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The Significance of Mode I Branch Cracks for Mixed Mode Fatigue Crack Growth Threshold Behaviour

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ABSTRACT In the general case of mixed mode loading what is meant by the fatigue crack growth threshold needs careful definition; in particular, it is necessary to distinguish between thresholds for Stage I and Stage II crack growth.

It is a matter of observation that, under essentially elastic conditions, Stage II fatigue cracks grow in Mode I. In general, for the case of mixed mode loading, Stage II fatigue crack growth is not in the plane of the initial crack. Prediction of Stage II threshold behaviour involves finding criteria for the formation and propagation of a Mode I branch crack (or cracks) at the tip of the initial (main) crack. No satisfactory criterion for the formation of Mode I branch cracks appears to exist. Through an examination of stress intensity factors for Mode I branch cracks an approximate lower bound envelope for mixed mode loading involving all three modes has been constructed and, for both mixed Mode I and II and mixed Mode I and III loading, confirmed experimentally. The lower bound is appropriate for design purposes, but in some circumstances may be unduly conservative.

Stage I fatigue growth may take place below the lower bound threshold for Stage II growth, but does not normally lead to complete failure. A tentative failure mechanism map for mixed Mode I and II fatigue crack growth threshold behaviour, which includes a lower bound for the Stage I threshold, has been constructed.

Introduction

For situations where the crack tip stress field can be characterised by stress intensity factors it is a matter of observation (1) that Stage II fatigue cracks in isotropic metallic materials tend to grow in Mode I. Conventional specimens used for the determination of fatigue crack growth properties are, therefore, designed so that only Mode I crack tip surface displacements are present, and crack growth rates are expressed in terms of ΔK_I, the range of Mode I stress intensity factor in the fatigue cycle. Crack growth does not take place unless a threshold value of ΔK_I, ΔK_{th}, is exceeded.

In the general case of a crack (or crack-like flaw) from which a service failure originates, Mode II or Mode III displacements (or both) may be present. One approach (2) to the prediction of threshold behaviour under such mixed mode loadings is to consider conditions for the formation and propagation of a small Mode I branch crack (or cracks) at the tip of the initial (main) crack.

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Clearly, a Mode I branch crack will not grow unless ΔK_{th} is exceeded, and by expressing branch crack stress intensity factors in terms of main crack stress intensity factors it is possible (see Appendix) to construct theoretical lower bound failure envelopes. These are compared with experimental mixed mode threshold data taken from the literature, and a tentative failure mechanism map is developed for mixed Mode I and II loading.

Discussion is from a macroscopic viewpoint; in particular, the fracture surfaces are generally assumed to be smooth, although on a microscopic scale they are actually very irregular (**1**). In mixed mode testing the direction of growth of a branch crack is often of interest, but the irregularity of actual cracks (and for curved branch crack paths the need to estimate a tangent at the start of the branch crack) makes it difficult to measure crack direction accurately, and there is always some uncertainty (**3**).

Subscripts I, II, III denote the mode of a stress intensity factor, K; lower case ks are used to distinguish stress intensity factors at the tip of a branched crack. The prefix Δ, indicating range in the fatigue cycle, is sometimes omitted for clarity. In tests at zero mean load ΔK_I and ΔK_{III} are calculated from the alternating load.

Anisotropic materials, situations where extensive plasticity prevents the use of stress intensity factors, and loadings other than constant amplitude, are not discussed.

Notation

a	Crack length
K	Main crack stress intensity factor, subscripts I, II, III denote mode
ΔK_{th}	Stage II fatigue crack growth threshold (critical value of ΔK_I for Stage II fatigue crack growth)
k	Branch crack stress intensity factor, subscripts I, II, III denote mode
k_I^*	Maximum value of k_I
k_{II}^*	Maximum value of k_{II}
N	Number of cycles
R	Stress ratio (ratio of minimum to maximum stress in fatigue cycle)
Δ	Prefix to indicate range in fatigue cycle
θ	Angle of branch crack
θ^*	Value of θ corresponding to k_I^*
ν	Poisson's ratio
ϱ_c	Critical value of notch tip radius
σ_Y	Yield stress (taken as 0.2 per cent proof stress)
ϕ	Inclination of branch crack element
ϕ^*	Value of ϕ corresponding to K_I^*

Definition of fatigue crack growth thresholds

There is no generally agreed definition of precisely what is meant by a fatigue crack growth threshold, either as a fundamental definition, or in terms of a practicable method of measurement, and various approaches are possible **(1)(4)**. A threshold exists because a crack cannot extend by less than about one lattice spacing per cycle ($\sim 4.10^{-10}$ m/cycle) **(1)**. Lower average crack growth rates are sometimes observed, this implies that the crack cannot be extending along the whole crack front on every cycle.

Consider first the fundamental definitions of ΔK_{th}. One possible definition is 'the minimum value of ΔK_I needed to cause Stage II fatigue crack growth at an initially stress free crack on the first loading cycle'. This is clearly impracticable. Specification of Stage II crack growth means that it must be in Mode I.

A growing fatigue crack leaves behind a wake of plastically deformed material which increases resistance to fatigue crack growth **(5)**, and an alternative definition is 'the minimum value of ΔK_I needed for continued Stage II fatigue crack growth under steady-state conditions'. The restriction to steady-state conditions implies both that a crack must have grown for some distance and that dK/da (a is crack length) must be small; both implications restrict the definition to 'long' cracks.

Because of crack closure due to the wake of plastically deformed material, the effective value of ΔK_I is below the nominal value, and so a better definition would be 'the minimum value of effective ΔK_I needed for continued Stage II crack growth under steady-state conditions'. However, measurement of effective ΔK_I presents practical problems.

