Development of Body Force Method and Its Application to Crack Problems

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1. Introduction

The body force method was originally proposed by H. Nisitani in 1967[1] as a versatile method of numerical stress analysis. The method has been applied to various notch[1-6] and crack problems[7-23]. In the early stage of the progress of the method, it was mainly applied to plane problems. Recently, various important three-dimensional problems have been solved by the method[11, 15-23].

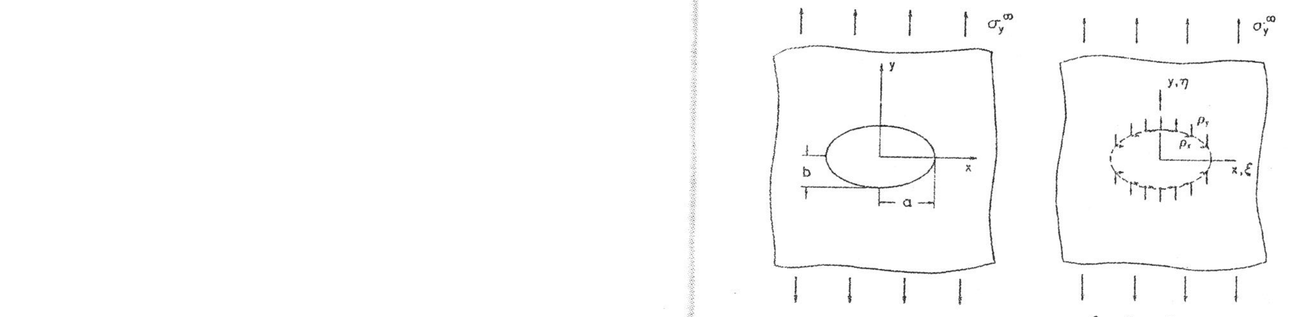
Although the basic concept of the body force method is analog to the method of Green's function (so-called B.E.M.), the important difference between the body force method and B.E.M. is that a unique idea of the body force density in notch problems and the density of the pair of body force in crack problems is introduced in the body force method. This idea enables one to obtain very accurate solutions. This is the reason why the body force method has both versatility and high accuracy.

In the present paper, first the basic concept of the method and the unique idea of the density of the pair of body force will be explained and then, several numerical solutions for crack problems, i.e. stress intensity factors will be shown.

2. Principle of the body force method

First of all, the solutions of an elliptic hole will be explained, because a crack is the limiting shape of a slender ellipse and therefore, the basic expression for crack problems is obtained from the solution of an elliptic hole [1].

Figure 1(a) shows an elliptic hole in uniformly stressed infinite plate. This problems can be solved by distributing point force continuously along the imagined ellipse in an infinite plate without any hole as shown in Fig. 1(b). The boundary conditions along the elliptic hole are completely satisfied as the result of the superposition of the stress fields due to uniform tension and



$\xi^2/a^2 + \eta^2/b^2 = 1.$

(a) (b)

Fig. 1

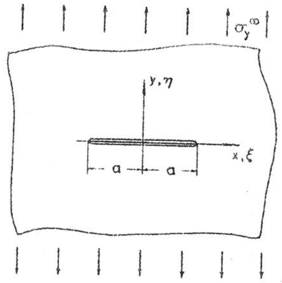


Fig. 2

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body force (continuously distributed point force), if the densities of body force have the following values in plane stress condition,

$$\rho_x = -\frac{\sigma_y^\infty}{1-\nu^2}\{1-\nu(1+2\frac{a}{b})\} \tag{1a}$$

$$\rho_y = \frac{\sigma_y^\infty}{1-\nu^2}\{(1+2\frac{a}{b})-\nu\} \qquad (\nu : \text{Poisson's ratio}) \tag{1b}$$

