

Estimating Multiaxial Elastic–Plastic Notch Stresses and Strains in Combined Loading

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ABSTRACT A method for estimating multiaxial elastic–plastic notch stresses and strains is presented and summarized in a solution scheme for routine application.

The method assumes proportional loading and is based on known approximation formulae, Hencky's equations, and notch element boundary conditions.

Application is illustrated by the example of a notched shaft under bending and torsional loading.

Introduction

In the last two decades a concept for crack initiation life prediction has gained importance namely one based on the assessment of the stress–strain path at the most stressed volume element of the structure under consideration (1–4). This concept – called the local strain approach – requires only a little experimental data, namely the cyclic stress–strain curve and the strain versus life curve, both obtained from smooth specimen tests. Hence, fatigue life predictions can be employed in the design stage.

Although some papers have recently been published dealing with multiaxial fatigue of notched components (e.g. (5)–(11)), the local strain approach for multiaxial situations is not yet a routine application. Besides selecting a suitable multiaxial fatigue theory, one reason is the lack of an easy to handle yet still sufficiently accurate procedure for determining the local elastic–plastic stresses and strains.

The finite element method is a powerful instrument but employment of this method for nonlinear problems is an exception, since large computer capacity and highly specialized engineers are required.

An analysis of approximation formulae proposed in the literature (1)(5)–(9) for multiaxially stressed notches shows that these methods follow a unique scheme based on reference (12).

- (i) Stresses and strains which cause failure (e.g., equivalent stress and strain or maximum stress and strain).
- (ii) An approximation formula (e.g., Neuber formula) to relate external load with notch stresses and strains causing failure.

* FG Werkstoffmechanik, Technische Hochschule Darmstadt, Petersenstrasse 13, D-1600 Darmstadt, FRG.

- (iii) A yield criterion (e.g., von Mises or Tresca).
- (iv) Additional assumptions (e.g., special loading cases).

Because of the particular assumptions associated with each of these items, a general application of these methods seems impossible. Consequently, as no consensus presently exists as to the most appropriate multiaxial fatigue theory, the use of a particular damage rule leads to restrictions.

In this paper a generalized method for estimating multiaxial elastic-plastic notch stresses and strains for components under proportional loading is presented and illustrated by the example of a notched shaft under bending and torsional loading. The procedure is constructed in a modular manner. Complete information is presented about the multiaxial stress and strain state at the notch (given in terms of all stress and strain components) enabling the employment of different or best-suited damage criteria.

Notation

B	Bending moment
E	Young's modulus
K'	Strength coefficient
K_{tq}	Equivalent stress concentration factor
S	Nominal stress
S^*	Modified nominal stress
T	Torque
a	Stress ratio σ_2/σ_1
e^*	Modified nominal strain
n'	Hardening exponent
α	Principal stress direction
γ	Shear strain
ε	Strain
$d\varepsilon$	Strain increment
ε_q	Equivalent strain
ν	Poisson's ratio
ν'	Variable Poisson's ratio
σ	Stress
$d\sigma$	Stress increment
σ_n, τ_n	Net section stresses
σ_q	Equivalent stress
σ_Y	Yield stress
$\sigma = g(\varepsilon)$	Stress-strain relationship
τ	Shear stress

Subscripts

1, 2, 3	Principal stresses, strains
e	Elastic quantities

- q Equivalent quantities
- p Value related to the plastic limit load

Main features of the approximate solution

The proposed method for estimating multiaxial elastic-plastic notch stresses and strains is based on the following conditions.

- (i) Knowledge of the complete state of elastic notch stresses.
- (ii) External loads are proportional in combined loading.
- (iii) As in the case of uniaxial notch stresses, cyclic loading can be reduced to a sequence of monotonic loadings by taking ‘Masing-’ (doubled cyclic stress-strain curve) and ‘Memory-behaviour’ into account.

Justification of the latter point is illustrated in Fig. 1 where two ways are followed for determining the maximum principal strain of a notched round bar under combined tensile and torsional loading (S : nominal stress ϵ_1 : maximum principal notch strain). First, the complete load sequence was followed by the finite element method using kinematic hardening. In the second solution only a finite element analysis for monotonic loading was carried out. The load-strain path was constructed using ‘Masing-’ and ‘Memory-behaviour’ from the load reversal to the prior load reversal point, leading to nearly the same results as with the cyclic finite element solution. A more detailed description of the analysis is given in (13).

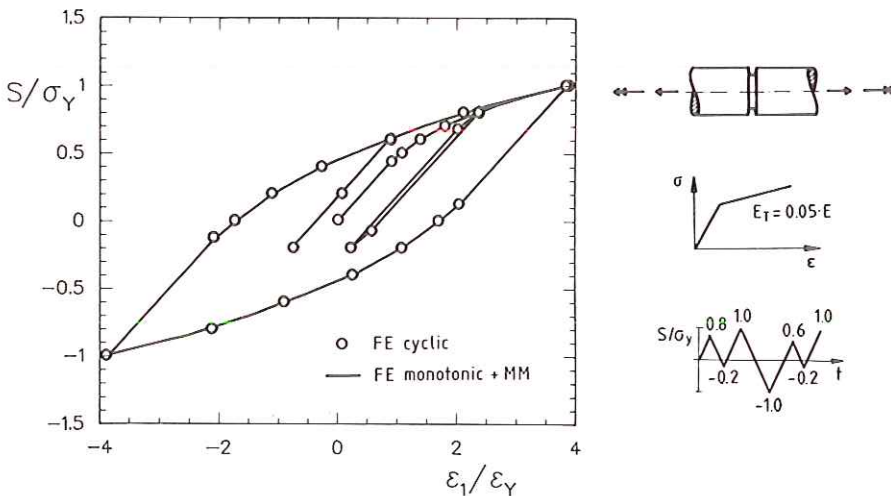


Fig 1 Applied nominal stress versus maximum strain curve for a notched round bar under combined tensile and torsional loading

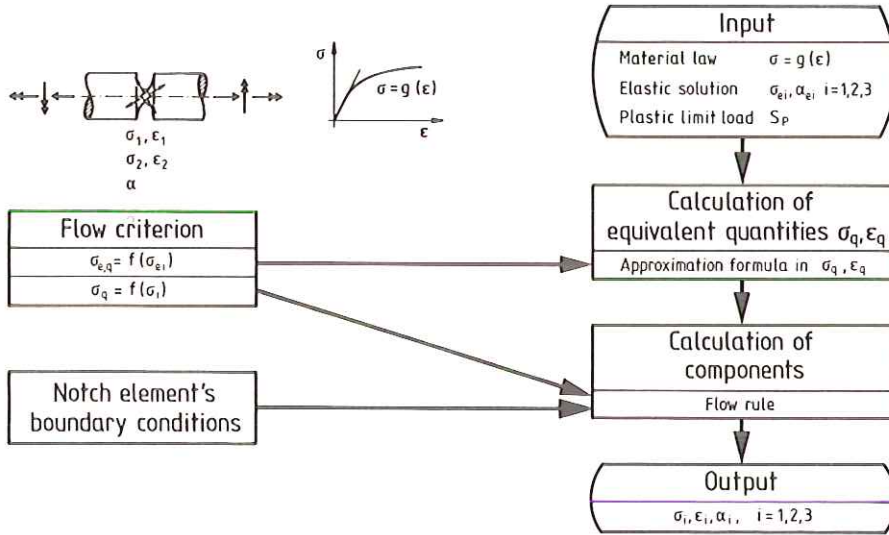


Fig 2 Structure of the approximate solution

Knowing that multi-axially stressed components under external proportional loading show ‘Masing-’ and ‘Memory-behaviour’ makes it a lot easier to find approximate solutions, as only monotonic loading has to be considered.