Practical methods of measuring ΔK_{th} fall into two main groups **(4)**; both are based on nominal rather than effective values of ΔK_I. In the first group, the da/dN versus ΔK_I (N is number of cycles) is followed downwards in a stepwise manner, with precautions to ensure steady-state conditions at each step, until the crack growth rate is substantially zero. Majority view appears to be **(4)** that 10^{-11} cycle (rather less than one lattice spacing per cycle) may be regarded as substantially zero, making a practical definition 'the value of ΔK_I corresponding to a steady-state crack growth rate of 10^{-11} m/cycle'.

In the second group, pre-cracked, stress relieved, specimens are subjected to increasing values of ΔK_I until crack growth occurs. Detection of crack growth may be either through the use of appropriate instrumentation **(4)**, or by the construction of S/N curves for cracked specimens **(6)**. In either case, experimental details are usually arranged so that a crack growth rate of around one lattice spacing per cycle is detected. This group might appear less precise, in that detecting crack growth does not ensure that steady-state conditions have been attained. In practice, an average crack growth rate over some finite distance is detected, and values of ΔK_{th} obtained are similar to those for the first group **(1)**.

Measured values of ΔK_{th} are dependent (4) on the stress ratio, R (ratio of minimum to maximum load in the fatigue cycle), environment, and metallurgical factors, especially grain size. At longer crack lengths, ΔK_{th} is largely independent of crack length, but for various reasons is usually lower for short cracks (7) (say less than 1 mm). In particular, $\mathrm{d}K/\mathrm{d}a$ is not small, so steady-state conditions may not be reached (1).

Measurement of the threshold through the detection of crack growth extends naturally to mixed mode loadings. An appropriate definition of the Stage II threshold for loadings where ratios between K_{I}, K_{II}, and K_{III} do not change during the fatigue cycle is 'the values of ΔK_{I}, ΔK_{II}, and ΔK_{III} corresponding to a steady-state Stage II fatigue crack growth rate of one lattice spacing per cycle'. In general, crack growth is not in the plane of the initial (main) crack. However, the use of initial (main) crack stress intensity factors is appropriate if the non-coplanar branch crack is small enough to be within the crack tip stress field of the initial (main) crack. The occurrence of Stage II crack growth may be demonstrated, either fractographically, or by ensuring that the initial direction of branch crack growth corresponds to that for a Mode I branch crack. Non-proportional loadings in which ratios between K_{I}, K_{II}, and K_{III} change during the fatigue cycle (out-of-phase loading) are outside the scope of this paper; there is then no branch crack direction for which K_{II} and K_{III} are zero throughout a fatigue cycle.

For both pure Mode I loading (7) and mixed Mode I and II loading (8)(9) it has been found that Stage I fatigue crack growth can take place at stress intensity factors lower than the Stage II threshold. For long cracks, where $\mathrm{d}K/\mathrm{d}a$ is small, such cracks usually arrest after a small amount of crack growth, often when an obstruction such as a grain boundary is encountered. At short crack lengths $\mathrm{d}K/\mathrm{d}a$ is large and Stage I cracks may transform into Stage II cracks, giving continuous crack growth (7). As Stage I crack growth is usually detected fractographically, one appropriate definition of the Stage I threshold is 'the minimum values of ΔK_{I}, ΔK_{II}, and ΔK_{III} needed to cause Stage I fatigue crack growth which can be detected fractographically'. In the remainder of the paper 'threshold' is taken to mean the Stage II threshold.

Experimental mixed mode fatigue crack growth threshold data

Nineteen experimental mixed mode fatigue crack growth threshold data sets taken from the literature are listed in Table 1. Some data were obtained using specimens pre-cracked in fatigue, and some sharply notched specimens. Unless a distinction is needed, the term 'crack' includes a notch. Except where noted, the data sets either conform for the definition of the mixed mode Stage II fatigue crack growth threshold given in the previous section, or include fractographic evidence of Stage I fatigue crack growth. Data were checked for validity using the criteria described below. Data set 11 was rejected and is considered in the discussion. The remaining data are shown in Figs 1–3; values

Table 1 Experimental mixed mode fatigue crack growth threshold data

Data set No.	*Material*	*Ref. No.*	*Notch type*	*Notch tip radius, ϱ (mm)*	*Stress ratio,* R	*Modes tested*
1	Commercially pure aluminium	18	Precracked	–	0.65	I, I and II
2	Mild steel	8	Precracked	–	0	I, I and II, II
3	Mild steel	2	Precracked	–	0.1	I, I and II, II
4			Spark eroded	0.02	0.1	II
5	Mild steel	2	Spot weld	–	0	I and II
6	Medium strength structural steel	2	Spot weld	–	0	I and II
7	Mild steel	22	Precracked	–	0.1	I, I and II, II
8	CMn steel BS 4360 50B	19	Precracked	–	0.05	I, II
9			Spark eroded	0.11	0.05	II
10	316 stainless steel	9	Precracked	–	0.2	I, I and II, II
11			Spark eroded	0.08	0.2	I, I and II, II
12			Precracked	–	0.5	I, I and II
13	Mild steel	21	Spark eroded	0.2	0.1	I, I and III
14	Mild steel	21	Machined	0.2	−1	III
15	0.4% C steel	11	Machined	≤0.008	−1	I, I and III
16	3% Ni steel	11	Machined	≤0.015	−1	I, I and III
17	3/3½% Ni steel	11	Machined	≤0.015	−1	I, I and III
18	CrV steel	11	Machined	≤0.023	−1	I, I and III
19	3½% NiCr steel	11	Machined	≤0.025	−1	I, I and III

of ΔK_{I}, ΔK_{II}, and ΔK_{III} are normalised by an appropriate value of ΔK_{th}. For data sets 5, 6, and 14 no values of ΔK_{th} were available for the actual material tested; values for similar material were used and the data sets are, therefore, somewhat less reliable.