The definitions of the densities of body force in Eq(1a) and (1b) are given by

$$\rho_x = \frac{dF_x}{d\eta} \quad , \quad \rho_y = \frac{dF_y}{d\xi} \tag{2}$$

where dF_x and dF_y denote the x- and y-components, respectively, of the body force of the element $dS = \sqrt{(d\xi)^2 + (d\eta)^2}$ of the imaginary ellipse: $\xi^2/a^2 + \eta^2/b^2 = 1$. The densities of body force for a crack (Fig. 2) are obtained as the limiting expression for Eq. (1a) and (1b), as $b \to 0$, i.e. as the ellipse reduces to the crack. However, both ρ_x and ρ_y become unbounded as $b \to 0$, the direct introduction of these quantities to crack problem is inadequate. In order to resolve this difficulty, the body force acting on the upper and lower part of the imaginary ellipse (Fig. 1(b)) is treated as a pair. Thus, the stress $\underset{\sim}{\sigma}$ due to the pair of the body force is given by the equation

$$\underset{\sim}{\sigma} = \int \frac{\partial \underset{\sim}{\sigma}^P}{\partial \xi}\bigg|_{P=1} \cdot 2\eta\rho_x \, d\xi + \int \frac{\partial \underset{\sim}{\sigma}^Q}{\partial \eta}\bigg|_{Q=1} \cdot 2\eta\rho_y \, d\xi \tag{3}$$

$$2\eta\rho_x = \frac{4\nu\sigma_y^\infty}{1-\nu^2}\sqrt{a^2-\xi^2} \quad , \quad 2\eta\rho_x = \frac{4\sigma_y^\infty}{1-\nu^2}\sqrt{a^2-\xi^2} \tag{4}$$

where, $\underset{\sim}{\sigma}^P$ and $\underset{\sim}{\sigma}^Q$ are Green's function for the point force P and Q in x and y direction, respectively.

As understood from the above discussion, the stress field induced by the body force in crack problems is characterized by the pair of body force and the derivatives of stress field due to body force, i.e. the derivatives of Green's function. The same characteristics are introduced also in three-dimensional crack problems, though only the expressions corresponding to Eq. (3) and (4) are changed.

The densities of body force for simple problems like Fig. 1(a) and Fig. 2 can be found in closed form. However, in general, weighting function f(ξ) must be multiplied to the basic expression of the pair of body force (Eq. (4)), because the densities of body force like Eq. (4) are insufficient to satisfy the boundary conditions that are different from Fig. 1(a). Since it is difficult to determine weighting functions in closed form in general problems, numerical analysis is used. The important characteristics of the body force method is to assume the expression for the pair of body force in crack problems as $f(\xi)\sqrt{a^2-\xi^2}$ where in general, weighting functions f(ξ) vary slightly within the defined area. In numerical analysis, the part of imaginary crack is divided into finite divisions N and the boundary conditions are satisfied at collocated points with regard to stress or within divisions with regard to resultant forces. Subsequently, discrete values of weighting functions are determined. However, considering that the numerical results for the systematic increase in the division number N vary systematically, we can obtain very accurate values by extrapolation procedure, i.e. $N \to \infty$. Stress intensity factors are determined from the values of the weighting function at crack tip.

Although based on the body force method, all problems can be solved in principle by using Green's function for an infinite or a semi-infinite plate,

the use of Green's function for an infinite plate containing a traction-free crack or hole makes sometimes the method much more effective[10, 13, 14].

3. Basic expressions for the pair of body force

In performing numerical analysis, the basic pairs of body force multiplied with weighting functions are used. Therefore, for the sake of convenience various basic pairs of body force for two- and three-dimensional crack problems are summarized in Table 1 and 2, respectively. Consequently, the solutions of all crack problems are reduced to solve the following integral equations numerically

$$\int (W \times B \times DG)\, dA + \sigma_{ext} = B.C. \tag{5}$$

where,
- W : weighting functions, unknowns in the problem,
- B : basic pair of body force (see Table 1 & 2),
- DG : derivatives of Green's functions,
- A : crack length (two-dimensional problems) or crack area (three-demensional problems),
- σ_{ext} : stress field due to external load,
- B.C. : boundary condition at crack.

