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Cyclic Strain Energy Density as a Criterion for Multiaxial Fatigue Failure

REFERENCE Ellyin, F., **Cyclic strain energy density as a criterion for multiaxial failure fatigue**, *Biaxial and Multiaxial Fatigue*, EGF 3 (Edited by M. W. Brown and K. J. Miller), 1989, Mechanical Engineering Publications, London, pp. 571–583.

ABSTRACT A form of the cyclic strain energy density is used as a criterion for multiaxial fatigue failure. The salient feature of this approach is based on the premise that the damage caused in the material due to cyclic loading is related to the mechanical energy input. The proposed criterion is hydrostatic pressure sensitive, and is consistent with the concept of crack initiation and subsequent propagation. The predictions of the proposed criterion are compared with the experimental results of biaxial fatigue tests, and are shown to be in good agreement.

Introduction

The fatigue failure of components has been studied intensively, and because of active industrial interest and the complexity of the phenomenon, the general topic has been subdivided into a number of inter-related fields. These divisions include high and low cyclic fatigue, fatigue of notched members, and fatigue crack initiation and propagation, among others. Furthermore, within each of these divisions, investigations have been carried out on specific loading conditions resulting in uniaxial or multiaxial states of stress. These investigations have led to a general understanding of the fatigue failure phenomenon to an extent that one may now attempt to propose some general theories concerning specific aspects. Contributions made by various investigators have been mentioned in a recent survey paper by Ellyin and Valaire (1). They could be subdivided into three categories: stress-based, strain-based; or energy-based criteria. No attempt will be made here to summarise these contributions, and the interested reader is encouraged to consult reference (1).

In general, the fatigue process may be divided into two phases, viz. initiation of a 'starting crack', and its subsequent propagation until failure results. It would be extremely useful if one could find a unifying damage parameter which can describe these two processes. It is the objective of this paper to show that the energy approach, advocated in 1974 by the present author for multiaxial stress-states (2) and in subsequent works (3)–(5) can lead to a unified approach. In this paper, a 'total' strain energy density concept is introduced for the multiaxial states of stress. This energy may be viewed as a composite measure of the amount of fatigue damage per cycle. Experimental data are presented to

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show the correlation between the total strain energy and cycles to failure. A fatigue failure criterion is then proposed for the multiaxial stress conditions.

Notation

C	Elastic strain energy at fatigue limit
C_u	Elastic strain energy at fatigue limit for uniaxial condition
E	Young's modulus
E_s	Secant modulus
E_t	Tangent modulus
J_2	Second invariant of deviatoric stress tensor
k	Material constant in equation (4)
K'	Strength coefficient in the cyclic stress–strain curve, equation (4)
K^*	Strength coefficient in the master curve, equation (1)
N_f	Number of cycles to failure
n^*	Cyclic strain hardening exponent of 'master' curve, equation (1)
n'	Cyclic strain hardening exponent of cyclic curve, equation (4)
s_{ij}	Deviatoric stress tensor
ΔW^{e+}	Positive elastic strain energy density per cycle
ΔW^p	Plastic strain energy density per cycle
ΔW^t	'Total' strain energy density per cycle
ΔW_{lim}	Positive elastic strain energy at the material fatigue limit
α.	Life exponent in equation (7)
ε_{ij}	Strain tensor
ε^e_{ij}	Elastic part of strain tensor
ε^p_{ij}	Plastic part of strain tensor
$\bar{\varepsilon}$	Effective strain (equation 23)
$\Delta\bar{\varepsilon}$	Range of effective strain
$\varkappa(\varrho)$	Energy coefficient in equation (35)
$\varkappa_u$	Energy coefficient for uniaxial stress condition, equation (7)
ϱ	Strain ratio $= 2\varepsilon_1/\gamma_{max}$
ϱ^*	Biaxial strain ratio, circumferential to axial strain ratio
σ_{ij}	Stress tensor
σ_{kk}	Trace of stress tensor $= I_1$
$\bar{\sigma}$	Effective stress, equation (20)
$\Delta\bar{\sigma}$	Range of effective stress
$\delta\sigma_0$	Increase in the proportional stress limit (Fig. 2)
ν	Poisson's ratio

The total strain energy density

The energy approach has been used in continuum mechanics to derive field equations and variational principles. In the area of materials science, the significance of the energy approach is in its ability to unify microscopic and macroscopic test data and to formulate multiaxial life prediction models. To demonstrate the concept, we will begin with the case of a uniaxial stress state.

