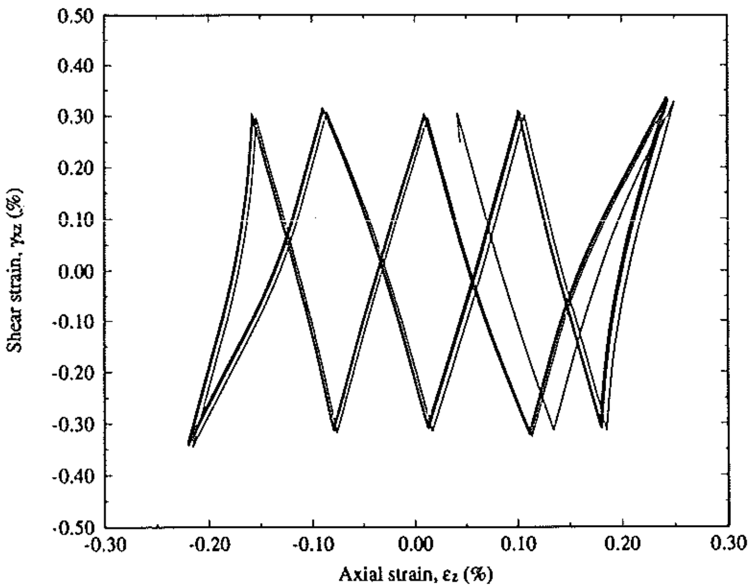


Fig 6(d) Calculation using the finite-element method for maximum nominal stresses of S_z = 296 MPa and S_{xz} = 193 MPa.

occurring during this loading path is indicated by the offset of the peaks in the strain response. If the material had remained elastic, the strain path would have taken the same shape as the loading path. The results of the simplified model and the finite-element model are presented in Figs. 5(c–d), for five repetitions of the loading path. Also note the transient strain response from the initial loading cycle of the experiment before the material continues in a cyclic path, and the similar initial response and the subsequent cyclic response of the models, as the yield surfaces translate in stress space.

A comparison of the experimental response to the methods of notch strain calculation are also presented for loading paths where the frequency of the applied loads S_z: S_{xz} is 1:5 (Fig. 6). For all of these paths, the calculations of the simplified model matches quite closely with that of the finite-element analysis, which in turn agrees well with the experimental results.

7 Summary

A simplified method of calculating notch-root strains from applied nonproportional nominal load has been developed, using the concept of a structural yield surface. The yield surface was assumed to be anisotropic to account for preferred directions of plastic flow at the notch root due to geometric constraint, and the coefficients of anisotropy were determined from elastic stress concentration factors of the notch and the isotropic material yield strength. The method compared well with experimental results and finite-element analysis of a notched shaft subjected to nonproportional tension and torsion loading paths with zero mean nominal load.

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