Table 1 Basic pair of body force for plane stress problems

Plane stress problem, line crack : $x = -a \sim a$				
Mode	Mode I		Mode II	
B	$2\eta\rho_x$	$2\eta\rho_y$	$2\eta\rho_x$	$2\eta\rho_y$
	$\nu \cdot 2\eta\rho_y$	$\frac{4\rho_y^\infty}{1-\nu^2}\sqrt{a^2-\xi^2}$	$\frac{2\tau_{xy}^\infty}{1+\nu}\sqrt{a^2-\xi^2}$	$2\eta\rho_x$
D	$\frac{\partial}{\partial\xi}$	$\frac{\partial}{\partial\eta}$	$\frac{\partial}{\partial\eta}$	$\frac{\partial}{\partial\xi}$

B : Basic pair, D : operator to Green's function

Table 2 Basic pair of body force for three-dimensional problems

Three-dimensional problem, elliptical crack : $\xi^2/a^2 + \eta^2/b^2 = 1$, $a > b$							
Mode	Mode I			Shear τ_{zy}^∞		Shear τ_{zx}^∞	
B	$2\zeta\rho_x$	$2\zeta\rho_y$	$2\zeta\rho_z$	$2\zeta\rho_y$	$2\zeta\rho_z$	$2\zeta\rho_x$	$2\zeta\rho_z$
	$\frac{\nu}{1-\nu}\cdot 2\zeta\rho_z$	$\frac{\nu}{1-\nu}\cdot 2\zeta\rho_z$	$\frac{(1-\nu)^2}{(1-2\nu)}\cdot\frac{4\sigma_z^\infty b\sqrt{1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}}}{E(k)}$	$\frac{2(1-\nu)bk^2\tau_{zy}^\infty\sqrt{1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}}}{(k^2+\nu k'^2)E(k)-\nu k'^2 K(k)}$	$2\zeta\rho_y$	$\frac{2(1-\nu)bk^2\tau_{zx}^\infty\sqrt{1-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}}}{(k^2-\nu)E(k)+\nu k'^2 K(k)}$	$2\zeta\rho_x$
D	$\frac{\partial}{\partial\xi}$	$\frac{\partial}{\partial\eta}$	$\frac{\partial}{\partial\zeta}$	$\frac{\partial}{\partial\zeta}$	$\frac{\partial}{\partial\eta}$	$\frac{\partial}{\partial\zeta}$	$\frac{\partial}{\partial\xi}$

$K(k)$, $E(k)$: The complete elliptic integral of first & second kind, $k=(1-b^2/a^2)^{1/2}$.

4. Examples of solutions

Because of the limitation of space, only selected solutions which may be interesting in relation to the recent topics in fracture mechanics and fatigue problems will be presented.

The all numerical results of stress intensity factors for the problems of Fig. 3-9 may be found in accordance with the indication of Table 3.

Table 3

Problems	Stress intensity factor
Fig. 3	Table 4 [12]
Fig. 4	Table 5 [12]
Fig. 5	Table 6 [21]
Fig. 6	Table 7 [21]
Fig. 7	Table 8 [18]
Fig. 8	Table 9 [22]
Fig. 9	Table 10 [23]

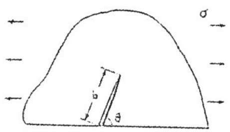


Fig. 3

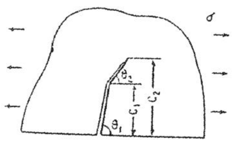


Fig. 4

Table 4

θ deg	$K_I/\sigma\sqrt{\pi b}$	$K_{II}/\sigma\sqrt{\pi b}$
0	0	0
15	0.229	0.227
30	0.461	0.336
45	0.705	0.364
60	0.920	0.306
75	1.068	0.174
90	1.121	0

Table 5

θ_1 deg	θ_2 deg	c_1/c_2	$K_I/\sigma\sqrt{\pi b}$	$K_{II}/\sigma\sqrt{\pi b}$
90	30	0.5	0.463	0.337
	45		0.704	0.365
	60		0.919	0.306
	30	0.9	0.468	0.342
	45		0.707	0.359
	60		0.921	0.296
30	90	0.5	1.122	~ 0
45			1.121	
60			1.121	
30		0.9	1.098	~ 0
45			1.121	
60			1.125	