The structure of the proposed approximate procedure is depicted in Fig. 2. The following input is required.

- (i) Elastic material constants and uniaxial stress–strain curve.
- (ii) Elastic stress state at the notch, e.g., described by the three principal stresses σ_{ei} and the three principal stress directions α_{ei} (subscript e denotes elastic quantities, $i = 1, 2, 3$).
- (iii) Plastic limit load level S_p for elastic–perfectly-plastic material.

The approximate solution consists of two steps. First, a relationship between applied load and equivalent notch stresses and strains is established. The known approximation formulas (5)(14)–(19) valid for uniaxially stressed notches are extended to multi-axial stress states by introducing a yield criterion.

In the second step, the principal stresses and strains at the notch root are correlated to the equivalent quantities σ_q and ϵ_q obtained from the first step. This correlation is established by a flow rule describing the plastic deformation of a multi-axially stressed material element.

In general, five boundary conditions of the notch element have to be known. The modular structure of the approximate procedure, therefore, consists of an approximation formula, a flow criterion, a flow rule, and, finally, the notch element’s boundary conditions, and these derive the specific behaviour resulting from loading type, component geometry, and material.

Description of the approximate solution

A notch at a traction-free surface is now considered (for an example of triaxially stressed notches see (20)). The approximate procedure is presented with fixed contents of the modules proposed in (21) for routine application; that is:

- (i) Neuber's rule as an approximation formula;
- (ii) von Mises flow criterion;
- (iii) Hencky's flow rule;
- (iv) boundary condition I: fixed principal stress direction;
- (v) boundary condition II: constant ratio $\varepsilon_2/\varepsilon_1$ of the surface strains.

Note that strains ε_2 and ε_3 are not ordered according to magnitude, and that ε_1 , ε_2 denote surface strains and ε_3 is the strain normal to the notch surface. Three more boundary conditions, namely $\sigma_3 = 0$ and two principal stress directions being fixed, are satisfied automatically. In the following the different modules are described in detail.

Neuber's rule for multiaxial stress states based on von Mises flow criterion

According to a proposal of Neuber (22)(23) the known approximation formulas are extended to multiaxial stress states by replacing the uniaxial notch stresses σ , and strains, ε , as well as the stress concentration factor, K_t , by the corresponding equivalent quantities σ_q , ε_q , and K_{tq} .

Estimation of the notch equivalent strain ε_q and stress σ_q requires knowledge of:

- (i) the material's uniaxial stress–strain curve

$$\sigma_q = g(\varepsilon_q) \quad (1)$$

- (ii) the theoretical elastic equivalent notch stress, $\sigma_{e,q}$, which is calculated using von Mises flow criterion, that is

$$\sigma_{e,q} = \sqrt{(\sigma_{e1}^2 - \sigma_{e1} \cdot \sigma_{e2} + \sigma_{e2}^2)} \quad (2)$$

- (iii) and the plastic limit load, S_p for elastic–perfectly-plastic material.

Values of $\sigma_{e,q}$ can be expressed in terms of the equivalent stress concentration factor, K_{tq} , and the corresponding nominal stress, S , or be related to the applied load, L

$$\sigma_{e,q} = K_{tq} \cdot S = c_q \cdot L \quad (3)$$

with free definition of S and L and c a constant. Introducing the abbreviation

$$a_e = \sigma_{e2}/\sigma_{e1} \quad (4)$$

the equivalent stress concentration factor is determined by

$$K_{tq} = \sigma_{e1} \sqrt{(1 - a_e + a_e^2)/S} \quad (5)$$

Neuber's rule, in its generalized form (17), reads

$$E \cdot \sigma_q \cdot \varepsilon_q = (K_{tq} \cdot S)^2 \cdot \frac{E \cdot e^*}{S^*}; \quad K_{tq} \cdot S > \sigma_Y \quad (6)$$

and

$$S^* = \frac{S}{S_p / \sigma_Y}; \quad e^* = g^{-1}(S^*) \quad (7)$$

where the term $E \cdot e^*/S^*$ takes the nonlinear net section behaviour into account (24).

For a given stress level, S , the right hand side of equation (6) is determined by equation (7), and the elastic-plastic notch stresses, σ_q , and strains, ε_q , can be calculated by consideration of the material's stress-strain curve, equation (1).

Application of Neuber's rule is independent from the chosen definition of nominal stress and corresponding stress concentration factor, K_{tq} since only the product $K_{tq} \cdot S$ enters equation (6). The value of K_{tq} does not have to be derived explicitly. For example, it is sufficient to know the equivalent notch stress, $\sigma_{e,q}$, from a single elastic finite element analysis.

In addition to the elastic equivalent notch stress, $\sigma_{e,q}$, Neuber's rule requires the knowledge of the plastic limit stress level, S_p (for elastic-perfectly-plastic material) if notch stresses and strains are to be calculated in the fully plastic range. It is noted, however, that for elastic nominal strains, $E \cdot e^*/S^*$ in equation (6) equals 1, i.e., Neuber's rule is independent of S_p .

Note that there are always plastic nominal strains for the material law

$$\varepsilon = \sigma/E + (\sigma/K')^{1/n'} \quad (8)$$

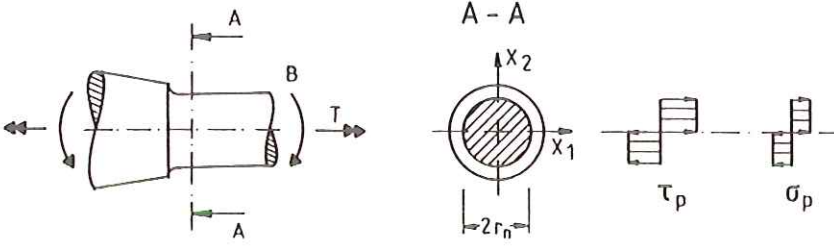
However, in most cases these strains are so small that a rough estimate of S_p is sufficient.

For a routine application of Neuber's rule it is proposed to estimate the plastic limit stress, S_p , by elementary equilibrium considerations, neglecting internal triaxiality, e.g., assume a constant normal and shear stress distribution at the cross section for combined bending and torsional loading, Fig. 3.

