GENERALIZED STRAIN FATIGUE CRITERION FOR MATERIALS UNDER MULTIAXIAL RANDOM LOADINGS

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Abstract:
A generalized fatigue criterion for materials under multiaxial random loadings has been presented. The criterion has been based on the assumption that quantities of shear and normal strains in the expected fracture plane - shear strain in one direction on this plane is considered - determine the fracture plane. It has been shown that the well known cyclic fatigue criteria of maximum normal strain, maximum shear strain and the criterion of maximum shear and normal strains on the critical shear plane result from the formulated criterion.

1. INTRODUCTION

Formulation of fatigue fracture criteria for materials under multiaxial random loadings by extension of the known criteria for cyclic loadings is connected with various limitations. In case of some criteria the limitations are theoretical. In papers [19,20] five criteria for random triaxial stress state have been proposed and some limitations making some other criteria under these conditions impossible to use have been proposed.

From these five criteria three are based on stresses, one on the elastic strain energy and one-on strains. Then the stress criteria were formulated as one generalized criterion of the maximum shear and normal stresses at the fracture plane [21]. Formulation of a similar generalized strain criterion is a subject of this paper.

2. STRAIN FATIGUE CRITERIA UNDER MULTIAXIAL CYCLIC LOADINGS

The fatigue criteria for materials under multiaxial cyclic loadings are discussed in many papers [2,5,6,7,8,9,12,15,22]. Let us consider the criteria based on strains that are most frequently verified by tests. The influence of the mean stress, notches and some other factors is neglected. The values assumed in these criteria as determining material fatigue are the most interesting for us.

Maximum normal strain criterion: it is assumed that ε_{amax}, i.e. the maximum amplitude of the normal strain determines fatigue fracture formation under both multiaxial and uniaxial loadings

$$\varepsilon_{amax}=\max_{t}\{\varepsilon_1(t)\}=\max_{i=\hat{x},\hat{y},\hat{z}}\ \max_{t}\{\varepsilon_{ii}(t)\}=\max_{i=\hat{x},\hat{y},\hat{z}}\{\varepsilon_{aii}\}=\varepsilon_{a\hat{x}\hat{x}}=\varepsilon_{af} \tag{1a}$$

According to this criterion $\Delta\varepsilon_{max}$ - the maximum range of the normal strain cycle can determine fatigue fracture initiation, as well.

$$\Delta\varepsilon_{max} = \max_{i=\hat{x},\hat{y},\hat{z}}\{\Delta\varepsilon_{ii}\} = \Delta\varepsilon_{\hat{x}\hat{x}} = \Delta\varepsilon_f \tag{1b}$$

where: $\varepsilon_1(t)$ - maximum principal strain (t-time), $\frac{1}{2}\Delta\varepsilon_f = \varepsilon_{af}$ - fatigue limit under tension-compression and $0\hat{x}\hat{y}\hat{z}$ - system of axes with extreme amplitudes and ranges of normal strains ($\varepsilon_{a\hat{x}\hat{x}} \geq \varepsilon_{a\hat{y}\hat{y}} \geq \varepsilon_{a\hat{z}\hat{z}}$)

According to this criterion a critical plane with the normal $\bar{\eta}$ and the normal strain determining fatigue can be found:

$$\varepsilon_\eta(t) = \varepsilon_{\hat{x}\hat{x}}(t) = \varepsilon_{a\hat{x}\hat{x}}\sin\omega t = \frac{\Delta\varepsilon_{\hat{x}\hat{x}}}{2}\sin\omega t$$

Maximum shear strain criterion has been formulated on the assumption that γ_{amax} - the maximum amplitude or $\Delta\gamma_{max}$ - the maximum range of the shear strain are critical quantities for fatigue of materials:

$$\gamma_{amax}=\max_{t}\{\varepsilon_1(t)-\varepsilon_3(t)\}=\max_{\substack{i,j=\hat{x},\hat{y},\hat{z}\\ i\neq j}}\ \max_{t}\{\varepsilon_{ii}(t)-\varepsilon_{jj}(t)\}=$$

$$=\max_{\substack{i,j=\hat{x},\hat{y},\hat{z}\\ i\neq j}}\{\varepsilon_{aii}\mp\varepsilon_{ajj}\}=(1+\nu)\varepsilon_{af} \tag{2a}$$

$$\Delta\gamma_{max}=\max_{\substack{i,j=\hat{x},\hat{y},\hat{z}\\ i\neq j}}\{\Delta\varepsilon_{ii}\mp\Delta\varepsilon_{jj}\}=(1+\nu)\Delta\varepsilon_f \tag{2b}$$

where $\varepsilon_3(t)$ - minimum principal strain, ν - Poisson's ratio.

The sign (-) in (2) concerns the case in which strains $\varepsilon_{ii}(t)$ and $\varepsilon_{jj}(t)$ are coincident in phase and the sign (+) concerns the case of antiphased strains.

For other phase shifts a selection of two components of strains $\varepsilon_{ii}(t_1)$ and $\varepsilon_{jj}(t_2)$ at different moments t_1 and t_2 in which $\Delta\gamma_{max}$ reaches its maximum value is proposed [11,13].

The idea of the criterion can also be expressed by assuming that fatigue fracture is determined by the shear strain:

$$\gamma_{\eta s}(t) = \gamma_{a\eta s}\sin\omega t = \frac{\Delta\gamma_{\eta s}}{2}\sin\omega t$$

in direction $\bar{s}$ on the critical plane with normal $\bar{\eta}$; it is one of two planes having the maximum amplitude $\gamma_{a\eta s} = \gamma_{amax}$ (formula (2a)) and the maximum range of changes $\Delta\gamma_{\eta s} = \Delta\gamma_{max}$ (formula 2 b)) of the shear strain. The unit vector $\bar{s}$ coincides with the direction of the maximum range of the shear strain $\Delta\gamma_{max}$.