The use of stress intensity factors is only appropriate if yielding is confined to a limited region at the crack tip. Two checks are needed: a global check to ensure that the nominal net section stress is not too close to general yield, and a local check to ensure that the crack tip plastic zone size is small compared with the crack length. For Mode I fatigue crack growth rate testing the global check is usually more discriminating, and limiting the nominal net section stress to 80 per cent of the yield stress (or the 0.2 per cent proof stress) is satisfactory **(1)**.

For mixed mode testing there is little evidence on which to base appropriate global or local checks, so where possible a different approach was used. At load levels approaching and above general yield, fatigue cracks tend to follow planes of maximum shear stress rather than growing in Mode I **(1)(10)**. The occurrence of Mode I branch crack growth was, therefore, taken as indicating that the use of stress intensity factors was acceptable up to the K levels concerned.

In some mixed Mode I and III tests **(11)** there was no information on initial crack direction so, arbitrarily, nominal net section stresses were limited to 80 per cent of general yielding based on Mises' criterion, and the plastic zone size,

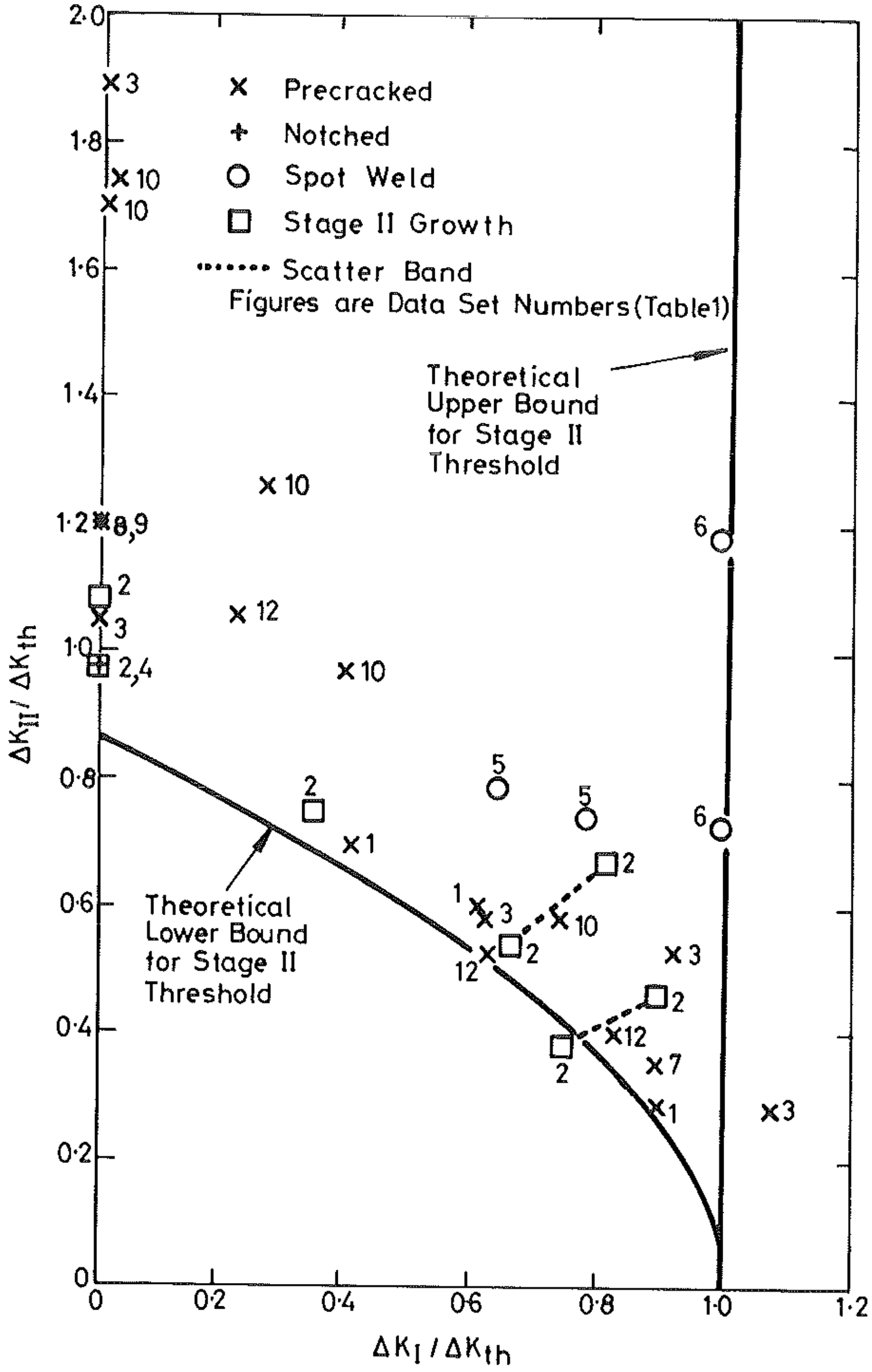

Fig 1 Mixed Mode I and II Stage II threshold and crack growth data

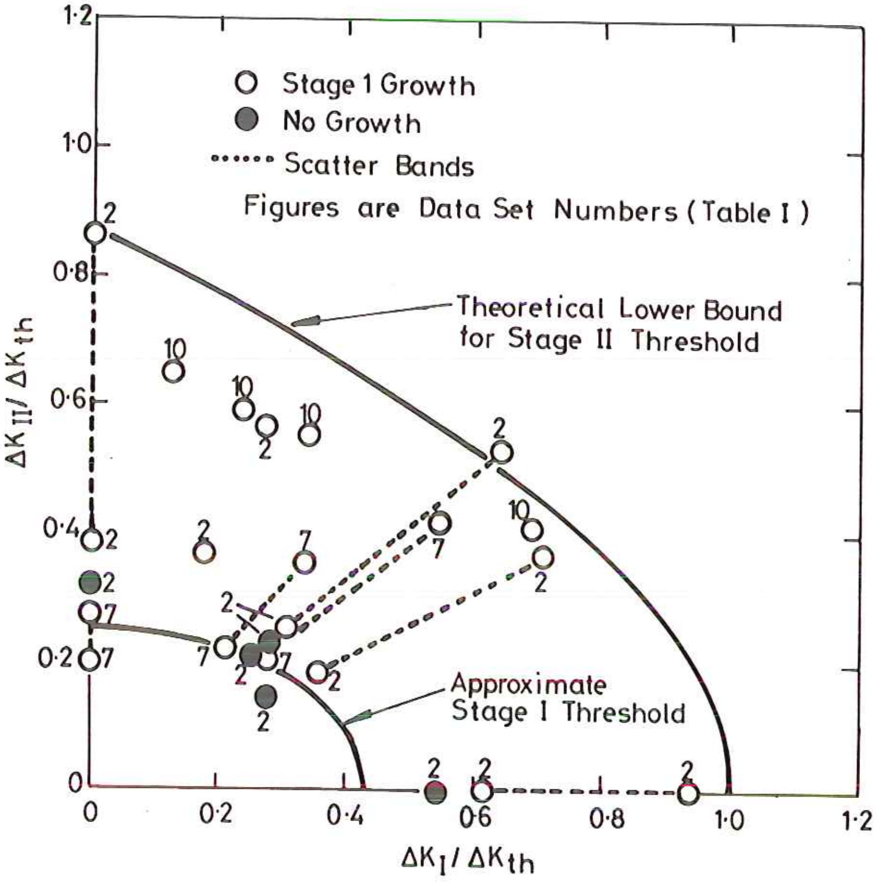

Fig 2 Mixed Mode I and II Stage I crack growth data

r_p, given by (12)