Flow rule

The normality rule, postulating that the vector of the plastic strain increment is perpendicular to the yield surface (25), represents the starting point for the formulation of the flow rule. Considering von Mises flow criterion and isotropic hardening leads to the Prandtl-Reuss equations, representing a differential stress-strain relationship, which, for example, gives the maximum principal strain

$$d\varepsilon_1 = \{d\sigma_1 - \nu \cdot (d\sigma_2 + d\sigma_3)\}/E + \{\sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3)\} \cdot d\varepsilon_q^p / \sigma_q \quad (9)$$



$$B_p = 4 \cdot \sigma_p \int_0^{r_n} \sqrt{r_n^2 - x_2^2} \cdot x_2 \, dx_2$$

$$\sqrt{\sigma_p^2 + 3\tau_p^2} = \sigma_Y$$

$$T_p = 2\pi \cdot \tau_p \cdot \int_0^{r_n} r^2 \cdot dr$$

Fig 3 - Estimation of plastic limit loads from basic equilibrium considerations

Calculation of the total principal strains requires an incremental solution. A close integration of these equations is only possible for the special case of proportional deviatoric stresses where the Prandtl-Reuss equations reduce to Hencky's rule representing a generalization of Hooke's law. As fixed principal stress directions are assumed (boundary condition I) formulation can be derived using principal quantities. Furthermore, $\sigma_3 = 0$ is taken into account in the following equations.

$$\varepsilon_1 = \frac{\varepsilon_q}{\sigma_q} (\sigma_1 - \nu' \cdot \sigma_2) \quad (10)$$

$$\varepsilon_2 = \frac{\varepsilon_q}{\sigma_q} (\sigma_2 - \nu' \cdot \sigma_1) \quad (11)$$

$$\varepsilon_3 = -\nu' \frac{\varepsilon_q}{\sigma_q} (\sigma_1 + \sigma_2) \quad (12)$$

$$\nu' = \frac{1}{2} - \left(\frac{1}{2} - \nu\right) \frac{\sigma_q}{E\varepsilon_q} \quad (13)$$

Comparative studies have shown (13)(21) that, for the boundary conditions: fixed principal stress direction and constant ratio of the surface strains, $\varepsilon_2/\varepsilon_1$, results from Hencky's rule are nearly identical with the exact solution (Prandtl-Reuss), although deviatoric stresses behave non-proportionally.

The Notch element boundary conditions

In general, five boundary conditions have to be known to allow calculation of the stress and strain components at the notch. Considering a notch at a traction-free surface, three conditions are given: namely, $\sigma_3 = 0$ and two principal stress directions fixed. The remaining equations are obtained by statements concerning the third principal stress direction (denoted by α), and the geometrical constraint at the notch.

Boundary condition I: fixed principal stress directions

It is assumed that possible changes of principal stress directions can be neglected. For a lot of structures and loading situations the principal axes are fixed because of symmetry conditions. Experimental and numerical investigations under combined loadings (see geometry of structures investigated in Fig. 4) show that, even in the cases considered, there is no rotation of the principal stress directions for local yielding, and only small changes for general yielding (21)(26).

It is expected that the assumption of fixed principal axes describes the real behaviour at the notch root with sufficient accuracy, as long as the external loads increase proportionally. However, the modular structure of the approximate procedure would allow for changing directions, too. Then, the flow rule is formulated in stress and strain components (21).

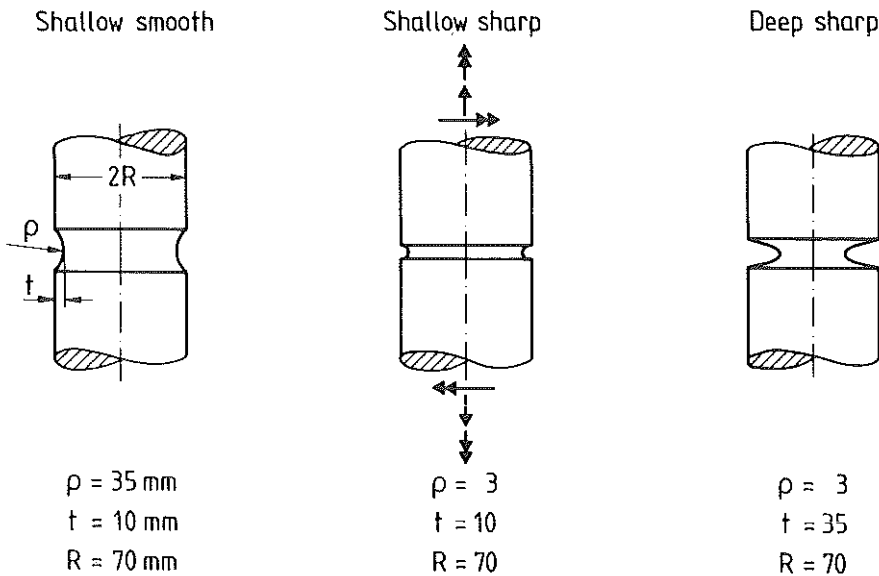


Fig 4 Structures and loading cases investigated in reference (26)

Boundary condition II: constraint at notches

As the elastic surrounding hinders deformation of the plasticized notch root, it seems reasonable to describe this geometrical constraint by a simple formulation. For example, notch circumferential strains are controlled by the circumferential strains at the gross area.

Based on experimental and numerical investigations it is assumed that the ratio of the surface strains, $\varepsilon_2/\varepsilon_1$, remains constant during loading. The special case of pure torsional loading ($\varepsilon_2/\varepsilon_1 = -1$) is satisfied exactly.

The constraint assumption should not be applied to structures with extremely shallow and mild notches, which resemble smooth specimens. For these cases an assumption concerning a stress ratio is advised in (21).

Calculation of the stress and strain components

Together with von Mises flow criterion

$$\sigma_q = \sqrt{(\sigma_1^2 - \sigma_1 \cdot \sigma_2 + \sigma_2^2)} \quad (14)$$

and Hencky's flow rule, equations (10)–(12), there are four equations for the five unknown quantities ε_1 , ε_2 , ε_3 , σ_1 , and σ_2 . Making use of the boundary condition II, $\varepsilon_2/\varepsilon_1 = \text{constant}$, the set of equations can be solved

$$a = \frac{\sigma_2}{\sigma_1} = \frac{\varepsilon_2/\varepsilon_1 + \nu'}{1 + \nu'\varepsilon_2/\varepsilon_1} \quad (15)$$

$$\frac{\varepsilon_3}{\varepsilon_1} = -\nu' \frac{1 + a}{1 - \nu'a} \quad (16)$$

$$\sigma_1 = \frac{1}{\sqrt{(1 - a + a^2)}} \sigma_q \quad (17)$$

$$\varepsilon_1 = \frac{1 - \nu'a}{\sqrt{(1 - a + a^2)}} \varepsilon_q \quad (18)$$

Note that boundary condition I, fixed principal stress directions, has already been employed in the formulation of the flow rule, equations (10)–(12).