<u>Criterion of maximum octahedral shear strains</u> is often applied to approximation of fatigue test results under in-phase and antiphased multiaxial loadings [14,25,26]. It is assumed that fatigue fracture is determined by $\gamma_{aoctmax}$ - the maximum amplitude - or $\Delta\gamma_{octmax}$ - the maximum range of octahedral shear strain

$$\gamma_{actmax} = \max_t \frac{1}{3}\{[\varepsilon_1(t)-\varepsilon_2(t)]^2+[\varepsilon_2(t)-\varepsilon_3(t)]^2+[\varepsilon_3(t)-\varepsilon_1(t)]^2\}^{1/2}$$

$$=\max_t \frac{1}{3}\{[\varepsilon_{xx}(t)-\varepsilon_{yy}(t)]^2+[\varepsilon_{yy}(t)-\varepsilon_{zz}(t)]^2+[\varepsilon_{zz}(t)-\varepsilon_{xx}(t)]^2 +$$

$$+6[\varepsilon_{xy}{}^2(t)+\varepsilon_{xz}{}^2(t)+\varepsilon_{yz}{}^2(t)]\}^{1/2} = \frac{1}{3}\{(\varepsilon_{a\hat{x}\hat{x}} \mp \varepsilon_{a\hat{y}\hat{y}})^2+(\varepsilon_{a\hat{y}\hat{y}} \mp \varepsilon_{a\hat{z}\hat{z}})^2+$$

$$+(\varepsilon_{a\hat{z}\hat{z}} \mp \varepsilon_{a\hat{x}\hat{x}})^2\}^{1/2} = \frac{(1+\nu)\sqrt{2}}{3}\varepsilon_{af} \qquad (4a)$$

or

$$\Delta\gamma_{octmax} = \frac{1}{3}\{(\Delta\varepsilon_{\hat{x}\hat{x}} \mp \Delta\varepsilon_{\hat{y}\hat{y}})^2+(\Delta\varepsilon_{\hat{y}\hat{y}} \mp \Delta\varepsilon_{\hat{z}\hat{z}})^2+(\Delta\varepsilon_{\hat{z}\hat{z}} \mp \Delta\varepsilon_{\hat{x}\hat{x}})^2\}^{1/2}$$

$$= \frac{(1+\nu)\sqrt{2}}{3}\Delta\varepsilon_f \qquad (4b)$$

Since octahedral shear strains are identified on the octahedral plane as those in the direction of octahedral shear stresses (in elastic range) it can be assumed that, according to this criterion, fatigue fracture is determined by shear strains

$$\gamma_{\eta s}(t) = \gamma_{a\eta s}\sin\omega t = \frac{\Delta\gamma_{\eta s}}{2}\sin\omega t$$

in direction $\bar{s}$ on the critical plane with the normal $\bar{\eta}$. The unit vector $\bar{\eta}$ is inclined at the same angles to the axes $0\hat{x}\hat{y}\hat{z}$ and the unit vector $\bar{s}$ lies along the direction of the maximum range of the octahedral shear strain $\Delta\gamma_{octmax}$, and

$\gamma_{a\eta s} = \gamma_{aoctmax}$ -formula (4a) $\Delta\gamma_{\eta s} = \Delta\gamma_{oct\ max}$ -formula (4b)

Criterion of maximum shear and normal strains on the critical shear plane is based on assumptions similar to those made by Stanfield [24] for his stress criterion. It is assumed that fatigue fracture is determined by $0.5\gamma_{a\eta max}$ - the maximum amplitude of shear strain and $\varepsilon_{a\eta}$ - the normal strain amplitude at the critical shear plane with the normal $\bar{\eta}$. Brown and Miller [5] assume that the plane on which shear strains reach their maximum amplitude and the range is the critical shear plane. According to Lohr and Ellison [16] the shear plane inclined at $\pi/4$ to the external surface of a material (its range of shear strain at a given point of the material is not always maximum) is the critical one. Brown and Miller propose a non-linear function

$$\frac{1}{2}\gamma_{a\eta max} = f(\varepsilon_{a\eta}) \tag{5}$$

for a canstant number of cycles to fracture. Function (5) is assumed as

$$\left(\frac{\frac{1}{2}\gamma_{a\eta max}}{g}\right)^j + \left(\frac{\varepsilon_{a\eta}}{h}\right)^j = 1 \tag{6a}$$

where g,h,j-constants ($0.5 \le j \le 3.7$) [7].

The criterion (6 a) can be written as

$$\max_t\{\gamma_\eta^j(t) + S\varepsilon_\eta^j(t)\} = \gamma_{a\eta max}^j + S\,\varepsilon_{a\eta}^j = \text{const} \tag{6b}$$

$$\text{where} \quad S = \left(\frac{2g}{h}\right)^j \quad \text{or} \quad \Delta\gamma_{\eta max}^j + S\Delta\varepsilon_\eta^j = \text{const} \tag{6c}$$

Lohr and Ellison [16] have obtained good approximation of experimental results with a linear relationship

$$\frac{1}{2}\gamma^*_{a\eta max} + k\,\varepsilon^*_{\eta a} = \text{const} \tag{7a}$$

where k-constant, $\frac{1}{2}\gamma^*_{a\eta max}$, $\varepsilon^*_{a\eta}$ -maximum amplitude of shear and normal strains respectively on the shear plane inclined at $\pi/4$ to the external surface of a material.

Similarly to (6 b) and (6 c) the criterion (7 a) can be written as

$$\max_t\left\{\frac{1}{2}\gamma^*_\eta(t) + k\,\varepsilon^*_\eta(t)\right\} = \frac{1}{2}\gamma^*_{amax} + k\,\varepsilon^*_{a\eta} = \text{const} \tag{7b}$$

$$\text{or} \quad \frac{1}{2}\Delta\gamma^*_{\eta max} + k\,\Delta\varepsilon^*_\eta = \text{const} \tag{7c}$$

The shear strain with the maximum range on the critical shearing plane can be understood as the strain along the direction $\bar{s}$ on the plane with normal $\bar{\eta}$, i.e.

$$\varepsilon_{\eta s}(t)=\varepsilon_{a\eta s}\sin\omega t=\frac{1}{2}\ \gamma_{a\eta s}\sin\omega t\ =\ \frac{\Delta\gamma_{\eta s}}{4}\ \sin\omega t$$

where $\gamma_{a\eta s}=\gamma_{a\eta max}$-according to (6b) or $\gamma_{a\eta s}=\gamma^{*}_{a\eta max_{*}}$ according to (7b) and $\Delta\gamma_{\eta s}=\Delta\gamma_{\eta max}$ according to (6c) or $\Delta\gamma_{\eta s}=\Delta\gamma^{*}_{\eta max}$ ac-ccording to (7c) respectively,depending on selection of the critical shearing plane.

The following linear dependence [10,23]

$$\frac{1}{2}\ \gamma_{a\eta max}\ +\ \varepsilon_{a\eta}=\ \varepsilon_{af} \tag{8}$$

is a specific form of (6).

The presented review of mathematical models of strain criteria is not complete;the discussed criteria are still verified by fatigue tests. But it results from the review that a critical plane and normal or/and shear strains connected with the plane may be observed in case of each criterion. The maximum ranges of these strains determine directions along which strains change sinusoidally. It can be said that strains normal to the critical plane and/or shear strains along one direction on the plane determine fatigue fracture.

3.GENERALIZED CRITERION OF MAXIMUM SHEAR AND NORMAL STRAINS ON THE FRACTURE PLANE

Let us assume that a random strain tensor is a six-dimensional stationary and ergodic Gaussian process with the expected values equal to zero ($\hat{\varepsilon}_{ij}=0$) and frequency spectra of a low -band type.