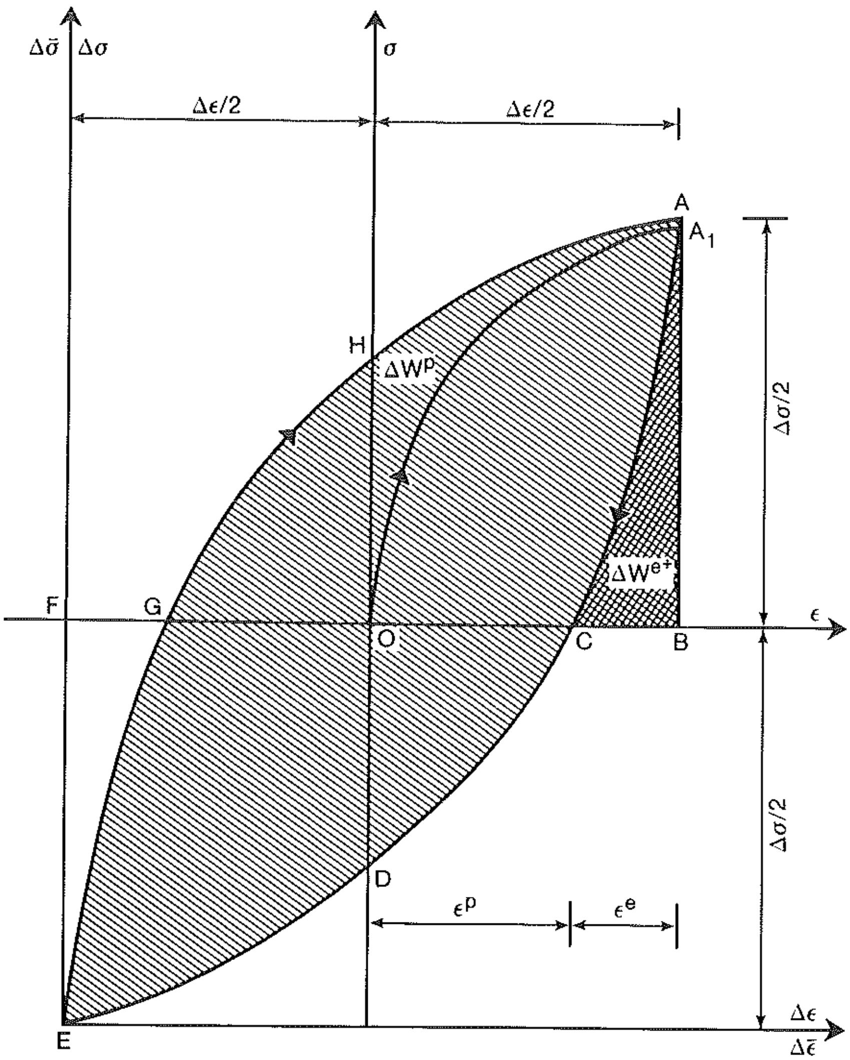

Fig 1 First loading and stable cyclic hysteresis loop; plastic and elastic strain energies. The effective-stress and effective-strain range coordinate system with origin at the tip of the minimum stress is also shown

A stable cyclic response of a specimen is shown in Fig. 1. Starting from a free stress-state, O, the first loading curve is OA_1, and the area under it represents the strain energy density in monotonic loading. The area OA_1C is the corresponding plastic strain energy density. In the case of cyclic loading, the strain energy density associated with the positive (tensile) stress is the area under the curve GHA. In a similar manner, the cyclic strain energy density associated with the compressive stress is the area under the curve CDE. The elastic parts