Scheme for routine application

The steps required to estimate multiaxial notch stresses and strains are listed below.

- (1) Definition of the material stress–strain curve $\sigma = g(\varepsilon)$ and selection of von Mises yield criterion.
- (2) Description of the elastic solution by (i) elastic material constants, (ii) principal stresses σ_{e1} , σ_{e2} ($= a_e \cdot \sigma_{e1}$), and (iii) principal stress direction.
- (3) Estimation of the plastic limit stress, S_p , for elastic–perfectly-plastic material based on elementary equilibrium considerations.

- (4) Calculation of the theoretical elastic equivalent notch stress, $\sigma_{e,q}$, based on von Mises yield criterion, equation (2). If nominal stress, S , is preferred as a variable, then calculation of the concentration factor, K_{tq} , equation (5).
- (5) Choice of Neuber's rule as approximation formulae, equations (6) and (7). In connection with the material stress-strain curve, equation (1), calculation of the equivalent notch stresses and strains for a given nominal stress S .
- (6) Notch element boundary conditions:
 I: fixed principal stress direction
- $$\alpha = \alpha_c = \text{constant} \quad (19)$$
- II: constant strain ratio
- $$\frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_{c2}}{\varepsilon_{c1}} = \frac{a_c - \nu}{1 - \nu \cdot a_c} = \text{constant} \quad (20)$$
- (7) Calculation of the stress and strain components according to equations (13) and (15)–(18).

Means for more accurate estimates

Extensive studies concerning the accuracy of the method presented show (21) that the solution mainly depends on the chosen approximation formula; here, Neuber's rule.

As Neuber's rule describes a non-tangential transition from the elastic into the elastic-plastic regime, it tends to overestimate notch strains (18)(27). This is especially true for low hardening materials. Therefore, employment of a formula proposed by Seeger (17)(27) (with tangential transition) is advised for low hardening materials, and if highly accurate notch stresses and strains are wanted for small plastic strains

$$\varepsilon_q = \frac{\sigma_q}{E} \cdot \left\{ \left(\frac{\sigma_{e,q}}{\sigma_q} \right)^2 \cdot \frac{2}{u^2} \cdot \ln \frac{1}{\cos u} + 1 - \frac{\sigma_{e,q}}{\sigma_q} \right\} \cdot \frac{E \cdot e^*}{S^*} \quad (21)$$

with

$$u = \frac{\pi}{2} \cdot \frac{\sigma_{e,q}/\sigma_q - 1}{K_p - 1}; \quad K_p = K_{tq} \cdot S_p/\sigma_Y; \quad \sigma_{e,q} = K_{tq} \cdot S$$

If the plastic deformation of the cross section is no longer small, the plastic limit level, S_p , should be calculated taking internal triaxiality into account. A relatively simple procedure is given in (21) enabling sufficiently accurate estimates of S_p by use of the elastic stress distribution.

Application

Application of the proposed procedure is illustrated by the example of the notched shaft investigated in a SAE research programme (28)(11). Geometry, type of loading, and material data are taken from (11).

Notch stress and strain estimates are compared with finite element results for bending and torsional loading, i.e., loading with one component as well as combined loading. The elastic-plastic finite element analyses were carried out using ADINA (29) with a non-linear material behaviour including von Mises flow criterion. The stress-strain curve, equation (22), was approximated by seven straight lines. The finite element mesh used for the combined loading case is depicted in Fig. 5. For a more detailed description of the finite element analyses and comparison with analyses published in literature, see reference (30).

The approximate solutions for the three loading cases have the following in common.

- (i) The material stress-strain law (SAE-1045 steel)

$$\varepsilon = \sigma/E + (\sigma/K')^{1/n'} \quad (22)$$

$$E = 202375 \text{ MPa}, \quad K' = 1283 \text{ MPa}, \quad n' = 0.211, \quad \nu = 0.30$$

- (ii) The theoretical elastic solution is taken from a linear finite element analysis.
- (iii) A nominal stress, S , is used as load variable. The equivalent stress at the notch obtained from common structural analysis, involving a linear distribution of net stresses, σ_n , τ_n , is chosen as the nominal stress

$$S = \sqrt{(\sigma_n^2 + 3\tau_n^2)} \quad (23)$$

where σ_n and τ_n represent net stresses for pure bending and torsional loading

$$\sigma_n = \frac{4B}{\pi r^3}, \quad \tau_n = \frac{2T}{\pi r^3} \quad (24)$$

Here B is the bending moment at the notch $B = V \cdot l_1 = V \cdot l_2 + B_0$ (see Fig. 5) and T is the torque. The plastic limit level is estimated by elementary equilibrium considerations according to Fig. 3 (21), neglecting the influence of shear stresses caused by the vertical force V

$$S_p/\sigma_Y = \frac{16}{3} \sqrt{\left\{ \frac{1 + 3(\tau_n/\sigma_n)^2}{\pi^2 + 48(\tau_n/\sigma_n)^2} \right\}} \quad (25)$$

The relationship between external loads and nominal stress reads

$$B = \frac{\pi \cdot r^3}{\sqrt{\{16 + 12(T/B)^2\}}} \cdot S, \quad T = \frac{\pi \cdot r^3}{\sqrt{\{16(B/T)^2 + 12\}}} \cdot S \quad (26)$$