$$r_p = \frac{1}{2\pi\sigma_Y^2}\{K_I^2(1-2\nu)^2 + 3K_{III}^2\} \quad (1)$$

(where ν is Poisson's ratio, taken as 1/3), was limited to 5 per cent of the notch depth. The latter criterion was much more discriminating.

Short crack limitations mean that stress intensity factors are not a valid basis for analysis of mixed mode fatigue crack growth thresholds unless the initial (main) crack has a minimum length of the order of a quarter millimeter (13). For notched specimens, to ensure that stress intensity factors were applicable, the notch root radius was limited, arbitrarily, to a maximum of 5 per cent of the notch depth.

Pre-cracked specimens were regarded as acceptable, provided that residual stresses arising from pre-cracking were minimized, either by a stress-relief

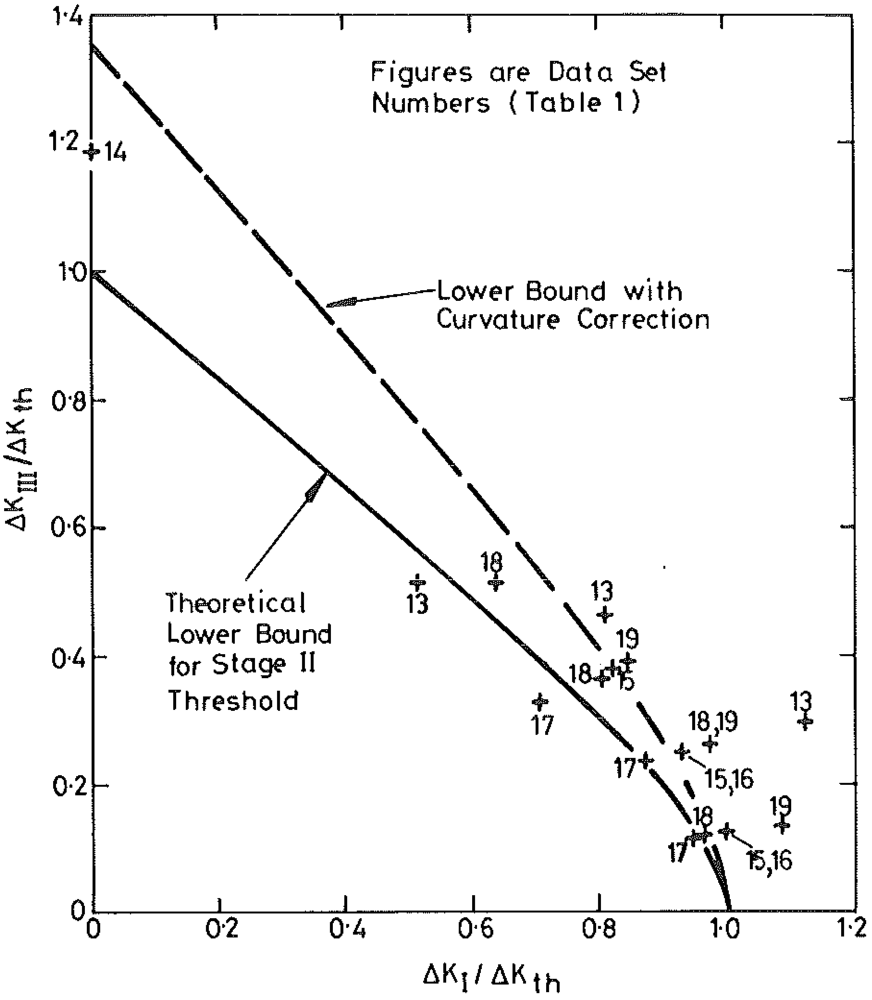

Fig 3 Mixed Mode I and III Stage II threshold data

treatment or by the use of a load shedding technique. Notched specimen data were discarded if there was evidence that ΔK_{th} obtained from notched specimens was substantially greater than that obtained from pre-cracked specimens. This was on the assumption that, as with fracture toughness testing (**14**), there is a material-dependent critical notch root radius, ϱ_{c}, below which results are independent of notch tip radius. The effective tip radius of a spark eroded notch is probably less than that of a machined notch of the same tip radius, because spark erosion introduces small, randomly-oriented radial cracks (**15**).