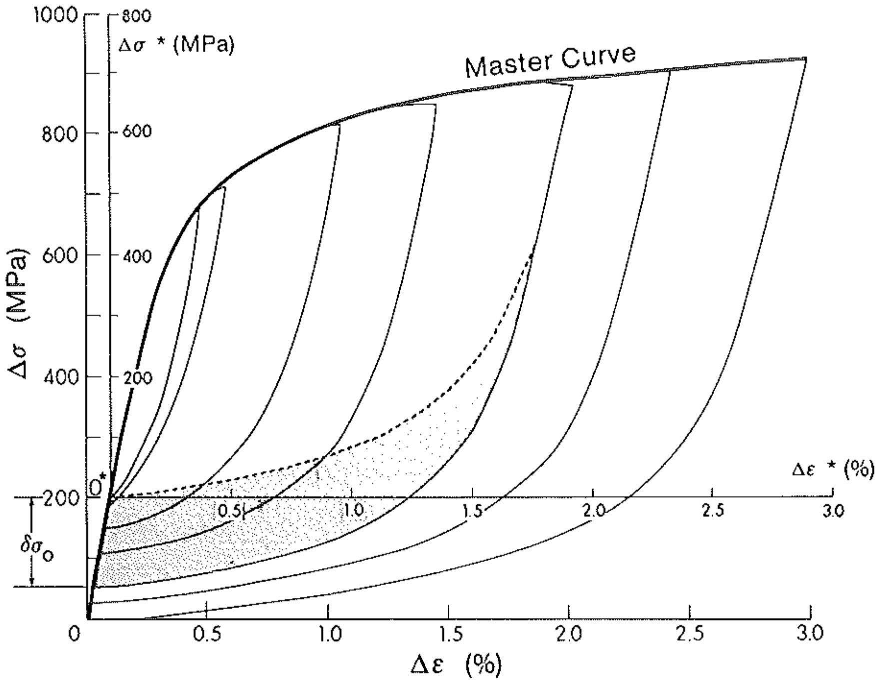

Fig 2 Stable hysteresis loops at half-life translated along the linear portion to match upper branches. The 'master' curve thus obtained is shown with the origin at O*

of the strain energy are recovered during a full cycle, and the plastic strain energy per cycle, ΔW^p, is the area of the hysteresis loop, EGHACDE. The damaging part of the above described input energies into material, are the plastic strain energy and the tensile part of the elastic strain energy.

The plastic strain energy per cycle, ΔW^p, can be calculated as described in reference (4). It will suffice to mention that the expression for calculating the cyclic plastic strain energy density applies for both non-Masing and ideal Masing material description. In general, a 'master curve' different from that of the cyclic curve is defined. This curve is obtained from matching upper branches of the hysteresis loops by translating each loop along its linear response portion, Fig. 2. The expression for the master curve with the origin at the lower tip of the smallest plastic strain hysteresis loop, O*, is

$$\Delta\varepsilon^* = \frac{\Delta\sigma^*}{E} + 2\left(\frac{\Delta\sigma^*}{2K^*}\right)^{1/n^*} \tag{1}$$

where a prefix Δ refers to the range of the quantity unless otherwise specified. The cyclic plastic strain energy is then calculated from

$$\Delta W^p = \frac{1 - n^*}{1 + n^*}\,\Delta\sigma\Delta\varepsilon^p + \frac{2n^*}{1 + n^*}\,\delta\sigma_0\Delta\varepsilon^p \tag{2}$$

where

$$\delta\sigma_0 = \Delta\sigma - \Delta\sigma^* = \Delta\sigma - 2K^*(\Delta\varepsilon^p/2)^{n^*} \quad (3)$$

is the increase in the proportional stress due to non-Masing behaviour of material, Fig. 2. It is to be noted that for the ideal Masing behaviour, the master curve and cyclic curve will coincide, i.e., $n^* = n'$; $\delta\sigma_0 = 0$; $K^* = K'$, and equation (2) reduces to

$$\Delta W^p = \frac{1 - n'}{1 + n'} \Delta\sigma\Delta\varepsilon^p = k \frac{1 - n'}{1 + n'} \Delta\sigma^{(1+n')/n'} \quad (4)$$

where

$$k = 2(2K')^{-1/n'}$$

The positive (tensile) elastic strain energy density for fully-reversed strain-controlled tests, area ABCA of Fig. 1, is given by

$$\Delta W^{e+} = \tfrac{1}{8}\Delta\sigma\Delta\varepsilon^e = \frac{1}{8E} \Delta\sigma^2 \quad (5)$$

Therefore, the total damaging energy input per unit material volume is

$$\Delta W^t = \Delta W^p + \Delta W^{e+} = \frac{1 - n^*}{1 + n^*} (\Delta\sigma - \delta\sigma_0)\Delta\varepsilon^p + \delta\sigma_0\Delta\varepsilon^p + \frac{1}{8E} \Delta\sigma^2 \quad (6)$$

which includes the plastic work and the positive part of the elastic strain energy.