The criterion is based on the assumption that a fatigue fracture plane is a result of the presence of random values and directions of the principal strains. The plane is a reaction of material to random fatigue loadings. Estimating fatigue life of a material we will use the term "expected fracture plane position". It is assumed that magnitudes of shear and normal strains on the fracture plane determine the occurrence of fracture but only shear strains in one direction on the plane are taken into consideration. Weight participation of normal and shear strains in fatigue fracture formation is dependent on a kind and condition of material (elastic-plastic or elastic-brittle state). In the case of elastic-brittle materials,a bigger contribution is due to normal strains.

Since directions of principal strains determine the fatigue fracture plane position and these directions randomly change, the mean directions of principal strains are introduced and the expected fracture plane position is determined in relation to them. Problems of averaging the principal directions and of estimating the expected fatigue fracture plane position are discussed in many papers [1,3,4,19]. Three methods are especially interesting: the weight method [1,19], the variance method [3,4] and the damage cumulation method [17]. At a given fracture plane the shear and normal strains are dependent on all random components of strain. The detailed assumptions of the proposed criterion are the following:

1. Fatigue fracture occurs under the influence of normal strain $\varepsilon_{\eta}(t)$ and shear strain $\varepsilon_{\eta s}(t)$ in the direction $\bar{s}$ on the fracture plane with the normal $\bar{\eta}$.

2. The direction $\bar{s}$ on the fracture plane coincides with the mean direction of the maximum shear strain $\varepsilon_{\eta s max}(t)$.

3. For a given fatigue life the maximum value of a linear combination of strains $\varepsilon_{\eta s}(t)$ and $\varepsilon_{\eta}(t)$ under multiaxial random loadings satisfies the equation

$$\max_{t}\{b\varepsilon_{\eta s}(t) + k\varepsilon_{\eta}(t)\} = \max_{t}\{ W(t)\}=q \tag{9}$$

where b,q,k - constants used to select a particular form of equation (9).

The normal and shear strains $\varepsilon_{\eta}(t)$ and $\varepsilon_{\eta s}(t)$ on the fracture plane are presented in Fig.1. Results of their action depending on the maximum values that occur with time are shown in Fig.2. The strain sum $W(t)$ is a stochastic process as well and can be understood as a function of the fatigue effort of material. The expression $\max_{t} \{ W(t)\}$ should be read as 100% quantile of a random variable W.

If the maximum value of the effort function $\max_{t} \{ W(t)\}$ exceeds the limit value q, then damage will accumulate in the material resulting in fatigue fracture. In the case of any fixed fatigue fracture plane position, directions of the unit vectors $\bar{\eta}$ and $\bar{s}$ can be defined by the respective mean direction cosines $\hat{\alpha}_{\eta i}$, $\hat{\alpha}_{\eta j}$ and $\hat{\beta}_{si}$, $\hat{\beta}_{sj}$, (i,j=x,y,z) in relation to the system 0xyz.

$$\bar{\eta} = \hat{\alpha}_{\eta x}\,\bar{i} + \hat{\alpha}_{\eta y}\,\bar{j} + \hat{\alpha}_{\eta z}\,\bar{k} \qquad \bar{s} = \hat{\beta}_{sx}\,\bar{i} + \hat{\beta}_{sy}\,\bar{j} + \hat{\beta}_{sz}\,\bar{k}$$

where $\bar{i}$, $\bar{j}$, $\bar{k}$ - versors of the axes 0xyz.

The strains $\varepsilon_{\eta s}(t)$ and $\varepsilon_{s}(t)$ are calculated according to transformation laws for components of the strain tensor:

$\varepsilon_{\phi v} = c_{\phi i}\ c_{vj}\ \varepsilon_{ij},\qquad (\phi, v = \eta,\ s,\ \zeta;\quad i,j = x,y,z)$

where c-cosines of rotation angles of axes.

$$\varepsilon_{\eta}(t) = \hat{\alpha}_{\eta i}\ \hat{\alpha}_{\eta j}\ \varepsilon_{ij}\ (t) \tag{10}$$

$$\varepsilon_{\eta s}(t) = \hat{\beta}_{\eta i}\ \hat{\beta}_{sj}\ \varepsilon_{ij}(t) \tag{11}$$

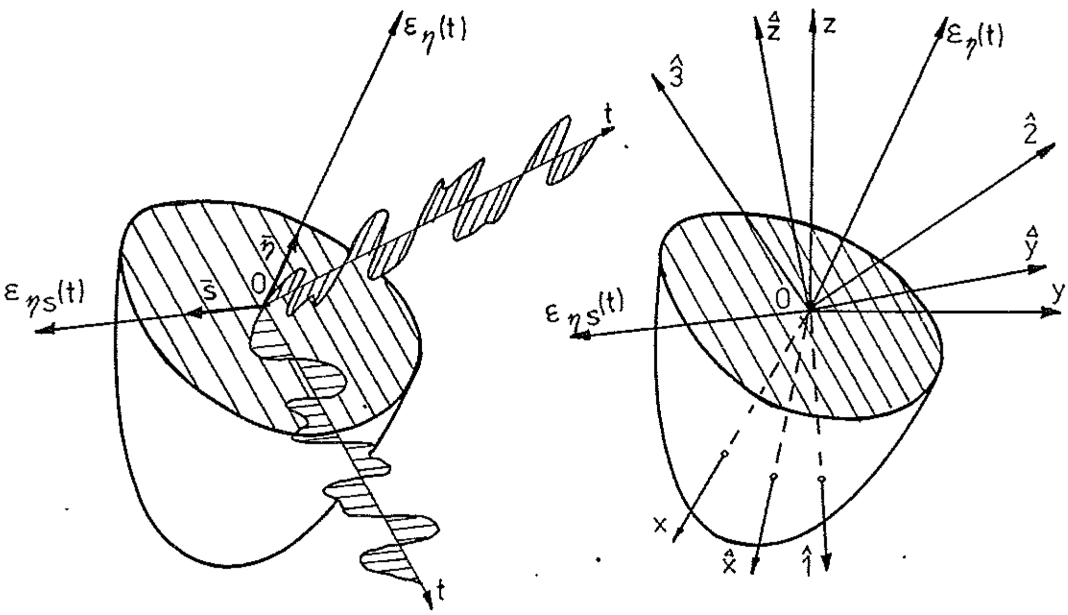

Fig.1 Normal and shear strains $\varepsilon_{\eta}(t)$ and $\varepsilon_{\eta s}(t)$ respectively in the fracture plane. Oxyz - system of axes connected with the material, $0\hat{x}\hat{y}\hat{z}$ - system of axes along which normal strains have extreme ranges $(\Delta\varepsilon_{xx} \geq \Delta\varepsilon_{yy} \geq \Delta\varepsilon_{zz})$, $0\hat{1}\hat{2}\hat{3}$ - system of axes determining mean position of principal strains $(\varepsilon_{1}(t) \geq \varepsilon_{2}(t) \geq \varepsilon_{3}(t))$.