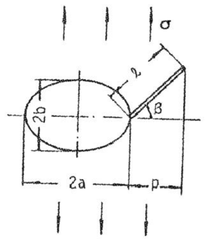


Fig. 5

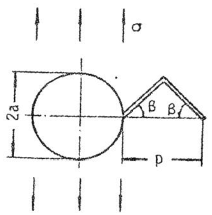


Fig. 6

Table 7 (β=45°)

P/a	$K_I/\sigma\sqrt{\pi p}$	$K_{II}/\sigma\sqrt{\pi p}$
0.01	2.462	-1.267
0.1	2.092	-1.034
0.2	1.803	-0.866
0.5	1.314	-0.624
1.0	0.970	-0.506

Table 6

b/a		0.5			1.0			2.0		
	β \ p/a	0.01	0.1	0.5	0.01	0.1	0.5	0.01	0.1	0.5
F_I	0°	5.181	3.256	1.640	3.293	2.772	1.728	2.230	2.120	1.726
	45°	4.017	2.559	0.999	2.388	2.203	1.182	1.671	1.624	1.331
F_{II}	0°	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	45°	1.938	1.074	0.818	1.262	0.990	0.677	0.860	0.802	0.616

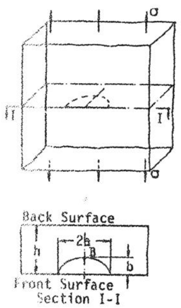


Fig. 7

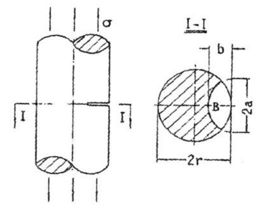


Fig. 8

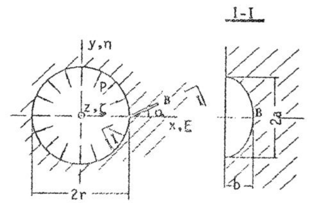


Fig. 9

Table 8 $M_B = K_{IB}/(\sigma\sqrt{\pi b}\,/\,E(k))$

b/a \ b/h	0	0.1	0.2	0.5
0	1.122	1.187	1.361	2.728
0.125	1.106	1.121	1.188	1.631
0.2	1.100	1.108	1.158	1.485
0.25	1.096	1.102	1.138	1.395
0.5	1.070	1.073	1.088	1.201
0.75	1.049	1.052	1.060	1.119
1.0	1.036	1.037	1.041	1.076

Table 9

	$K_{IB}/p\sqrt{\pi b}$						$K_{II\,B}/p\sqrt{\pi b}$			
b/r	0.1			0.5			0.1		0.5	
b/a \ α	0°	30°	45°	0°	30°	45°	30°	45°	30°	45°
1.0	1.202	1.020	0.827	0.946	0.885	0.808	0.421	0.552	0.223	0.313
0.5	1.588	1.373	1.137	1.214	1.138	1.038	0.448	0.570	0.244	0.340
0.25	1.817	1.567	1.300	1.365	1.276	1.161	0.467	0.608	0.258	0.354
0.0	1.989	1.703	1.378	1.481	1.372	1.238	0.506	0.636	0.284	0.392

Table 10

b/a=1	a/r	0.125	0.250	0.375	0.500
	$K_{IB}/\sigma\sqrt{\pi b}$	0.665	0.683	0.714	0.758
$K_{IB}/\sigma\sqrt{\pi b}$	a/r \ b/a	0.25	0.50	0.75	1.00
	0.25	1.003	0.890	0.780	0.683
	0.50	0.996	0.920	0.840	0.758

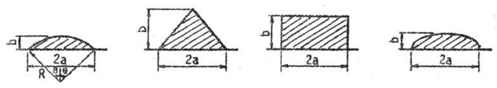
A couple of other examples are illustrated in Fig. 10 and 11.