A major part of this energy input is dissipated into heat, and the remaining mechanical energy causes dislocation movements along slip lines and volumetric change, and eventual crack propagation. Thus, the fatigue life is related to the above defined 'total' energy input, i.e., $\Delta W^t = g(N_f)$. In particular, a power law type relation of the form

$$\Delta W^t = \varkappa_u N_f^{\alpha} + C_u \quad (7)$$

is suggested by the experiments. The constant C_u is the elastic energy input which causes no perceivable damage. It is related to the strain energy density at the material fatigue limit, i.e.

$$C_u = \Delta W_{lim} = \frac{1}{8E} (\Delta\sigma^2)_{\text{fatigue limit}} \quad (8)$$

Figure 3 shows the total strain energy density, ΔW^t, plotted against fatigue life, N_f, for various types of tests. The material used in these tests was a low alloy carbon steel employed in the construction of modern pressure vessels. It is designated as ASTM A-516 Gr. 70. The test procedure; chemical composition and mechanical properties are given in references (4)–(6). Also, the conventional S–N curves can be found in the above references. Herein we are concerned with a unifying damage parameter, and those material properties

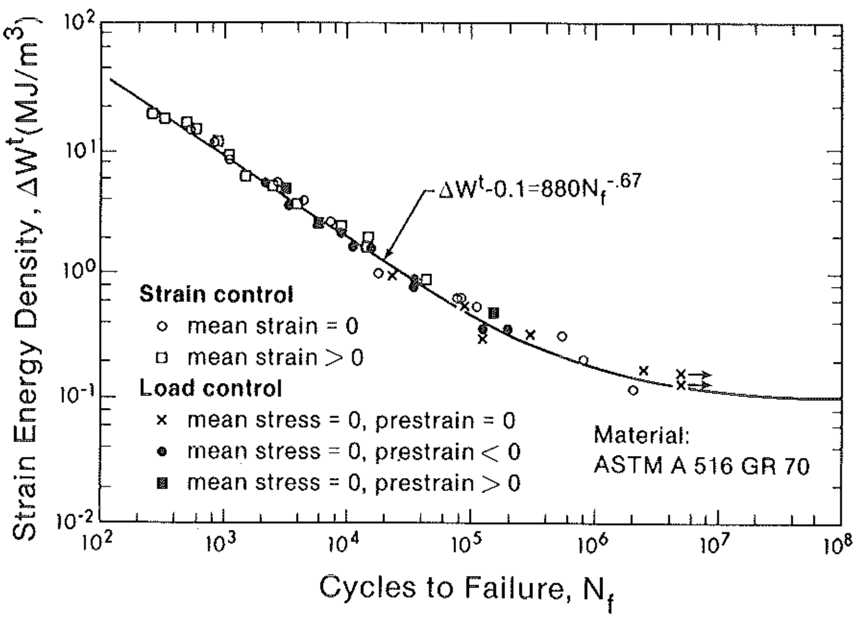

Fig 3 Total strain energy density per cycle, ΔW^t in uniaxial tests vs number of cycles to failure, N_f

entering into the determination of the total strain energy density are listed in Table 1.

It is noted from Fig. 3, that the correlation with the uniaxial experimental data is indeed good over the entire range of fatigue lives. We thus observe the unifying nature of the proposed cyclic strain energy density. It is worth emphasizing that the salient feature of this approach is that the damage, caused in the material due to cyclic loading, is related to the mechanical energy input. The cyclic strain energy density ΔW^t given by equation (6) has a physical interpretation, and it is also shown to be a parameter which can describe both the initiation and subsequent propagation of fatigue cracks (7). In the following, this idea is generalized to multiaxial cyclic stress states.

Table 1 Material properties of ASTM A-516 Gr. 70 low alloy carbon steel

Properties	E (MPa)	ν	K' (MPa)	K^* (MPa)	n'	n^*
Values	204 000	0.27	1067	630	0.193	0.144
Properties	σ_f (MPa)	σ'_f (MPa)	α	σ_0 (MPa)	σ_{lim} (MPa)	$(\sigma'_y)_{0.2\%}$ (MPa)
Values	993	842	−0.67	180	205	325

Multiaxial cyclic loading

The concept of elastic and plastic strain separation, as demonstrated in Fig. 1 and equation (1), is generalized to the multiaxial states in an incremental sense, i.e.

$$d\varepsilon_{ij} = d\varepsilon^{e}_{ij} + d\varepsilon^{p}_{ij} \qquad (i, j = 1, 2, 3) \tag{9}$$

The total strain energy density of a material element subjected to a cyclically varying stress and strain history σ_{ij} and ε_{ij} is given by

$$W = \int \sigma_{ij}\, d\varepsilon_{ij} \tag{10}$$

where the integral has to be evaluated in an appropriate manner.