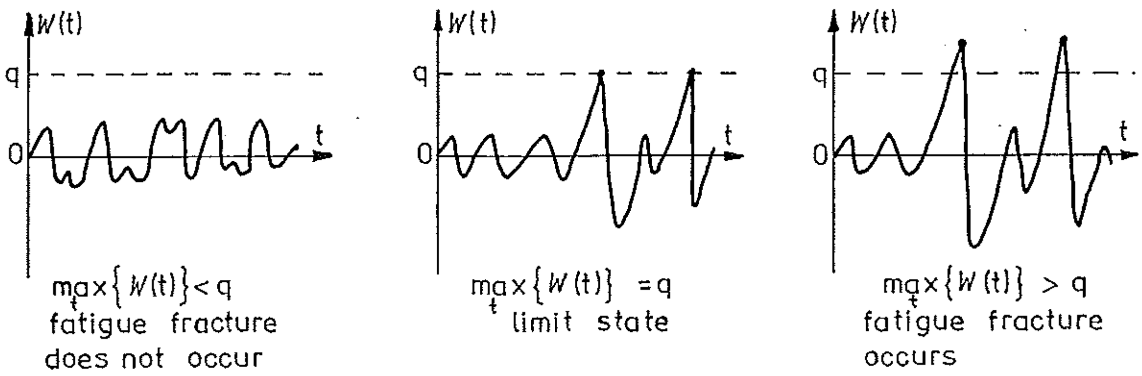

Fig.2 Effect of strain $W(t)$ with various maximum values.

The criterion (9) can be expressed as

$$\max_t \{b\ \hat{\beta}_{\eta i}\ \hat{\beta}_{sj}\ \varepsilon_{ij}(t) + k\ \hat{\alpha}_{\eta i}\ \hat{\alpha}_{\eta j}\ \varepsilon_{ij}\ (t)\} = q \tag{12}$$

Assuming specific fracture plane positions and values of the constants b,k,q we obtain various particular forms of (9) or (12)

1^o If $b=0, k=1$ and $q=\varepsilon_{af}$ and if we assume that the normal to the expected fracture plane $\bar{\eta}$ coincides with a mean direction $\hat{1}$ along which the maximum principal strain occurs (Fig.3),i.e.

$$\bar{\eta} = \hat{l}_1\ \bar{i} + \hat{m}_1\ \bar{j} + \hat{n}_1\ \bar{k}$$

the criterion (9) becomes

$$\max_t \{\ \varepsilon_\eta\ (t)\ \} = \varepsilon_{af} \tag{13}$$

where

$$\varepsilon_\eta(t) = \hat{l}_1^{\,2}\varepsilon_{xx}(t)+\hat{m}_1^{\,2}\varepsilon_{yy}(t)+\hat{n}_1^{\,2}\varepsilon_{zz}(t)+ 2\hat{l}_1\hat{m}_1\varepsilon_{xy}(t) +$$

$$+ 2\ \hat{l}_1\hat{n}_1\varepsilon_{xz}(t) + 2\hat{m}_1\hat{n}_1\varepsilon_{yz}(t) \tag{14}$$

$\hat{l}_1$, $\hat{m}_1$, $\hat{n}_1$ - expected values of direction cosines of strain $\varepsilon_1(t)$.

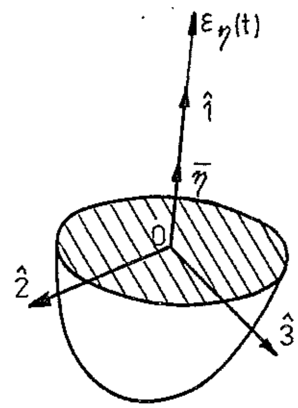

Fig.3 Directions of unit vector $\bar{\eta}$ and normal strain $\varepsilon_\eta(t)$ when the fatigue fracture plane is perpendicular to the mean direction of the maximum principal strain $\varepsilon_1(t)$.

If under multiaxial sinusoidal loading we denote strain components to have the direction of the $0\hat{x}$ axis prescribed to the longitudinal strain with maximum amplitude,i.e.

$$\varepsilon_{\hat{x}\hat{x}}(t) = \varepsilon_{a\hat{x}\hat{x}}\sin\omega t$$

and further if we assume $\hat{l}_1 = 1$,then,according to (13) and (14),we obtain

$$\max_t\{\varepsilon_\eta(t)\} = \max_t\{\varepsilon_{a\hat{x}\hat{x}}\sin\omega t\} = \varepsilon_{a\hat{x}\hat{x}} = \varepsilon_{af} \tag{15}$$

Thus it is shown that under sinusoidal loading the maximum normal strain criterion (1) results from (9).

2^{o} For $b=1, k=0$ and $q=1/2\ (1+\nu)\ \varepsilon_{af}$ we assume that the expected fracture plane is determined by the mean position of one of two planes of the maximum shear strain $\gamma_1(t)=\varepsilon_1(t)-\varepsilon_3(t)$. On this plane we choose a direction $\bar{s}$ coincident with the mean position of strain $\gamma_1(t)$. The unit vectors $\bar{\eta}$ and $\bar{s}$ - according to Fig.4 - are equal to

$$\bar{\eta} = \frac{1}{\sqrt{2}}[(\hat{l}_1+\hat{l}_3)\ \bar{i} + (\hat{m}_1+\hat{m}_3)\ \bar{j} + (\hat{n}_1+\hat{n}_3)\ \bar{k}] \qquad (16)$$

$$\bar{s} = \frac{1}{\sqrt{2}}\ [(\hat{l}_1-\hat{l}_3)\ \bar{i} + (\hat{m}_1-\hat{m}_3)\ \bar{j} + (\hat{n}_1-\hat{n}_3)\ \bar{k}] \qquad (17)$$

where $\hat{l}_3$, $\hat{m}_3$, $\hat{n}_3$- expected values of direction cosines of strain $\varepsilon_3(t)$.

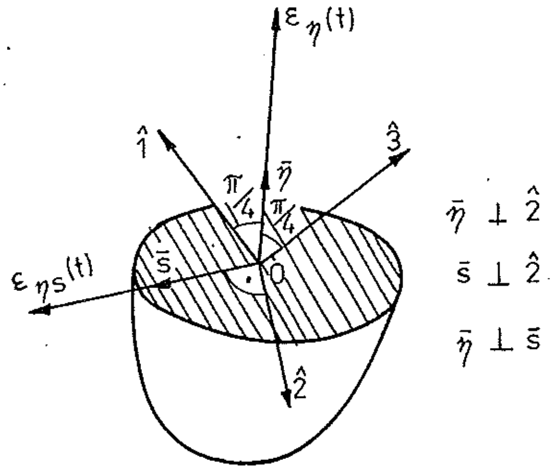

Fig.4.Directions of unit vectors $\bar{\eta}$, $\bar{s}$ and normal and shear strains $\varepsilon_\eta(t)$, $\varepsilon_{\eta s}(t)$ in a plane determined by a mean position of one of two planes of the maximum principal shear strain $\gamma_1(t)$.