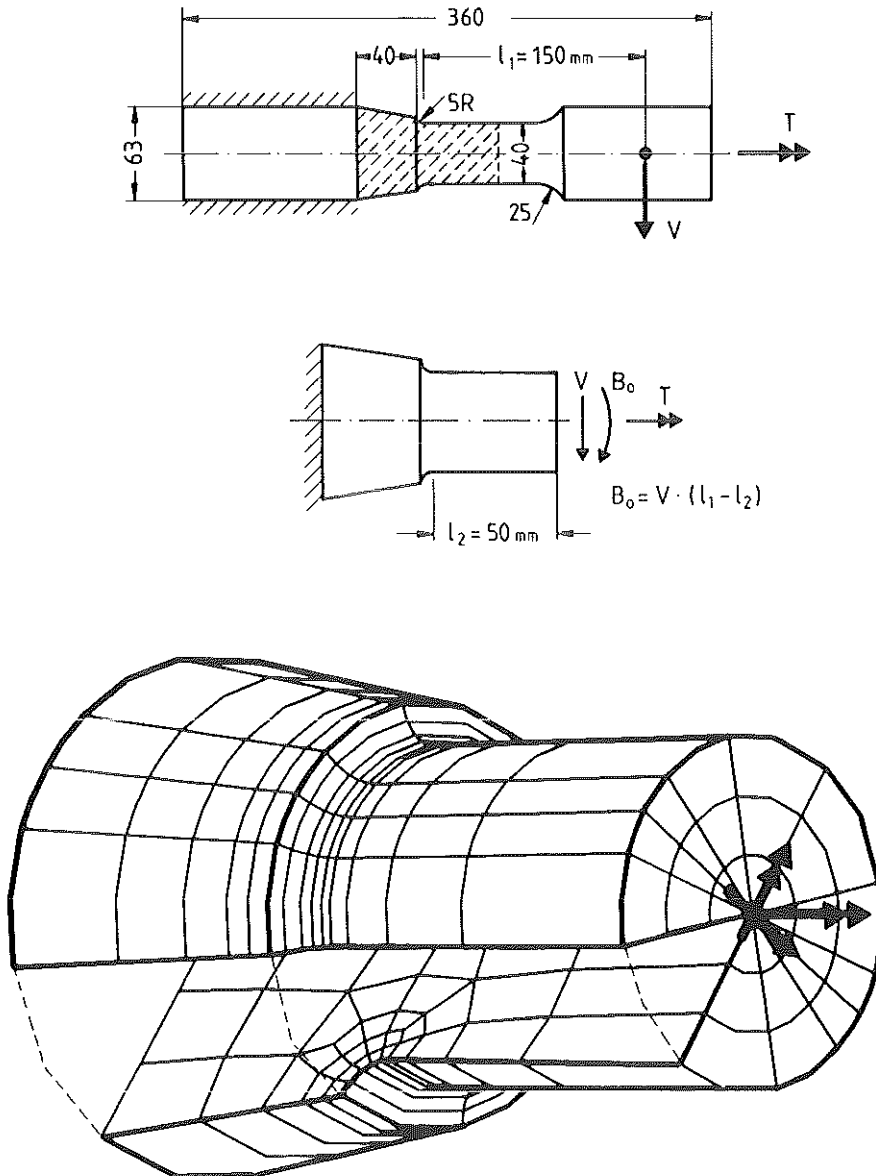


Fig 5 Component geometry, loading conditions, and the finite element mesh. Dimensions in mm

Bending

The solution follows the application scheme given previously.

- (1) Material stress-strain curve according to equation (22).
- (2) Poisson's ratio, $\nu = 0.30$. Elastic analysis for bending gives $\sigma_{e1} = \sigma_{e2} = 1.64\sigma_n$ and $\sigma_{e3} = \sigma_{e\theta} = 0.208\sigma_{e2}$, where z denotes the longitudinal and θ the circumferential direction.
- (3) $\tau_n/\sigma_n = 0$ in equation (25) leads to $S_p/\sigma_Y = 1.7$.
- (4) For pure bending the chosen nominal stress, S , is equal to the net stress σ_n . The value of K_{tq} is obtained according to equation (5)

$$K_{tq} = 1.64 \cdot \sqrt{(1.0 - 0.208 + 0.208^2)} = 1.50 \quad (5a)$$

- (5) With these values of K_{tq} and S_p/σ_Y then Neuber's rule reads

$$E \cdot \sigma_q \cdot \varepsilon_q = 2.25 \cdot S^2 \cdot \frac{E \cdot e^*}{S^*} \quad (6a)$$

$$S^* = 0.588 \cdot S \quad (7a)$$

- (6) For pure bending the principal stress directions remain fixed. Hence, only boundary condition II has to be introduced as an assumption. From equation (20)

$$\frac{\varepsilon_2}{\varepsilon_1} = \frac{0.208 - 0.30}{1 - 0.30 \cdot 0.208} = -0.10 = \text{constant} \quad (20a)$$

- (7) Stress and strain components at the notch are calculated with equations (13), and (15)–(18).

The load versus equivalent strain curves calculated by finite element analysis and Neuber's rules are depicted in Fig. 6, together with the material law. Logarithmic axes are used because linear scales over-emphasize high load levels and suppress the range of small plastic strains, which in most cases is the range of application.

As mentioned earlier, Neuber's rule tends to overestimate notch strains. The deviations increase with higher load levels and maximum deviation amounts to 30 per cent for strains or 10 per cent for loads (see Fig. 6).

The load-strain curves for the principal quantities are depicted in Fig. 7. The left diagram shows a plot of the nominal stress, S , versus the maximum strain, ε_1 . Additionally, in this diagram, the corresponding stress σ_1 is plotted versus ε_1 . The complete information about the stress and strain state is given by the right-hand diagram showing the relationship between nominal stress, with strain and stress ratios, $\varepsilon_2/\varepsilon_1$, $\varepsilon_3/\varepsilon_1$, and σ_2/σ_1 , respectively.

Accuracy studies reveal that the differences between finite element analysis and the approximate solution are mainly caused by the inaccuracy of Neuber's rule. Note that the $\varepsilon_2/\varepsilon_1$ ratio remains nearly constant for the whole loading range.

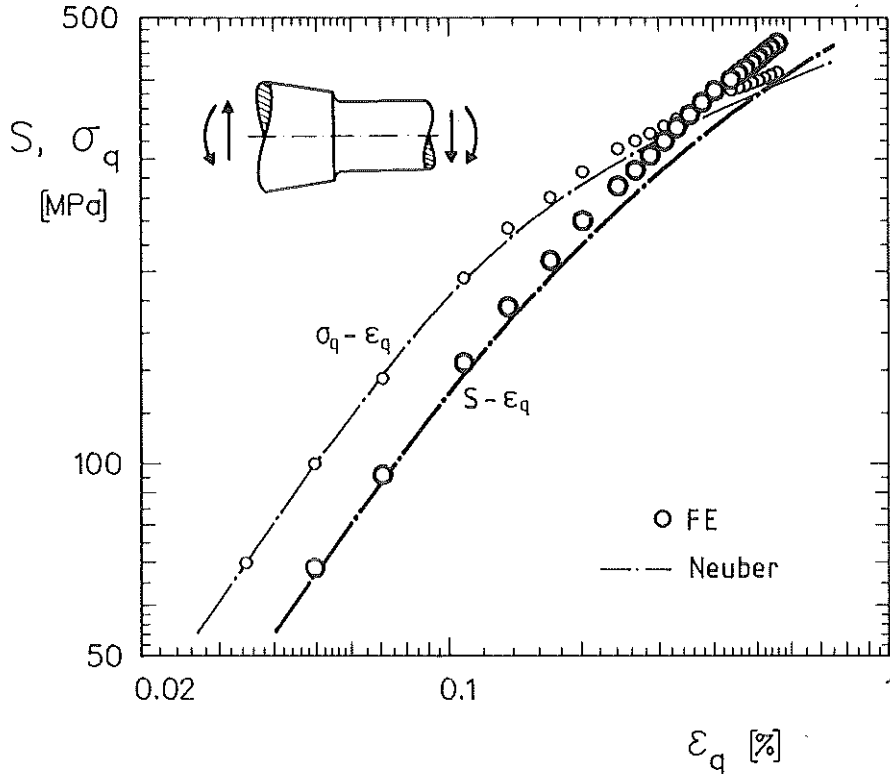


Fig 6 Notched shaft under bending. Equivalent notch stresses and strains calculated by finite element analysis and Neuber's rule

Torsion

Pure torsional loading represents a special multiaxial problem. The complete stress and strain state is given by a description of shear stress, τ , and shear strain, γ . Therefore, it is reasonable, although slightly different from the proposed solution scheme, to use shear stress and strain instead of the principal stresses and strains. Von Mises flow criterion simplifies to

$$\sigma_q = \sqrt{3} \cdot \tau \quad (14b)$$

The steps of the application scheme are as follows.