Discussion

Putting together the information in Figs 1 and 2 leads (**16**) to a tentative failure mechanism map for mixed Mode I and II fatigue crack growth threshold behaviour. Within experimental error all fractographically determined Stage II crack growth data and measured Stage II threshold data lie between a theoretical upper and lower bound (Fig. 1). Once Stage II crack growth starts it normally continues to cause eventually complete failure (**2**). The upper bound is based on the idea (**17**) that a Mode I branch crack must form and propagate when ΔK_{I} exceeds ΔK_{th}. Between the upper and lower bounds, Stage II crack growth, by the formation and propagation of a Mode I branch crack, is possible. Once a Mode I branch crack has formed it will grow, provided that ΔK_{I} for a Mode I branch crack, ΔK_{I} exceeds ΔK_{th}, and the lower bound is given by equation (5) (see Appendix). The conditions for the formation of a Mode I branch crack are unclear, but it appears to be facilitated (**2**) by metallurgical discontinuities and stray Mode III crack tip surface displacements, and is sometimes preceded by a limited amount of Stage I crack growth (**9**)(**18**)(**19**). In the vicinity of pure Mode II, pre-cracked specimen data (but not notched specimen data) tend to be well above the lower bound, and failure may take place away from the pre-crack tip (**2**). This has been ascribed (**19**) to interaction between asperities on opposing pre-crack surfaces.

The fractographic data for Stage I crack growth (Fig. 2) show that such growth is possible at substantially below the lower bound for the Stage II threshold. However, it usually arrests after a limited amount of crack growth (**8**). The results suggest the existence of a Stage I threshold, given approximately by equation (7), but the conditions for Stage I crack growth to take place are unclear. The precise status of information from data set 7, shown in Fig. 2, is not made clear by the authors.

In examining mixed mode fatigue crack growth threshold behaviour, difficulties in interpretation arise if there is uncertainty in the value of the Mode I Stage II threshold ΔK_{th}. Consider, for example, the mixed Mode I and II threshold data for 316 stainless steel, shown in Fig. 4 (Table 1, data sets 10 and 11). The value of ΔK_{th} obtained from specimens containing spark eroded notches with a notch tip radius of 0.08 mm is 1/3 greater than that for pre-cracked specimens. Mixed Mode I and II thresholds for pre-cracked specimens, normalised by the corresponding value ΔK_{th}, fall above the theoretical lower bound, whereas data for notched specimens fall below, apparently invalidating it as a lower bound. However, if the notched specimen data are normalised by the pre-crack value of ΔK_{th} they fall above the lower bound. The probable explanation is that, for Mode I loading, the critical notch root radius, ϱ_{c}, is less than 0.08 mm, so that a high value is obtained for ΔK_{th}, but as $K_{\mathrm{II}}/K_{\mathrm{I}}$ increases, ϱ_{c} increases to above 0.08 mm and valid thresholds are obtained for $K_{\mathrm{II}}/K_{\mathrm{I}}$ greater than about one.