A required component of the tensor $\varepsilon_{\eta s}(t)$, according to (11),(16) and (17) is given by

$$\varepsilon_{\eta s}\ (t) = 0.5[(\hat{l}_1^{\,2}-\hat{l}_3^{\,2})\varepsilon_{xx}(t) + (\hat{m}_1^{\,2}-\hat{m}_3^{\,2})\varepsilon_{yy}(t) + (\hat{n}_1^{\,2}-\hat{n}_3^{\,2})$$
$$\varepsilon_{zz}(t) + 2\ (\hat{l}_1\hat{m}_1 - \hat{l}_3\hat{m}_3)\varepsilon_{xy}(t) + 2\ (\hat{l}_1\hat{n}_1-\hat{l}_3\hat{n}_3)\varepsilon_{xz}(t) +$$
$$+ 2\ (\hat{m}_1\hat{n}_1 - \hat{m}_3\hat{n}_3)\ \varepsilon_{yz}\ (t)] \qquad (18)$$

In the limit state corresponding to fatigue strength the maximum value of the shear strain $\max_t\{\varepsilon_{\eta s}(t)\}$ is equal to a corresponding amplitude of the maximum shear strain under sinusoidal tension-compression with the amplitude ε_{af}. Hence, according to (9) we obtain the following form of the criterion of the maximum shear strain in the fracture plane

$$\max_t\{\varepsilon_{\eta s}(t)\} = 0.5\ (1+\nu)\varepsilon_{af} \quad \text{or} \quad \max_t\{\gamma_{\eta s}(t)\} = (1+\nu)\varepsilon_{af} \tag{19}$$

where $\varepsilon_{\eta s}(t)$ is determined from (18) and $\gamma_{\eta s}(t)=2\varepsilon_{\eta s}(t)$.

In the case of multiaxial sinusoidal loadings let us denote the components of strain state so that the direction of axis $0\hat{x}$ is assigned to the normal strain with the maximum amplitude and the direction of axis $0\hat{z}$ - to the strain with the minimum amplitude. Let us also assume that $\hat{l}_1=\hat{n}_3=1$. Then, according to (18) and (19), we obtain

$$\max_t\{\gamma_{\eta s}(t)\} = \max\{\varepsilon_{\hat{x}\hat{x}}(t)-\varepsilon_{\hat{z}\hat{z}}(t)\} = \max_t\{\varepsilon_{a\hat{x}\hat{x}}\sin\omega t - \varepsilon_{a\hat{z}\hat{z}}\sin\omega t\} =$$

$$= \varepsilon_{a\hat{x}\hat{x}} - \varepsilon_{a\hat{z}\hat{z}} = (1+\nu)\ \varepsilon_{af} \tag{20}$$

When strain $\varepsilon_{\hat{z}\hat{z}}(t)$ is antiphased in relation to $\varepsilon_{\hat{x}\hat{x}}(t)$ we have

$$\max_t\{\gamma_{\eta s}(t)\} = \varepsilon_{a\hat{x}\hat{x}} + \varepsilon_{a\hat{z}\hat{z}} = (1+\nu)\varepsilon_{af} \tag{21}$$

From (20) and (21) it results that the known maximum shear strain criterion under sinusoidal loadings (2) results from the criterion of the maximum shear strain in the fracture plane (9).

3^{o} For b=1, k=1 and $q=\varepsilon_{af}$ we assume, as in 2^{o}, that the expected fracture plane is determined by a mean position of one of two planes of the maximum principal shear strain $\gamma_1(t)$ and the direction $\bar{s}$ agrees with the mean direction $\gamma_1(t)$ (Fig.4). The normal strain $\varepsilon_\eta(t)$ can be calculated - like $\varepsilon_{\eta s}(t)$ - with (10) and (16)

$$\varepsilon_\eta(t) = 0.5\ [(\hat{l}_1+\hat{l}_2)^2\ \varepsilon_{xx}(t) + (\hat{m}_1+\hat{m}_3)^2\ \varepsilon_{yy}(t) + (\hat{n}_1+\hat{n}_3)^2$$
$$\varepsilon_{zz}(t) + 2\ (\hat{l}_1+\hat{l}_3)(\hat{m}_1+\hat{m}_3)\varepsilon_{xy}(t) + 2\ (\hat{l}_1+\hat{l}_3)(\hat{n}_1+\hat{n}_3)\varepsilon_{xz}(t) +$$
$$+ 2\ (\hat{m}_1+\hat{m}_3)(\hat{n}_1+\hat{n}_3)\ \varepsilon_{yz}\ (t)] \tag{22}$$

The criterion (9) has now the following form:

$$\max_t\ \{\ \varepsilon_{\eta s}(t) + \varepsilon_\eta(t)\ \} = \varepsilon_{af} \tag{23}$$

where $\varepsilon_{\eta s}(t)$ is expressed by (18) and $\varepsilon_\eta(t)$ - by (22).

Under multiaxial sinusoidal loading and on the assumption that $\hat{l}_1=\hat{n}_3=1$ from formulae (18), (22) and (23) it results that

$$\max_t\{\varepsilon_{\eta s}(t) + \varepsilon_\eta(t)\} = \max_t\left\{\frac{\varepsilon_{\hat{x}\hat{x}}(t) - \varepsilon_{\hat{z}\hat{z}}(t)}{2} + \frac{\varepsilon_{\hat{x}\hat{x}}(t) - \varepsilon_{\hat{z}\hat{z}}(t)}{2}\right\}$$

$$= \max_t\ \{\varepsilon_{a\eta s}\sin\omega t + \varepsilon_{a\eta}\ \sin\omega t\} = \varepsilon_{a\eta s} + \varepsilon_{a\eta}\ =$$

$$= \frac{\varepsilon_{a\hat{x}\hat{x}} - \varepsilon_{a\hat{z}\hat{z}}}{2} + \frac{\varepsilon_{a\hat{x}\hat{x}} + \varepsilon_{a\hat{z}\hat{z}}}{2} = \varepsilon_{af} \qquad (24)$$

Relationship (24) will also be right when $\varepsilon_{zz}(t)=\varepsilon_{azz}\sin(\omega t-\pi)$. Thus it has been shown that under sinusoidal loading a linear form of the criterion proposed by Brown and Miller (8) results from the criterion (9).

4^o Let us assume that b=1, k=1 and the expected fracture plane is equally inclined to the mean directions of the principal strains, i.e. coincides with a mean position of the octahedral plane (Fig.5).