Elastic strain energy density

The elastic strain increment is related to the stress increment through the generalized Hooke's law

$$d\varepsilon^{e}_{ij} = \frac{1+\nu}{E}\, d\sigma_{ij} - \frac{\nu}{E}\, d\sigma_{kk}\delta_{ij} \tag{11}$$

The elastic strain energy density can be calculated from

$$W^{e} = \int_{0}^{\varepsilon_{ij}} \sigma_{ij}\, d\varepsilon^{e}_{ij} \tag{12}$$

Substituting from (11) into (12) we get

$$W^{e} = \frac{1+\nu}{2E}\, \sigma_{ij}\sigma_{ij} - \frac{\nu}{2E}\, (\sigma_{kk})^{2} \tag{13}$$

The stress tensor σ_{ij} may now be decomposed into deviatoric and spherical parts, i.e.

$$\sigma_{ij} = s_{ij} + \tfrac{1}{3}\sigma_{kk}\delta_{ij} \tag{14}$$

Substituting (14) into (13) we obtain

$$W^{e} = \frac{1}{6E}\, \{3(1+\nu)s_{ij}s_{ij} + (1-2\nu)(\sigma_{kk})^{2}\} \tag{15}$$

The above equation can be written as

$$W^{e} = W^{D} + W^{V} \tag{16}$$

where the first term on right hand side is the strain energy density due to the distortion of an infinitesimal element, and the second term is the strain energy due to volumetric change.

Tests carried out by White *et al.* (8) and Morrison *et al.* (9) have indicated that the fatigue life is a pressure sensitive process. It was reported that superposed hydrostatic tension decreased the fatigue life, whereas that of compression increased it. Therefore, the hydrostatic part of the stress tensor, equation (14), has an influence on the fatigue life. When the fatigue damage is related to the plastic strain energy density (**2**)–(**6**), the effect of the hydrostatic pressure cannot be taken into account, since $d\varepsilon_{kk}^{p} = 0$, because there is no change in volume during the plastic flow process. While the contribution of tensile hydrostatic pressure may be small in the low cycle fatigue regime, this may not be true for long lives.

In the case of cyclic loading the elastic strain energy is recovered during a complete cycle. One way to account for the hydrostatic pressure effect in energy calculations is to consider the positive elastic strain energy density during cyclic loading – similar to the uniaxial case, Fig. 1 – that is

$$\Delta W^{e+} = \int_{H(\sigma_i^{min})\sigma_i^{min}}^{H(\sigma_i^{max})\sigma_i^{max}} \sigma_i \, d\varepsilon_i \qquad (i = 1, 2, 3) \tag{17}$$

where σ_i terms are components of the principal stress, and H is the Heaviside function with the following properties

$$\begin{aligned} H(\sigma_i) &= 1 \quad \text{for} \quad \sigma_i \geq 0, \\ H(\sigma_i) &= 0 \quad \text{for} \quad \sigma_i < 0 \end{aligned} \tag{18}$$

Substituting from (11) into (17) and carrying out the integration between $\sigma_i^{min} < 0$ and $\sigma_i^{max} > 0$, we get

$$\Delta W^{e+} = \frac{1 + \nu}{3E} (\bar{\sigma}^{max})^2 + \frac{1 - 2\nu}{6E} (\sigma_{kk}^{max})^2 \tag{19}$$

where the first term on the right hand side of (19) is the distortion energy and the second one is the energy associated with the volumetric change. In equation (19) the effective stress, $\bar{\sigma}$, is defined as

$$\bar{\sigma}^2 = \tfrac{3}{2} s_{ij} s_{ij} \tag{20}$$

Plastic strain energy density

Assuming a small-strain deformation theory of plasticity as the material model, the plastic strain–deviatoric stress relationship is (**10**)

$$\varepsilon_{ij}^{p} = \frac{3}{2} \frac{s_{ij}}{E_t} = \frac{3}{2} \left(\frac{1}{E_s} - \frac{1}{E} \right) s_{ij} \tag{21}$$

where E_t is the tangent modulus of the stress strain curve and E_s is the secant modulus of it. Accordingly

$$E_t = d\bar{\sigma}/d\bar{\varepsilon} \quad \text{and} \quad E_s = \bar{\sigma}/\bar{\varepsilon} \tag{22}$$

where $\bar{\sigma}$ and $\bar{\varepsilon}$ are effective-stress and effective-strain, respectively. The effective-stress is given by equation (20), whereas the effective-strain is defined as

$$\bar{\varepsilon} = (2\varepsilon_{ij}\varepsilon_{ij}/3)^{1/2} \tag{23}$$

It is to be noted that for stable cyclic behaviour, the use of the J_2 deformation theory is justified on the account of the proportional or nearly proportional loading. The incremental form of equation (21) is

$$\mathrm{d}\varepsilon_{ij}^{p} = \frac{3}{2}\left(\frac{1}{E_s} - \frac{1}{E}\right)\mathrm{d}s_{ij} + \frac{3}{2}\left(\frac{1}{E_t} - \frac{1}{E_s}\right)s_{ij} \tag{24}$$