- (1) Material stress-strain curve according to equation (22).
- (2) Elastic analysis gives $\tau_c = 1.29 \cdot \tau_n$.
- (3) $\sigma_n = 0$ in equation (25) leads to $S_p/\sigma_Y = 4/3$.
- (4) For pure torsional loading the chosen nominal stress, S , is equal to $S = \sqrt{3} \cdot \tau_n$. With equation (5) it follows

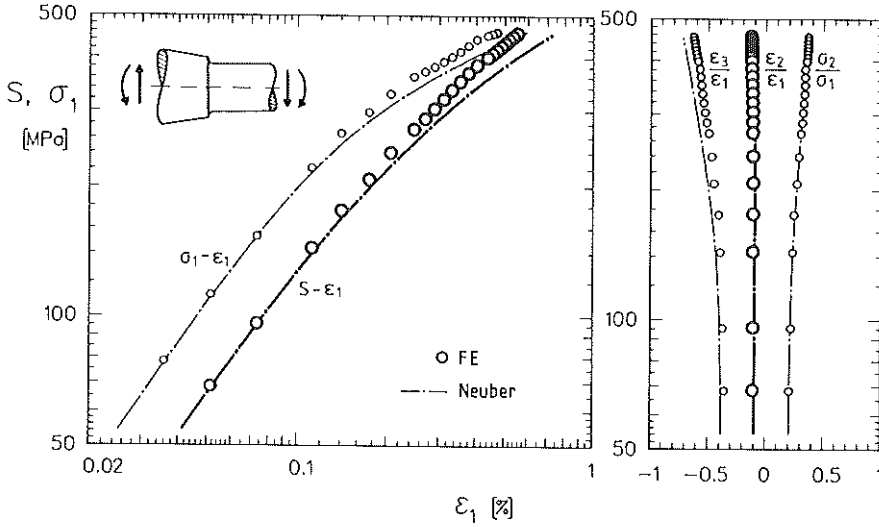


Fig 7 Notched shaft under bending. Principal notch stresses and strains calculated by finite element analysis and approximate solution using Neuber's rule

$$K_{tq} = \sqrt{3} \cdot 1.29 / \sqrt{3} = 1.29 \quad (5b)$$

where K_{tq} is the same as the classical definition of the stress concentration factor for torsional loading.

- (5) With these values of K_{tq} and S_p/σ_Y Neuber's rule reads

$$E \cdot \sigma_q \cdot \varepsilon_q = 1.66 \cdot S^2 \cdot \frac{E \cdot e^*}{S^*} \quad (6b)$$

$$S^* = 0.75 \cdot S \quad (7b)$$

- (6) There are no assumptions necessary. The boundary conditions $\alpha = \text{constant}$ and $\varepsilon_2/\varepsilon_1 = -1 = \text{constant}$ are satisfied exactly.
 (7) With $\sigma_1 = -\sigma_2 = \tau$ and $\varepsilon_1 = -\varepsilon_2 = \gamma/2$, Hencky's equations, equations (17) and (18), read

$$\tau = \sigma_q / \sqrt{3} \quad (17b)$$

$$\gamma = \frac{2}{\sqrt{3}} \cdot (1 + \nu') \cdot \varepsilon_q \quad (18b)$$

Due to the special case of torsional loading, complete information about the notch stress and strain state is given by plotting nominal stress, S , versus shear strain, γ , and shear stress, τ (Fig. 8).

For this type of loading Neuber's rule leads to better strain estimates. Maximum deviations are about 15 per cent for strains and 5 per cent for stresses.

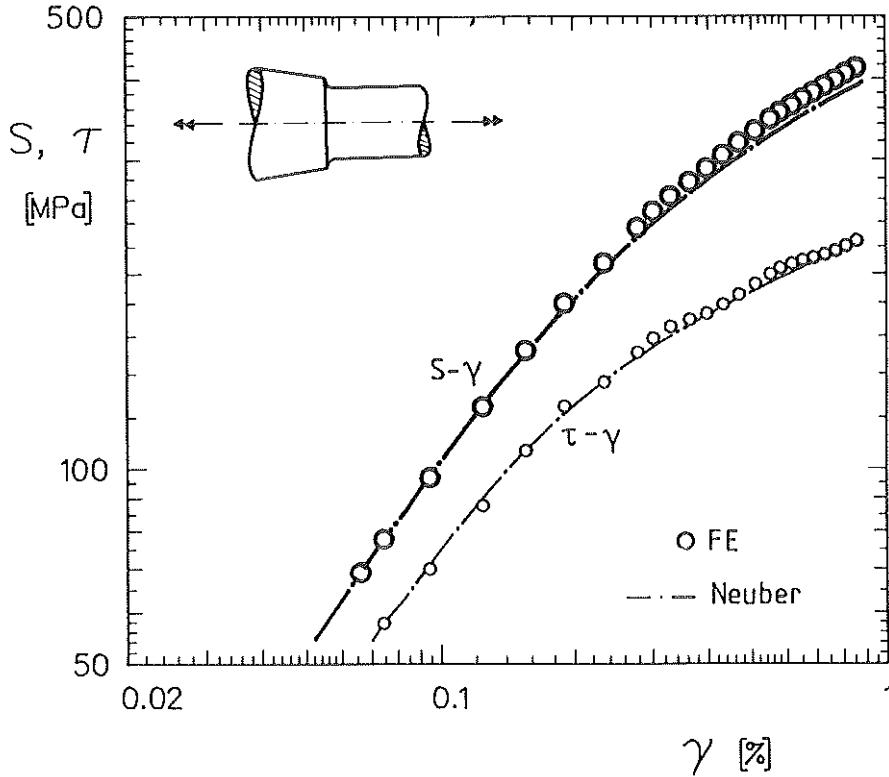


Fig 8 Notched shaft under torsion. Notch shear stresses and strains calculated by finite element analysis and approximate solution using Neuber's rule

Combined bending and torsion

The chosen ratio of torque to bending moment is $T/B = 1.43$, i.e., $\tau_n/\sigma_n = 0.7$, equation (24).