The unit vector $\bar{\eta}$, inclined at identical angles to the axes $0\hat{1}\hat{2}\hat{3}$ in the coordinate system 0xyz can be written as

$$\bar{\eta}= \frac{1}{\sqrt{3}} [(\hat{l}_1+ \hat{l}_2+ \hat{l}_3)\,\bar{i} + (\hat{m}_1+ \hat{m}_2+ \hat{m}_3)\,\bar{j} + (\hat{n}_1+ \hat{n}_2+ \hat{n}_3)\,\bar{k}] \qquad (25)$$

Hence, the normal strain $\varepsilon_\eta(t)$ is equal to

$$\varepsilon_\eta(t)= \tfrac{1}{3} [(\hat{l}_1+ \hat{l}_2+ \hat{l}_3)^2\varepsilon_{xx}(t) + (\hat{m}_1+ \hat{m}_2+ \hat{m}_3)^2\varepsilon_{yy}(t) + (\hat{n}_1+\hat{n}_2+\hat{n}_3)^2 \varepsilon_{zz}(t) + 2(\hat{l}_1+ \hat{l}_2+ \hat{l}_3)(\hat{m}_1+ \hat{m}_2+ \hat{m}_3)\varepsilon_{xy}(t) + 2(\hat{l}_1+ \hat{l}_2+ \hat{l}_3)(\hat{n}_1+ \hat{n}_2+ \hat{n}_3)\,\varepsilon_{xz}(t) + 2(\hat{m}_1+ \hat{m}_2+ \hat{m}_3)(\hat{n}_1+ \hat{n}_2+ \hat{n}_3)\varepsilon_{yz}(t)] \qquad (26)$$

Let us assume that in the plane ABC (Fig.5) the unit vector $\bar{s}$ is inclined at an angle ϕ to the direction of axis $0'$ $(\hat{1})$.

In the coordinate system 0xyz the vector $\bar{s}$ can be written as [18]

$$\bar{s} = [\sqrt{\tfrac{2}{3}}\, (\hat{l}_1+\hat{l}_2+\hat{l}_3)\cos\phi]\,\bar{i} - \sqrt{\tfrac{2}{3}}\,(\tfrac{1}{2}\cos\phi - \tfrac{\sqrt{3}}{2}\sin\phi)(\hat{m}_1+\hat{m}_2+\hat{m}_3)\bar{j} - \sqrt{\tfrac{2}{3}}\, (\tfrac{1}{2} \cos \phi + \tfrac{\sqrt{3}}{2} \sin \phi)(\hat{n}_1+ \hat{n}_2+ \hat{n}_3)\,\bar{k} \qquad (27)$$

Therefore, the strain $\varepsilon_{\eta s}(t)$ is equal to

$$\varepsilon_{\eta s}(t) = \frac{\sqrt{2}}{3}\{[(\hat{l}_1+ \hat{l}_2 + \hat{l}_3)^2 \cos\phi]\, \varepsilon_{xx}(t) + 0.5\, (\hat{m}_1+ \hat{m}_2+ \hat{m}_3)^2 (\sqrt{3}\sin\phi - \cos\phi)\, \varepsilon_{yy}(t) - 0.5\, (\hat{n}_1+ \hat{n}_2+ \hat{n}_3)^2 (\sqrt{3}\sin\phi + \cos\phi)\, \varepsilon_{zz}(t) + 0.5(\hat{l}_1+ \hat{l}_2+ \hat{l}_3)(\hat{m}_1+ \hat{m}_2+ \hat{m}_3)(\sqrt{3}\sin\phi + \cos\phi)\varepsilon_{xy}(t) + 0.5\, (\hat{l}_1+ \hat{l}_2+ \hat{l}_3)(\hat{n}_1+ \hat{n}_2+ \hat{n}_3)(\cos\phi - \sqrt{3}\sin\phi)\varepsilon_{xz}(t) - (\hat{m}_1+ \hat{m}_2+ \hat{m}_3)(\hat{n}_1+ \hat{n}_2+ \hat{n}_3)(\cos\phi)\varepsilon_{yz}(t)\} \qquad (28)$$

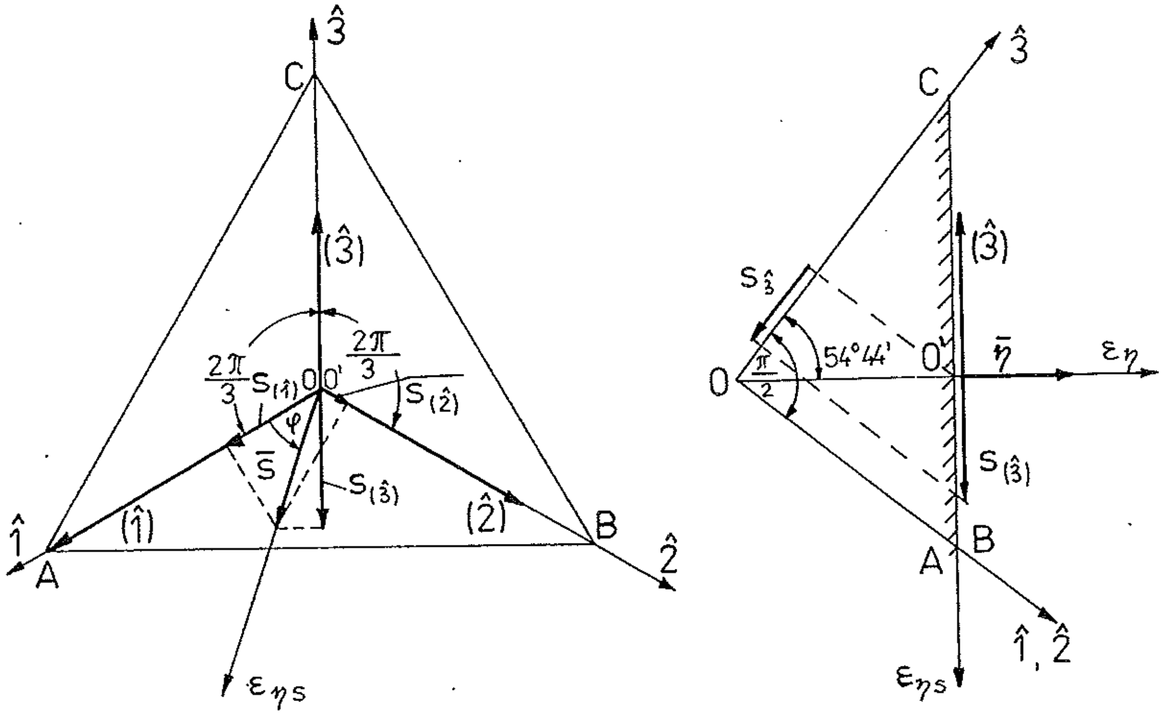

Fig.5 Directions of unit vectors $\bar{\eta}$ and $\bar{s}$,normal and shear strains $\varepsilon_{\eta}(t)$ and $\varepsilon_{\eta s}(t)$ respectively in the fracture plane determined by a mean location of the octahedral plane.