The plastic strain energy density can now be calculated by evaluating the integral

$$W^{p} = \int \sigma_{ij}\,\mathrm{d}\varepsilon_{ij}^{p} = \int s_{ij}\,\mathrm{d}\varepsilon_{ij}^{p} \tag{25}$$

Introducing equations (20), (22) and (23) into (24), we obtain

$$W^{p} = \int \left(\frac{1}{E_t} - \frac{1}{E}\right)\bar{\sigma}\,\mathrm{d}\bar{\sigma} = \int \bar{\sigma}\,\mathrm{d}\bar{\varepsilon}^{p} \tag{26}$$

To evaluate the tangent modulus, E_t, it is generally assumed that the effective stress–effective plastic strain ($\bar{\sigma}$ vs $\bar{\varepsilon}^{p}$) relationship is similar to that of uniaxial σ vs ε^{p} curve. (See for example, Ellyin (**11**) for the experimental corroboration.) From equation (1) we thus get

$$\bar{\varepsilon}^{*} = \frac{\bar{\sigma}^{*}}{E} + k^{*}\bar{\sigma}^{*\,1/n^{*}} \tag{27}$$

and

$$\frac{1}{E_t} = \frac{\mathrm{d}\bar{\varepsilon}^{*}}{\mathrm{d}\bar{\sigma}^{*}} = \frac{1}{E} + \frac{k^{*}}{n^{*}}\bar{\sigma}^{*(1-n^{*})/n^{*}} \tag{28}$$

where $k^{*} = 2(2K^{*})^{-1/n^{*}}$. Note that equations (27) and (28) are written with respect to the origin O* of the master curve, Fig. 2. For the sake of simplicity of demonstrating the concept, for the time being we will assume that the origins O and O* in Fig. 2 coincide, i.e., the material behaves as an ideal Masing model.

With reference to Fig. 1 and coordinates $\Delta\bar{\sigma}$ and $\Delta\bar{\varepsilon}$, the plastic strain energy per cycle, ΔW^{p}, can be evaluated from equation (26) by noting

$$\Delta W^{p} = \Delta\bar{\sigma}\Delta\bar{\varepsilon} - 2\int_{0}^{\Delta\bar{\sigma}} \Delta\bar{\varepsilon}\,\mathrm{d}(\Delta\bar{\sigma}) \tag{29}$$

Substituting from (27) into (29) and integrating, we obtain

$$\Delta W^{p} = k\,\frac{1-n'}{1+n'}\,\Delta\bar{\sigma}^{(1+n')/n'} \tag{30}$$

For a uniaxial stress state, $\bar{\sigma} = \sigma$ and (30) reduces to (4). Comparing (4) with (2) we note that, for the non-Masing material, an additional term, $(2n^{*}/1 + n^{*})\delta\sigma_0\Delta\varepsilon^{p}$, has to be added to that of the Masing material. In the case of the proportional or nearly proportional multiaxial loading, the additional term due to the non-Masing behaviour is

$$\Delta W^{\rm p}_{\rm RM} = \frac{4n^*}{1 + n^*}\left(\frac{\Delta\bar{\sigma}^*}{2K^*}\right)^{1/n^*}\delta\bar{\sigma}_0 \tag{31}$$

The effect of this term becomes important as the range of plastic strain (or stress) increases.