Figure 3 shows mixed Mode I and III Stage II fatigue crack growth threshold

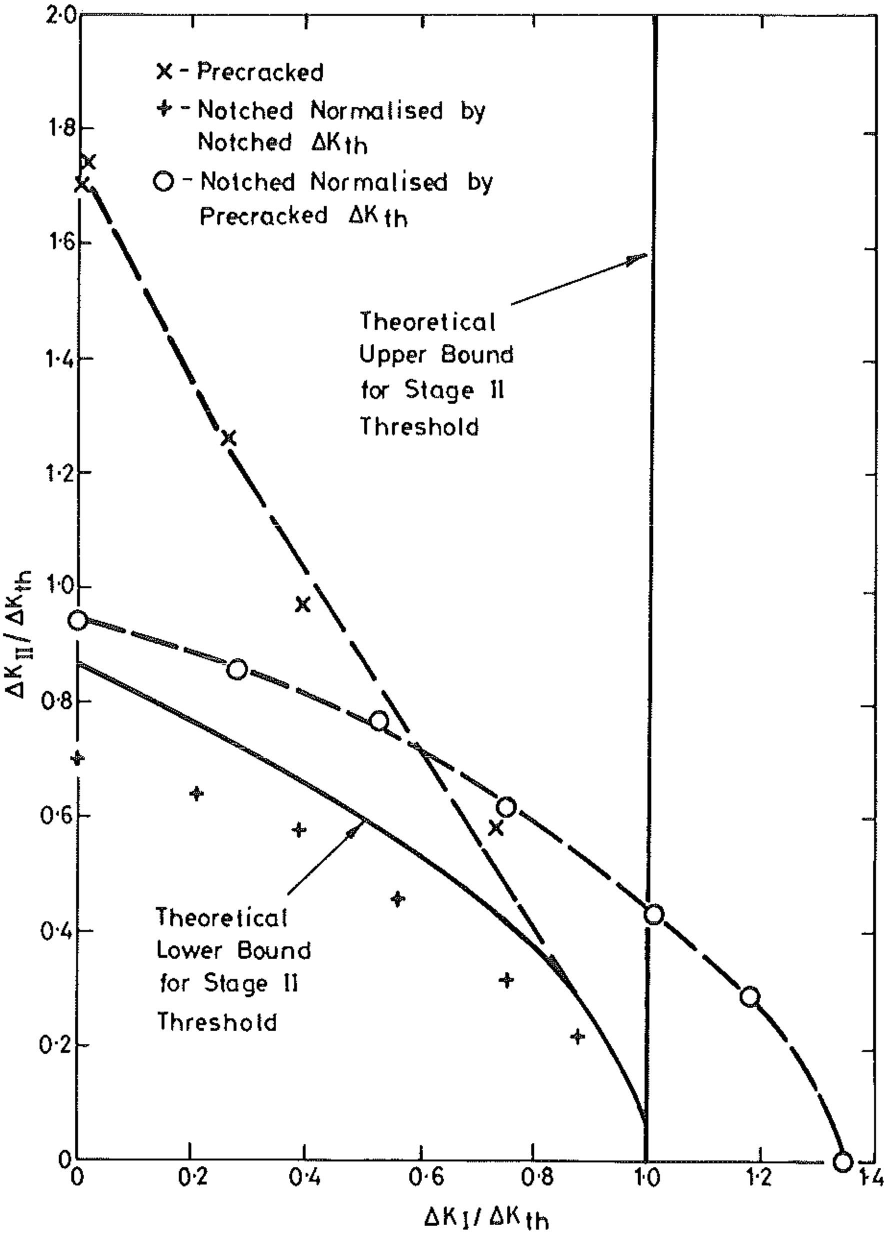

Fig 4 Mixed Mode I and II Stage II threshold data for 316 stainless steel, $R = 0.2$

data. Within experimental error all the data lie above the theoretical lower bound for the Stage II threshold (equation (10)). Many of the experimental data are for $R = -1$, whereas the theoretical lower bound was derived (see Appendix) on the assumption that loads do not pass through zero. However, the direction of a Mode I branch crack changes as the load passes through zero, and two sets of branch cracks appear **(10)(20)**, each of which is open on only one half of the load cycle. Comparison with the theoretical lower bound is, therefore, appropriate. The theoretical lower bound provides one of the boundaries on the mixed Mode I and III failure mechanism map. A lower bound suggested earlier **(3)(13)**, which incorporates a curvature correction based on possible Mode I branch crack geometries, is clearly unconservative. There is, as yet, no clear evidence on the positions of the other two boundaries, but the upper bound for the Stage II threshold is possibly the same as for mixed Mode I and II loading, that is a Mode I branch crack must form and propagate when ΔK_{I} exceeds ΔK_{th}.

Under mixed Mode I and III, and pure Mode III loadings, a Mode I branch crack may arrest after up to several millimetres of crack growth. This is because of interference between opposing crack surfaces in pure Mode III loading **(20)**, and the complex pattern of crack growth under mixed Mode I and III loading **(21)**; it may be necessary **(21)** to distinguish between Stage II thresholds for the start of crack growth, complete failure, and crack arrest. The crack growth threshold is relatively easy to measure, but because it can be substantially lower than the complete failure threshold, may be unduly conservative when applied to practical problems. The complete failure threshold is also relatively easy to measure **(21)**, but the value obtained is geometry dependent and, therefore, not readily applied to practical problems. The crack arrest threshold could, in principle, be measured and applied directly, but, in practice, it is very difficult to determine the required stress intensity factor for the complex crack shapes involved **(21)**. A similar effect has been observed for pure Mode I loading **(5)**, but the three thresholds did not differ greatly.

The theoretical lower bound for the Stage II fatigue crack growth threshold based on the propagation of a Mode I branch crack is confirmed by experimental data for mixed Mode I and II, and mixed Mode I and III loadings. It is reasonable to extend it to the general case of mixed Mode I, II, and III loading, and the resulting lower bound envelope (equations (11) and (12)) is shown in Fig. 5 for $\nu = 1/3$. Unless the initial crack is very short, Stage I crack growth does not lead to complete failure. It is, therefore, possible to use the envelope in Fig. 5 for design purposes, although at times it will be very conservative.

Conclusions

It is a matter of observation that Stage II fatigue cracks grow in Mode I under essentially elastic conditions. In the general case of mixed mode loading, Stage II fatigue crack growth may not be in the plane of the initial crack. For mixed