Assuming that under uniaxial sinusoidal loading we have

$\varepsilon_{xx}(t)=\varepsilon_{af}\sin\omega t$; $\quad \varepsilon_{yy}(t)= -\nu\, \varepsilon_{xx}(t)$; $\quad \varepsilon_{zz}(t)= -\nu\, \varepsilon_{xx}(t)$;

$\varepsilon_{xy}(t)=\varepsilon_{xz}(t)=\varepsilon_{yz}(t) = 0$

and that $\hat{l}_1=\hat{m}_2=\hat{n}_3=1$, from (26) and (28) we obtain

$$\varepsilon_{\eta}(t) = \frac{1-2\nu}{3}\,\varepsilon_{xx}(t) = \frac{1-2\nu}{3}\,\varepsilon_{af}\sin\omega t \qquad (29)$$

$$\varepsilon_{\eta s}(t) = [\frac{\sqrt{2}}{3}(1+\nu)\cos\phi]\,\varepsilon_{xx}(t)=[\frac{\sqrt{2}}{3}(1+\nu)\cos\phi\,]\,\varepsilon_{af}\sin\omega t \qquad (30)$$

It is clear that expression (30) reaches its maximum amplitude when $\phi=0$. Therefore,the maximum value of the strength function according to (9) is

$$\max_t\{\varepsilon_{\eta s}(t) + \varepsilon_{\eta}(t)\} = \max_t\{ \frac{\sqrt{2}}{3}(1+\nu)\varepsilon_{af}\sin\omega t + \frac{1-2\nu}{3}\varepsilon_{af}\sin\omega t\} = \frac{1+(1+\nu)\sqrt{2}-2\nu}{3}\,\varepsilon_{af} \qquad (31)$$

For a general case of loading the criterion (9) has the following form

$$\max_t \{ \varepsilon_{\eta s}(t) + \varepsilon_{\eta}(t) \} = \frac{1-2\nu + (1+\nu)\sqrt{2}}{3} \varepsilon_{af} \qquad (32)$$

where $\varepsilon_{\eta s}(t)$ is expressed by formula (28) and $\varepsilon_{\eta}(t)$ - by (26). The angle ϕ determines a mean position of the maximum strain $\varepsilon_{\eta s max}$ in the discussed fracture plane.

Under multiaxial sinusoidal loading when normal and shear strains are coincident in phase and on the assumption that $\hat{l}_1=\hat{m}_2=\hat{n}_3=1$, from (26) and (28) it results that

$$\varepsilon_{\eta}(t)=\frac{1}{3} \{[\varepsilon_{xx}(t) + \varepsilon_{yy}(t) + \varepsilon_{zz}(t)] + 2[\varepsilon_{xy}(t) +\varepsilon_{xz}(t)+\varepsilon_{yz}(t)]\} =$$

$$=\frac{1}{3} \{ [\varepsilon_{axx}+ \varepsilon_{ayy}+ \varepsilon_{azz}] + 2[\varepsilon_{axy}+ \varepsilon_{axz}+ \varepsilon_{ayz}]\}\sin\omega t = \varepsilon_{a\eta}\sin\omega t \qquad (33)$$

$$\varepsilon_{\eta s}(t) = \frac{\sqrt{2}}{3}[(\cos\phi)\varepsilon_{xx}(t) + 0.5(\sqrt{3}\sin\phi - \cos\phi)\varepsilon_{yy}(t) -$$

$$- 0.5 (\sqrt{3}\sin\phi + \cos\phi) \varepsilon_{zz}(t) + 0.5(\sqrt{3}\sin\phi + \cos\phi)\varepsilon_{xy}(t) -$$

$$- 0.5 (\sqrt{3} \sin\phi - \cos\phi) \varepsilon_{xz}(t) - (\cos\phi) \varepsilon_{yz}(t)] =$$

$$= \frac{\sqrt{2}}{3} [(\cos\phi) \varepsilon_{axx} + 0.5(\sqrt{3}\sin\phi - \cos\phi) \varepsilon_{ayy} -$$

$$- 0.5 (\sqrt{3}\sin\phi + \cos\phi) \varepsilon_{azz} + 0.5(\sqrt{3}\sin\phi + \cos\phi) \varepsilon_{axy} -$$

$$- 0.5 (\sqrt{3}\sin\phi - \cos\phi) \varepsilon_{axy} -(\cos\phi)\varepsilon_{ayz}]\sin\omega t = \varepsilon_{a\eta s}\sin\omega t \qquad (34)$$

From (33) it results that the strain $\varepsilon_{\eta}(t)$ includes two terms: one is the mean or octahedral normal strain which is usually assumed as equal to zero on reaching the yield limit; the second one includes components of shear strains which are not equal to zero even after exceeding the yield limit. This is a result of choosing one mean position of the octahedral plane. In fatigue tests the octahedral plane usually changes its position and the second term of expression (33) is omitted when describing test results.

Formula (34) is very interesting, too. From this formula it results that the strain $\varepsilon_{\eta s}(t)$ is considerably dependent on a value of ϕ determining a mean direction $\vec{s}$ of the maximum shear strain $\varepsilon_{\eta s max}$ in this plane.

Under multiaxial sinusoidal loading and with formulae (33) and (34) taken into consideration equation (32) can be written as

$$\max_{\phi,t} \{[1+\sqrt{2}\ \cos\phi]\ \varepsilon_{xx}(t) + [1+0.5\ \sqrt{2}\ (\sqrt{3}\sin\phi - \cos\phi)]$$

$$\varepsilon_{yy}(t) + [1 - 0.5\ \sqrt{2}\ (\sqrt{3}\ \sin\phi + \cos\phi)]\ \varepsilon_{zz}(t) + [2 +$$

$$+ 0.5\ \sqrt{2}(\sqrt{3}\ \sin\phi + \cos\phi)\ ]\ \varepsilon_{xy}(t) + [2 - 0.5\ \sqrt{2}\ (\sqrt{3}\ \sin\phi - \cos\phi]$$

$$\varepsilon_{xz}(t) + [2 - \sqrt{2}\ \cos\phi]\ \varepsilon_{yz}(t)\} = [1-2\nu + (1+\nu)\ \sqrt{2}\ ]\varepsilon_{af} \qquad (35)$$

Now it is clear that the maximum value of the left side at a constant frequency of loading is dependent on ϕ, amplitudes of ε_{aij} and phase angles of strains.