Combining (19) and (30), and noting that for the fully-reversed tests $\bar{\sigma}^{\rm max} = \Delta\bar{\sigma}/2$, and $k = 2(2K')^{-1/n'}$, we get

$$\Delta W^{\rm t} = \Delta W^{\rm e+} + \Delta W^{\rm p} = \frac{1 + \nu}{12E}(\Delta\bar{\sigma})^2 + \frac{1 - 2\nu}{6E}(\sigma^{\rm max}_{\rm kk})^2 + \frac{2(1 - n')}{1 + n'}(2K')^{-1/n'}(\Delta\bar{\sigma})^{(1+n')/n'} \tag{32}$$

In the case of the non-Masing material, the last term on the right hand side of (32) has to be modified by changing the cyclic stress–strain constants to those of the master curve. That is, replacing n', K', and $\Delta\bar{\sigma}$ with n^*, K^*, and $\Delta\bar{\sigma}^*$ and adding equation (31). Thus, in either case, equation (32) has the general form of

$$\Delta W^{\rm t} = \Delta W^{\rm t}(J_2, I_1^{\rm max}) \tag{33}$$

where $J_2 = \bar{\sigma}^2/3$ is the second invariant of the deviatoric stress associated with the distortion energy, and $I_1^{\rm max}$ is the positive (tensile) hydrostatic pressure.

Multiaxial failure criterion

A failure criterion for the multiaxial fatigue can now be proposed relating $\Delta W^{\rm t}$ to number of cycles to failure, $N_{\rm f}$, and imposed triaxiality condition, ϱ

$$\Delta W^{\rm t} = G(N_{\rm f}, \varrho) \tag{34}$$

In earlier works, we have suggested a particular relation of the power law type

$$\Delta W^{\rm t} = \varkappa(\varrho)N_{\rm f}^{\alpha} + C \tag{35}$$

In the above equation $\varkappa(\varrho)$ is a function of the imposed strain triaxiality. Equation (35) is similar to that of equation (7) for uniaxial loading. As a first approximation, we may take a linear function for $\varkappa(\varrho)$, i.e.

$$\varkappa(\varrho) = A\varrho + B \qquad \varrho = 2\varepsilon_2/(\varepsilon_1 - \varepsilon_3) \tag{36}$$

where A and B are material constants to be evaluated. Substituting in (35) from (33) and (36), we get the following multiaxial fatigue failure criterion

$$\Delta W^{\rm t}(J_2, I_1^{\rm max}) = (A\varrho + B)N_{\rm f}^{\alpha} + C \tag{37}$$

It is noted that the criterion (37) is hydrostatic pressure sensitive and has an invariant property, i.e., it is a frame indifferent criterion.

Comparison with experimental results

To compare the predictions of the proposed criterion, equation (37), with experimental results, we must first determine the material properties contained in the left hand side of (37). The elastic properties E, and ν are readily available from material tables. The cyclic properties n', K' (or n^*, K^*) are obtained from the cyclic stress–strain (or master) curve. The constants on the right hand side of (37) are determined as follows. From the uniaxial stress results, $\rho = -2\nu/(1+\nu)$, we obtain the value of the exponent α and the relations