- (1) Material stress-strain curve according to equation (22).
- (2) From pure bending and torsional loading

$$K_{ts} = \frac{\sigma_z}{\sigma_n} = 1.64, \quad \bar{a}_e = \frac{\sigma_\theta}{\sigma_z} = 0.208, \quad K_{tr} = \frac{\tau_{z\theta}}{\tau_n} = 1.29$$

With the chosen nominal stress definition, equation (23), the maximum stress, σ_{e1} , the stress ratio, $a_e = \sigma_{e2}/\sigma_{e1}$, and the principal stress direction, α_c (see Fig. 9) are determined as

$$\sigma_{e1} = \left[1 + \bar{a}_e + \sqrt{\left\{ (1 - \bar{a}_e)^2 + 4 \left(\frac{K_{tr} \cdot \tau_n}{K_{ts} \cdot \sigma_n} \right)^2 \right\}} \right] \cdot \frac{K_{ts}}{2\sqrt{1 + 3(\tau_n/\sigma_n)^2}} \cdot S$$

$$= 1.34 \cdot S \quad (27)$$

$$a_e = \frac{1 + \bar{a}_e - \sqrt{\left\{ (1 - \bar{a}_e)^2 + 4 \left(\frac{K_{tr} \cdot \tau_n}{K_{ts} \cdot \sigma_n} \right)^2 \right\}}}{1 + \bar{a}_e + \sqrt{\left\{ (1 - \bar{a}_e)^2 + 4 \left(\frac{K_{tr} \cdot \tau_n}{K_{ts} \cdot \sigma_n} \right)^2 \right\}}} = -0.06 \quad (28)$$

$$\tan 2\alpha_e = \frac{2 \cdot K_{tr} \cdot \tau_n}{(1 - \bar{a}_e) \cdot K_{ts} \cdot \sigma_n}; \quad \alpha_e = 27.2 \text{ degrees} \quad (29)$$

(3) $\tau_n/\sigma_n = 0.7$ in equation (25) leads to $S_p/\sigma_Y = 1.45$.

(4) Calculation of K_{tq} according to equation (5)

$$K_{tq} = 1.34 \cdot \sqrt{1 + 0.06 + 0.06^2} = 1.38 \quad (5c)$$

(5) Neuber's rule reads

$$E \cdot \sigma_q \cdot \varepsilon_q = 1.90 \cdot S^2 \cdot \frac{E \cdot e^*}{S^*} \quad (6c)$$

$$S^* = 0.69 \cdot S \quad (7c)$$

(6) Two assumptions are necessary

$$\alpha = \alpha_e = 27.2 \text{ degrees} = \text{constant}; \quad \text{i.e., } \alpha = 0.475 \text{ radians} \quad (19c)$$

$$\frac{\varepsilon_2}{\varepsilon_1} = \frac{-0.06 - 0.30}{1 + 0.30 \cdot 0.06} = -0.35 = \text{constant} \quad (20c)$$

(7) Stress and strain components at the notch are calculated with equations (13), and (15)–(18).

Figure 10 reveals that Neuber's rule enables correlation between load and equivalent notch strain for combined loading cases as well. Maximum devi-

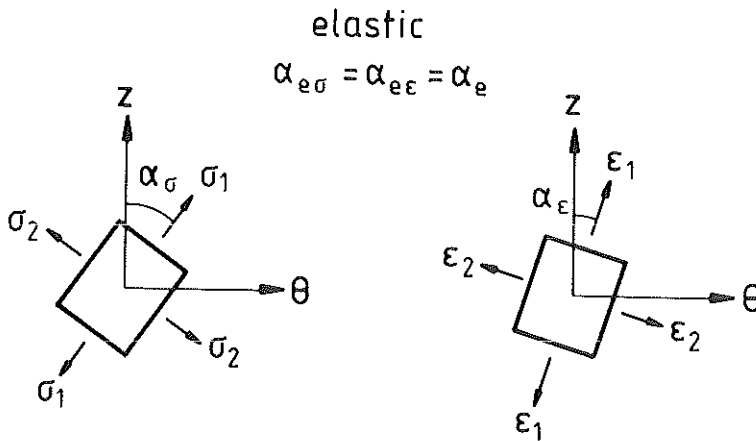


Fig 9 Definition of principal stresses and principal stress direction

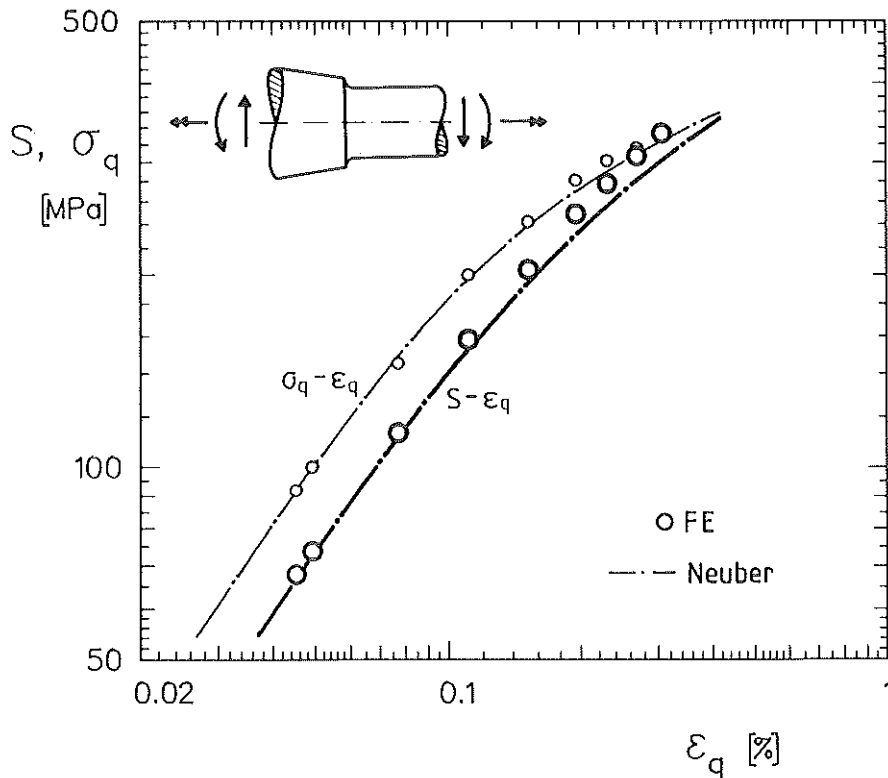


Fig 10 Notch shaft under combined bending and torsion ($T/B = 1.43$). Equivalent notch stresses and strains calculated by finite element analysis and Neuber's rule

ations are about 20 per cent for strains and 10 per cent for loads. Principal notch stresses and strains are depicted in Fig. 11 in the same manner as for bending. In addition to the stress and strain ratios the angle (radian) of the principal stress direction is plotted in the right-hand diagram. Finite element results indicate that the assumptions, fixed principal stress directions, and constant ratio of the surface strains, ϵ_2/ϵ_1 , describe the actual behaviour at the notch with sufficient accuracy.