4. CONCLUSIONS

From the analysis presented, the following conclusions can be drawn:

1. From a review of strain criteria for multiaxial cyclic fatigue it results that the critical plane and normal or/and shear strains connected with the plane may be observed in case of each criterion. The maximum ranges of these strains determine directions along which strains change sinusoidally.
2. A new fatigue criterion for multiaxial random loading is presented on the assumption that an expected fatigue fracture plane is a result of the occurence of random values and directions of principal strains. The fracture plane is determined by the maximum value of a linear combination of shear and normal strains in this plane. Only shear strain in one direction $\bar{s}$ on the discussed plane is considered. The equation

$$\max_{t} \{b\ \varepsilon_{\eta s}(t) + k\ \varepsilon_{\eta}(t)\} = q$$

is a mathematical expression of the criterion.
A location of the expected fracture plane is determined by mean direction cosines of principal strains.
3. In special cases from the formulated generalized criterion of maximum shear and normal strains in the fracture plane three classical criteria for multiaxial cyclic loadings result as special cases: criterion of the maximum normal strain, criterion of the maximum shear strain, criterion of the maximum shear and normal strains in the critical shearing plane.
4. The criterion is theoretical and it will be verified by experiments.

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5.REFERENCES

[1] Achtelik H.,Macha E.,Jakubowska I., Actual and estimated directions of fatigue fracture plane in Zl 250 grey cast iron under combined alternating bending and torsion, Studia Geotechnica et Mechanica, Vol.V, No.2, 1983, pp.9-30.

[2] Beavert P.W., Biaxial fatigue and fracture of metals: a review, Metals Forum, Vol.8, No.1,1985, pp.14-29.

[3] Będkowski W.,Lachowicz C.,Macha E., The expected fatigue fracture plane according to variance of normal stress under random triaxial state of stress,in: Proc.of the 8th Inter.Colloq. MECHANICAL FATIGUE OF METALS,Sept.1985,Gdańsk Poland, Scientific Papers of the Gdańsk Tech.Univ. No. 403, 1986, pp.155-162.

[4] Będkowski W., Macha E., Fatigue criterion of the maximum strain in the direction perpendicular to a fracture plane. The expected fatigue failure planes under random triaxial state of stress, Fortschr.-Ber. VDI Reihe 5, No.97, VDI-Verlag, Dusseldorf 1985, p.82.

[5] Brown N.W., Miller K.J., A theory for fatigue failure under multiaxial stress-strain conditions, Proceedings, Institution of Mechanical Engineering, Vol.187, 1973, pp.745-755.

[6] Brown M.W., Miller K.J., Two decades of progress in the assessment of multiaxial low-cycle fatigue life, in:Low-Cycle Fatigue and Life Prediction, ASTM STP 770, C.Amzallag, B.N. Leis and P.Rabbe, Eds.,American Society for Testing and Materials, 1982, pp.482-499.

[7] Brown M.W., Multiaxial fatigue testing and analysis, in:Fatigue at High Temperature, R.P.Skelton,Ed., Applied Science Pub., 1983, Chapter 3, pp.97-133.

[8] Deja J., Kuźnicka B., Oświęcimski W., On the possibilities of determining the limit state of materials under biaxial state of fatigue stresses. Scientific Papers of the Institute of Mat.Scien.and Appl.Mech. of Wrocław Technical University No.42, Monographs No.14,1980, p.54 (in Polish)

[9] Ellyin F., Valaire B., Development of fatigue failure theories for multiaxial high strain conditions, Solid Mechanics Archives, Vol.10,No.1, 1985, pp.45-85.

[10] Fash J.W.,Socie D.F.,Mc Dowell D.L., Fatigue life estimates for a simple notched component under biaxial loading, in: Multiaxial Fatigue, ASTM STP 853, K.J. Miller and M.W.Brown Eds., American Society for Testing and Materials, Philadelphia, 1985, pp.497-513.

[11] Fuchs H.O., Stephens R.I., Metal fatigue in engineering, Chapter 9, Multiaxial stresses and strains, John Wiley and Sons,New York, 1980,pp.173-187.

[12] Garud Y.S., Multiaxial fatigue: a survey of the state of the art,Journal of Testing and Evaluation, JTEVA, Vol.9,No. 3, 1981, pp.165-178.

[13] Kelly F.S., A general fatigue evaluation method elastic stress or plastic strain with constant or varying principal direction,ASME Journal of Pressure Vessel Technology, Vol. 102, No.3, 1980, pp.287-293.

[14] Kostyuk A.G., Trukhnii A.D., Michulin V.N., Strength criteria of materials under low-cycle fatigue with complex stress states, Machinovedenie,No.5,1974, pp.62-67.

[15] Krempl E., The influence of state of stress on low-cycle fatigue of structural materials: a literature survey and interpretive report, ASTM STP 549, American Society for Testing and Materials, Philadelphia, 1974, p.46.

[16] Lohr R.D., Ellison E.G., A simple theory for low cycle multiaxial fatigue,Fatigue Eng.Mater.Struct.,No.3,1980,pp.1-17

[17] Macha E., The expected position of fatigue fracture planes according to analytical methods,6th European Conference on Fracture (ECF 6), 1986,Amsterdam (poster session),p.8.

[18] Macha E., Generalization of strain criteria of multiaxial

cyclic fatigue to random loadings, Fortschr.-Ber.VDI, Reihe 18, Nr.52, Dusseldorf, VDI-Verlag 1988, p.102.

[19] Macha E., Mathematical models of the life to fracture for materials subject to random complex stress systems, Scientific Papers of the Institute of Mater.Scien.and Appl.Mech. of Wrocław Technical University, No.41, Mographs No.13,Wrocław 1979, pp. 99 (in Polish).

[20] Macha E.,Fatigue failure criteria for materials under random triaxial state of stress, in: Advances in Fracture Research, Proc. of the 6th Inter.Conf. on Fracture ICF6, New Delhi, 4-10 Dec.1984, Eds.S.R.Valluri et al, Pergamon Press Oxford, Vol.3, pp.1895-1902.

[21] Macha E., Generalization of fatigue fracture criteria for multiaxial sinusoidal loadings for the range of random loadings, Biaxial/Multiaxial Fatigue EGF 3, edited by K.J.Miller and M.W.Brown , 1988, Mechanical Engineering Publications, London, pp. 425-436.

[22] Movchan A.A., Different criteria of equivalent plastic strain range in the theory of low-cycle fatigue, Problemy Prochnosti, No.12, 1982, pp.7-10.

[23] Socie D.F., Waill L.A., Dittmer D.F., Biaxial fatigue of Inconel 718 including mean stress effects, in: Multiaxial Fatigue, ASTM STP 853, K.J.Miller and M.W.Brown,Eds.,American Society for Testing and Materials, Philadelphia 1985,pp 463-481.

[24] Stanfield G., Contribution to discussion, Proceedings, Institution of Mechanical Engineering, Vol.131, 1935, p.93.

[25] Tulyakov G.A., Metelkov V.A., Plekhanov V.A., Thermal fatigue testing of materials subjected to complex stressing, Problemy Prochnosti, No.6, 1972, pp. 109-113.

[26] Tulyakov G.A., Metelkov V.A., Investigation of the thermal fatigue of type Kh 18N10T steel under complex stress distributions, Problemy Prochnosti, No.7, 1972, pp.33-37.