$$\varkappa_u = \{-2\nu/(1+\nu)\}A + B \qquad C_u = C \tag{38}$$

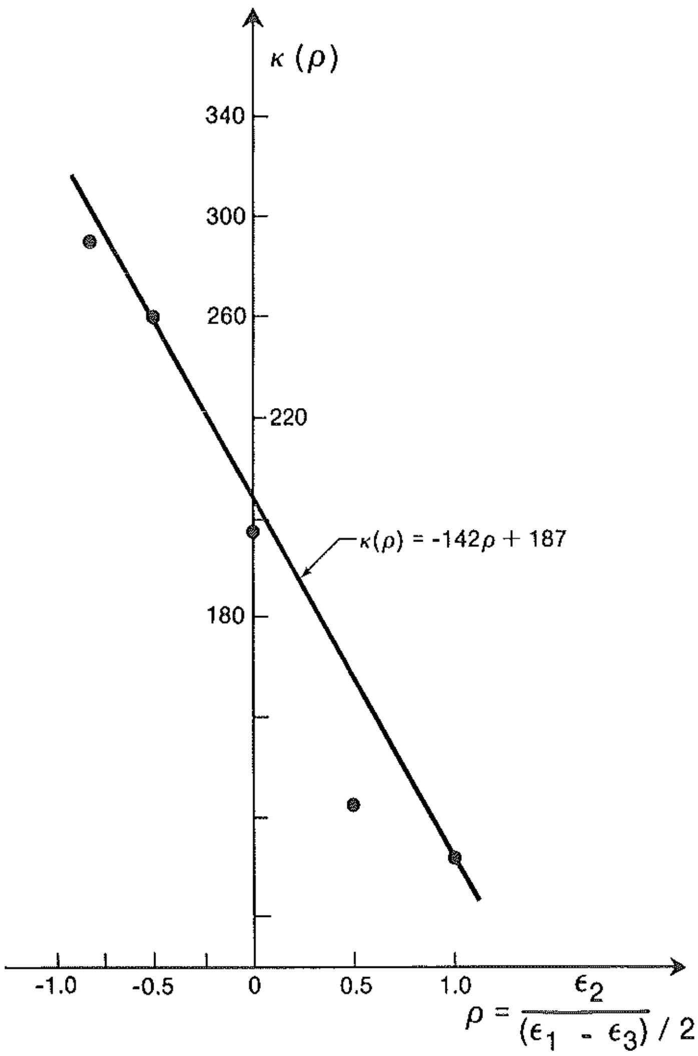

Fig 4 Strain triaxiality function $\kappa(\rho)$ vs ρ: predicted line and experimental values (circles) are shown on the graph

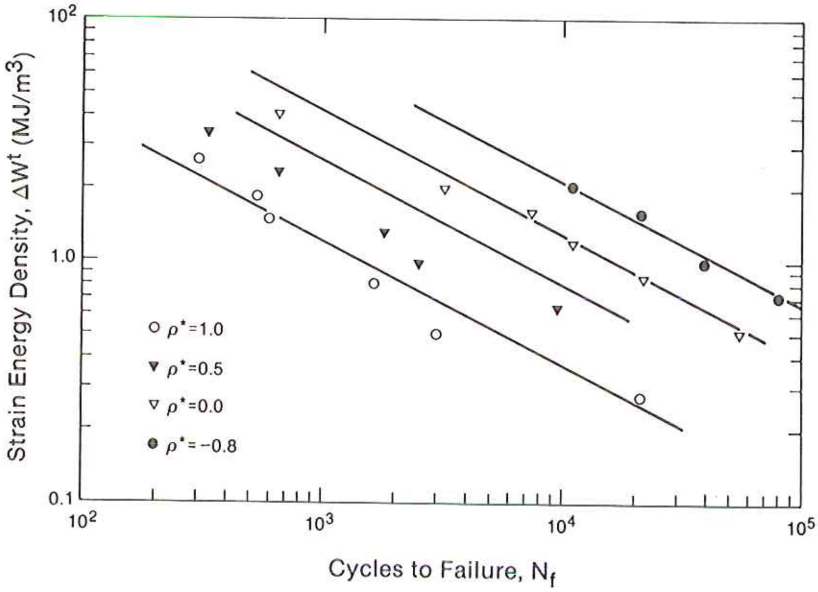

Fig 5 **Predicted cyclic strain energy density, ΔW^t, vs number of cycles to failure, N_f, and experimental data of reference (12)**

Results of tests for another strain ratio are required to provide two equations for the two unknowns A and B. Note that in the case of low cycle fatigue, the constant C associated with the elastic strain energy at the fatigue limit may be neglected (*cf* Fig. 3).

Experimental data for fully-reversed strain-controlled tubular test specimens under biaxial stress conditions were reported by Lefebvre *et al.* (**12**). From the uniaxial stress tests we get $\alpha = -0.54$, and the results of strain ratio $\varrho^* = 1$, combined with the uniaxial case yield, $A = -142$ MJ/m^3 and $B = 187$ MJ/m^3. The linear equation (36) with these coefficients is traced in Fig. 4, and is compared with the best fit experimental data. It is noted that with the exception of one point, $\varrho^* = 0.5$, all other data fall very close to the predicted values.

The predictions of equation (37) are compared with the experimental data in Fig. 5. As noted earlier, there is considerable scatter in the results of strain ratio $\varrho^* = 0.5$. The predicted values and experimental data are in good agreement for other strain ratios. It is worth mentioning that the results reported in reference (**12**) may have been subject to secondary effects, thus resulting in lower fatigue lives. For example, the slope of the uniaxial curve given by Fig. 3 for the same material is $\alpha = 0.67$ as compared with $\alpha = 0.54$ in reference (**12**). However, the purpose of this comparison is to show the predicted trend (from the given data of uniaxial condition), rather than precise life prediction values.

Conclusions

A multiaxial fatigue failure criterion is proposed based on the cyclic strain energy density. It has an invariant property, it is hydrostatic pressure sensitive, and it is a function of the imposed strain triaxiality condition. The criterion can be used for life predictions. The majority of the material parameters can be determined from the uniaxial test data. Predictions of the proposed criterion are in good agreement with biaxial fatigue test results. The proposed criterion is also consistent with the concept of crack initiation and crack propagation, and has a unifying feature for short and long life regimes.

Acknowledgement

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