As mentioned earlier, the accuracy of the method proposed depends mainly on the chosen approximation formula. To improve the estimates of notch equivalent strains, Neuber's rule has been replaced by Seeger's formula, equation (21), in Fig. 12, showing the stress versus notch equivalent strain curves for bending, torsional, and combined bending and torsional loading. The theoretical elastic equivalent notch stress, $\sigma_{e,q}$, has been used as a load variable allowing a better comparison than nominal stress of the different loading types. Maximum deviations are always smaller than 15 per cent for

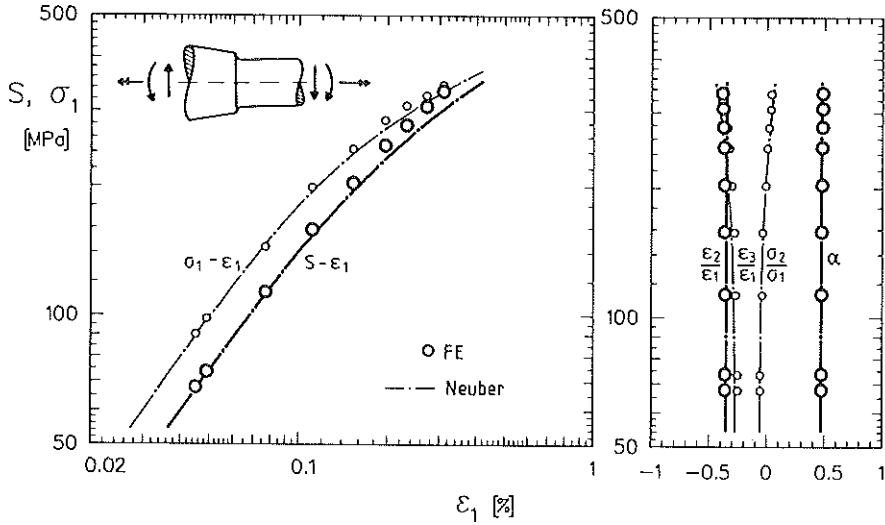


Fig 11 Notch shaft under combined bending and torsion ($T/B = 1.43$). Principal notch stresses and strains calculated by finite element analysis and approximate solution using Neuber's rule

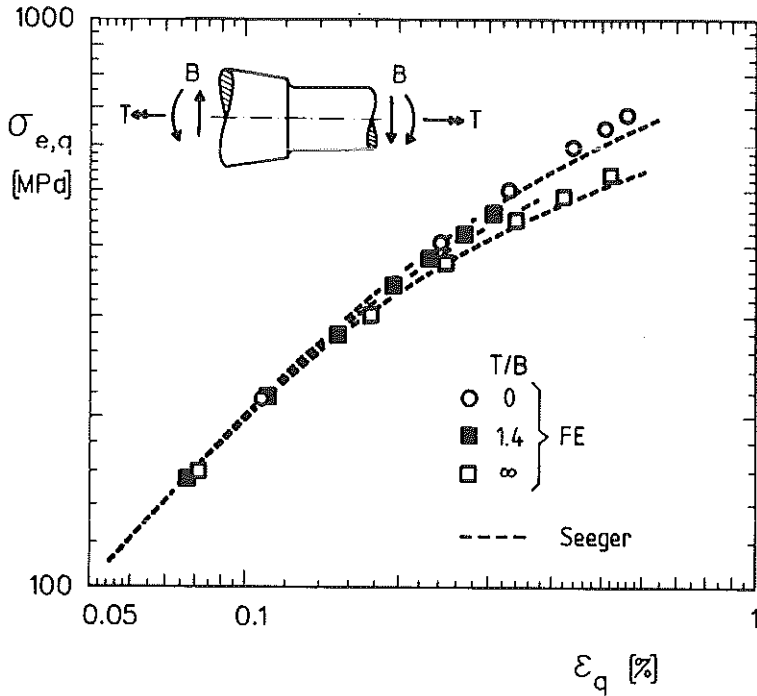


Fig 12 Stress versus equivalent notch strain for the notched shaft under different loading types. Comparison of finite element analysis with Seeger's formula

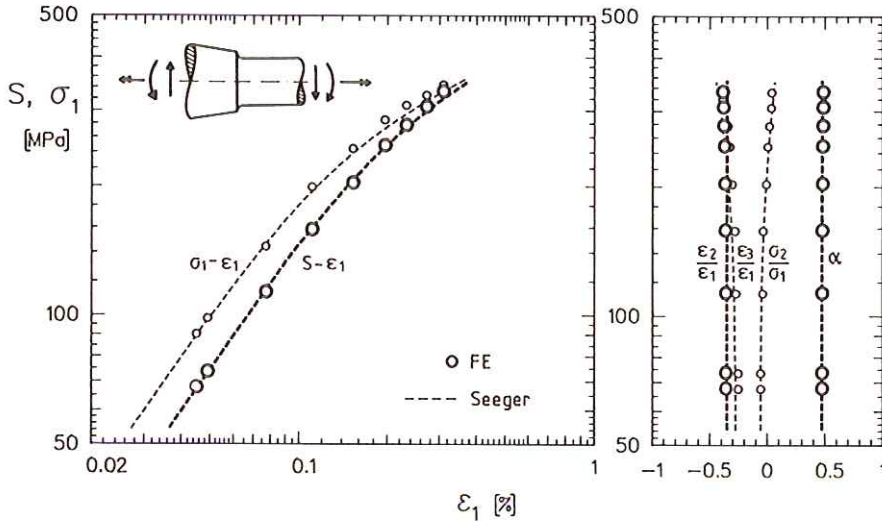


Fig 13 Notch shaft under combined bending and torsion ($T/B = 1.43$). Principal notch stresses and strains calculated by finite element analysis and approximate solution using Seeger's formula

strains and 10 per cent for loads. Figure 13 reveals that the principal notch stresses and strains can now be more accurately calculated.

Conclusions

- (1) Components under multiaxial, proportional loading show 'Masing-' and 'Memory-behaviour', as in uniaxial situations.
- (2) The load versus equivalent notch strain relationships can be established by adopting the known approximation formulae (developed for uniaxial notch stresses) and replacing the uniaxial quantities σ , ε , and K_t by equivalent quantities σ_q , ε_q , and K_{tq} .
- (3) Numerical and experimental investigations reveal that the notch element boundary conditions can be described by assuming fixed principal stress directions and a constant ratio, $\varepsilon_2/\varepsilon_1$ of the surface strains.
- (4) The proposed procedure delivers the complete state of elastic-plastic notch stresses and strains. Incorporating a multiaxial damage parameter it can be employed for fatigue life predictions (for an example, see (30)) with no need of expensive non-linear finite element analyses.
- (5) The method is restricted to proportional loading. It seems that, for arbitrarily non-proportional loading, an incremental formulation of the approximation formulae and flow rule is necessary.

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