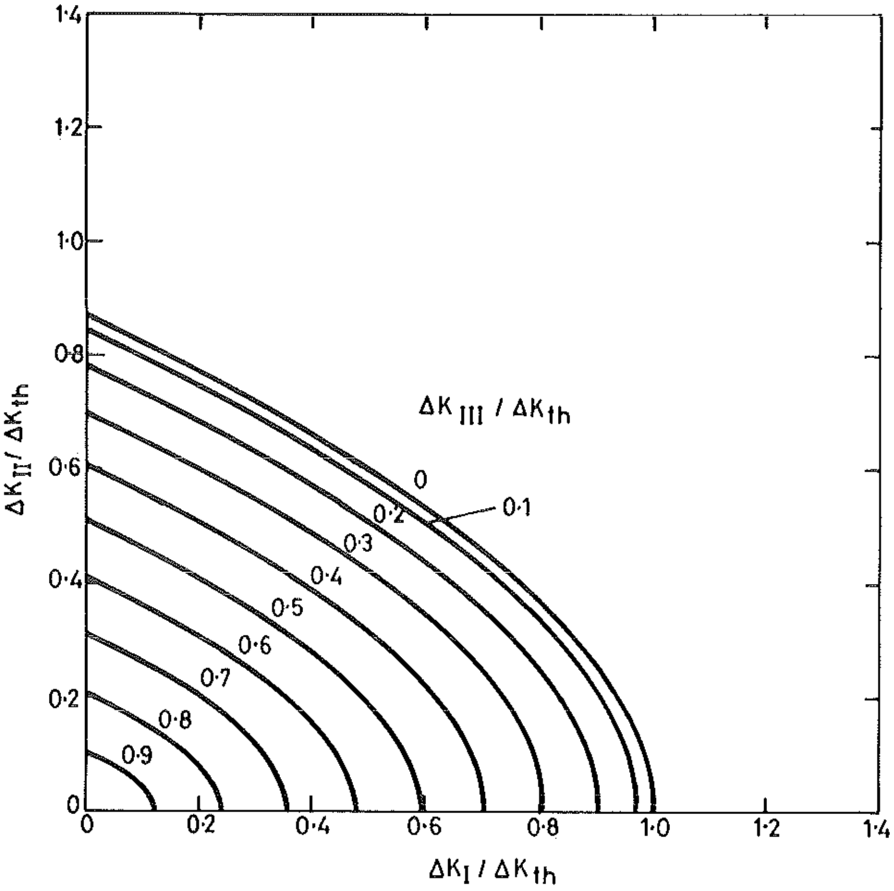

Fig 5 Theoretical lower bound for Stage II threshold for mixed Mode I, II, and III loading

Mode I and III loading it may be necessary to distinguish between thresholds for the start of crack growth, crack arrest, and growth to complete failure.

Prediction of mixed mode Stage II threshold behaviour involves finding criteria for the formation and propagation of a Mode I branch crack (or cracks) at the tip of the initial (main) crack. No satisfactory criterion for the formation of Mode I branch cracks appears to exist. Through an examination of stress intensity factors for Mode I branch cracks an approximate lower bound failure envelope for mixed mode loading involving all three modes has been constructed and, for both mixed Mode I and II and mixed Mode I and III loading, confirmed experimentally. The lower bound is appropriate for design purposes, but in some circumstances may be unduly conservative.

Stage I fatigue crack growth may take place below the lower bound for Stage II fatigue crack growth, but does not normally lead to complete failure. A tentative failure mechanism map for mixed Mode I and II fatigue crack growth threshold behaviour has been constructed.

Acknowledgements

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Appendix

Lower bound failure envelopes

The lower bound failure envelope given in references (**12**) and (**20**) was derived using several assumptions which for the Stage II fatigue crack growth threshold may be stated as follows.

(a) Failure cannot take place unless the stress intensity factor ($\Delta k_{\mathrm{I}}^{*}$) for a Mode I branch crack at the tip of the initial (main) crack exceeds the critical value for crack growth (ΔK_{th}).
(b) Main crack stress intensity factors do not pass through zero during the fatigue cycle.
(c) Ratios between K_{I}, K_{II}, and K_{III} do not vary during the fatigue cycle; that is, cyclic loads are in phase, and the stress ratio, R, for each mode is the same.
(d) The main crack is initially stress free; in particular there are no residual stresses associated with crack tip plastic zones developed by a different prior loading.
(e) For two-dimensional analysis the main crack is straight, or for three-dimensional analysis the main crack is flat and the crack front straight, or in either case, appropriate radii of curvature are large compared with branch crack length.
(f) There are no 'short crack' limitations; this implies (**13**) that the main crack must not be less than about a quarter millimetre long.

For the quasi-two dimensional case of mixed Mode I and II loading (Fig. 6), approximate Mode I and II stress intensity factors for the branch crack, k_{I} and k_{II}, are given by (**3**)(**13**)

$$k_{\mathrm{I}} = \cos\frac{\theta}{2}\left(K_{\mathrm{I}}\cos^{2}\frac{\theta}{2} - \frac{3}{2}K_{\mathrm{II}}\sin\theta\right) \qquad (2)$$

$$k_{\mathrm{II}} = \frac{1}{2}\cos\frac{\theta}{2}\,\{K_{\mathrm{I}}\sin\theta + K_{\mathrm{II}}(3\cos\theta - 1)\} \qquad (3)$$

where K_{I} and K_{II} are main crack stress intensity factors and θ is the inclination of the branch crack direction relative to the main crack plane. It follows that the

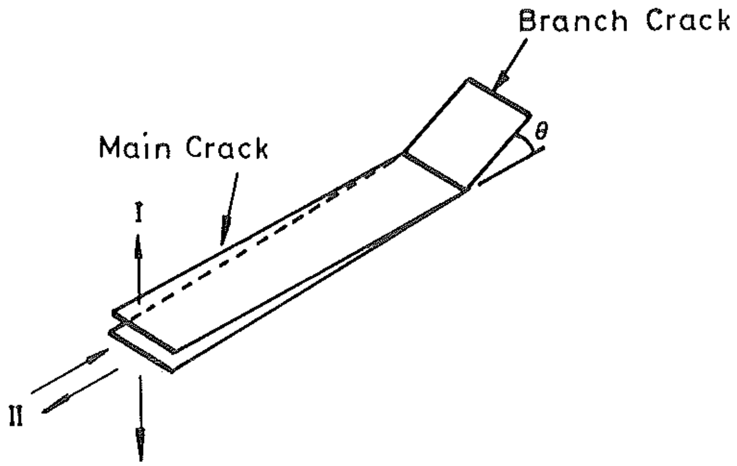

Fig 6 Quasi-two-dimensional crack with branch crack

direction, θ^*, for K_{II} to be zero and k_{I} to have its maximum value, k_{I}^*, is given by

$$K_{\text{I}} \sin \theta^* = K_{\text{II}}(3 \cos \theta^* - 1) \qquad (70.5° \ldots \leq \theta^* \leq -70.5° \ldots) \tag{4}$$

The failure envelope obtained is the theoretical lower bound for the Stage II threshold shown in Figs 1, 2, and 4. It does not have a convenient explicit representation, but the parabola

$$\frac{K_{\text{II}}}{k_{\text{I}}^*} = \left\{0.08\left(\frac{K_{\text{I}}}{k_{\text{I}}^*}\right)^2 - 0.83\,\frac{K_{\text{I}}}{k_{\text{I}}^*} + 0.75\right\}^{1/2} \tag{5}$$

is within one per cent. In the figures, Δs are added, and Δk_{I}^* is replaced by the corresponding critical value, ΔK_{th}.

Otsuka *et al.* (**8**) proposed a lower bound failure envelope for the Stage I threshold. Their derivation is mathematically equivalent to finding the value of θ for which k_{II} has its maximum value k_{II}^*, and assuming that Stage I crack growth is possible when Δk_{II}^* exceeds a threshold value. Note that when k_{II} has its maximum $k_{\text{I}} \neq 0$. Their envelope does not have a convenient explicit representation, but the ellipse

$$\frac{K_{\text{II}}}{k_{\text{II}}^*} = \left\{1 - \left(\frac{K_{\text{I}}}{2.6k_{\text{I}}^*}\right)^2\right\}^{1/2} \tag{6}$$

is a reasonable fit. However, the empirical ellipse

$$\frac{K_{\text{II}}}{k_{\text{I}}^*} = \left\{1 - \frac{1}{3}\left(\frac{K_{\text{I}}}{k_{\text{I}}^*}\right)^2\right\}^{1/2} \tag{7}$$

scaled such that $\Delta K_{\text{II}}/\Delta K_{\text{th}}\ (\Delta K_{\text{I}} = 0) = 0.25$ is a better fit to experimental data for the Stage I threshold (Fig. 2).

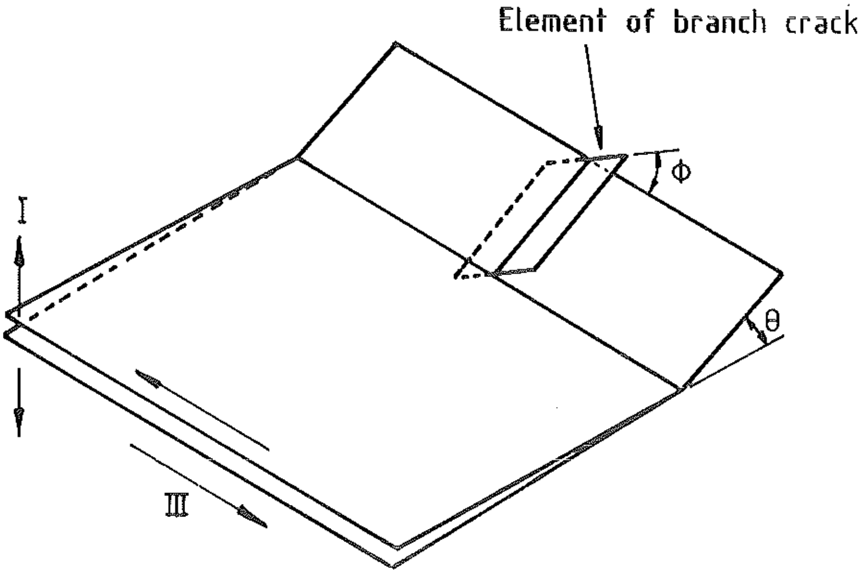

Fig 7 Mixed Mode I and III crack with element of branch crack

For mixed Mode I and III loading it can be shown (**3**)(**13**), by consideration of a flat element of a branch crack (Fig. 7), that the maximum value of k_I is given approximately by

$$k_I^* = \frac{K_I(1+2\nu) + \{K_I^2(1-2\nu)^2 + 4K_{III}\}^{1/2}}{2} \tag{8}$$

where ν is Poisson's ratio. This maximum occurs when both k_{III}, the branch crack Mode III stress intensity factor, and θ (Fig. 7) are zero. The corresponding value of ϕ, ϕ^*, is given by

$$\tan 2\phi^* = \frac{2K_{III}}{K_I(1-2\nu)} \qquad (45° \leqslant \phi^* \leqslant -45°) \tag{9}$$

The failure envelope obtained is the theoretical lower bound for the Stage II threshold shown in Fig. 3 for $\nu = 1/3$. It is given by

$$\frac{K_{III}}{k_I^*} = \left\{1 - \frac{(1+2\nu)K_I}{k_I^*} + 2\nu\left(\frac{K_I}{k_I^*}\right)^2\right]^{1/2} \tag{10}$$

Choosing a lower value of ν gives a slightly less conservative envelope.

In references (**3**) and (**13**), K_{III} in equations (8) and (10) was replaced by $K_{III}/0.74$ as a curvature correction to allow for the fact that the Mode I branch crack elements cannot be assembled into a flat crack. Experimental data (Fig. 3) show that the use of this correction is unconservative, so the correction is now omitted.

The theoretical lower bound failure envelope for the Stage II threshold for mixed Mode I, II, and III loading may be obtained by combining the results for mixed Mode I and III loading (equation (10)) and mixed Mode I and II loading (equation (5)) as indicated in references (**3**) and (**13**). This leads to

$$\frac{K_{\mathrm{II}}}{k_{\mathrm{I}}^{*}} = F\left\{0.08\left(\frac{K_{\mathrm{I}}/k_{\mathrm{I}}^{*}}{F}\right)^{2} - 0.83\,\frac{K_{\mathrm{I}}/k_{\mathrm{I}}^{*}}{F} + 0.75\right\}^{1/2} \qquad 0 \leqslant K_{\mathrm{I}}/k_{\mathrm{I}}^{*} \leqslant F \tag{11}$$

$$F = \frac{1 + 2\nu - \{(1 - 2\nu)^{2} + 8\nu(K_{\mathrm{III}}/k_{\mathrm{I}}^{*})^{2}\}^{1/2}}{4\nu} \qquad 0 \leqslant K_{\mathrm{III}}/k_{\mathrm{I}}^{*} \leqslant 1 \tag{12}$$

and is shown in Fig. 5 for $\nu = 1/3$.

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