From numerical results [17, 20] for Fig. 10, the maximum stress intensity factors $K_{I\,max}$ along crack front for various surface cracks can be approximated by

$$K_{I\,max} \cong 0.629\,\sigma_o\sqrt{\pi\sqrt{\mathrm{area}}} \quad \text{for Poisson's ratio } \nu = 0.0 \tag{6a}$$

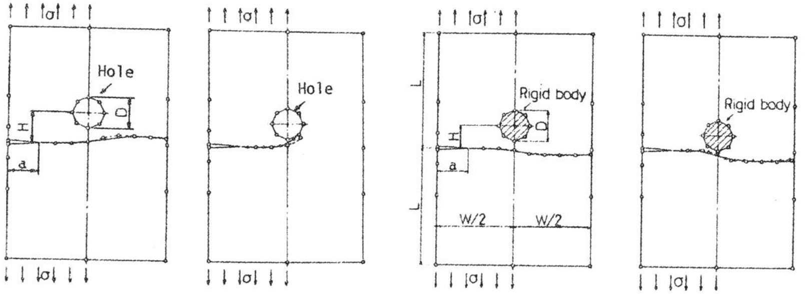
$$K_{I\,max} \cong 0.650\,\sigma_o\sqrt{\pi\sqrt{\mathrm{area}}} \quad \text{for Poisson's ratio } \nu = 0.3 \tag{6b}$$

, where σ_o is uniform tensile stress in infinity and area is the area of surface crack.



Partial circle area = $\theta R^2 - aR\cos\theta$ (a) — Triangle area = ab (b) — Rectangle area = 2ab (c) — Semi-ellipse area = $\frac{1}{2}\pi ab$ (d)

Fig.10



(a) H/a = 1.0 (b) H/a = 0.75 (c) H/a = 0.75 (d) H/a = 0.5

Fig. 11 Analysis of Crack Propagation Path
(a/W = 0.2, L/W = 0.75, D/a = 1.0, ν = 0.33)

— Experiment
o Prediction
Δ Prediction by small increment of propagation

Figure 11 illustrates the prediction by the body force method and the experimental result for the path of crack propagation in plexigrass[14]. In the prediction the maximum tangential stress criterion [24] was used.

References

[1] Nisitani, H., Two-dimensional problem solved using a digital computer, J. Japan Soc. Mech. Engrs., Vol. 70, No. 580, pp. 627-635 (1967), Nisitani, H., Stress analysis of notch problems, Mechanics of Fracture 5, Ed. by Sih, G.C., pp. 1-68 (1978), Noodhoff Int. Pub.

[2] Nisitani, H., Suematsu, M. and Saito, K., Tension of a semi-infinite palte with a row of elliptic holes (including cracks) or a infinite plate with two rows of elliptic holes, Trans. JSME, Vol. 39, No. 324, pp. 2323-2330 (1973).

[3] Nisitani, H., Saito, K. and Hara, N., Stress concentration due to an elliptic hole or crack existing near a notch under tension or longitudinal shear, Trans. JSME, Vol. 39, No. 324, pp. 2312-2322.

[4] Nisitani, H. and Hashimoto, K., Stress concentration factors for the torsion of shafts with a circumferential semi-elliptic groove, Trans. JSME, Vol. 43, No. 374, pp. 3642-3650 (1977).

[5] Murakami, Y., Tani-ishi, H. and Nisitani, H., Stress concentration analysis of an axisymmetrical surface pit in a semi-infinite body under biaxial tension, Trans. JSME, Vol. 48, No. 426, pp. 150-159.

[6] Murakami, Y., Noda, N. and Nisitani, H., Stress concentration analysis of an axisymmetrical body under bending by body force method (1st & 2nd Report), Trans. JSME, Vol. 48, No. 427, pp. 301-320 (1982).

[7] Nisitani, H. and Murakami, Y., Interaction of elasto-plastic cracks subjected to a uniform tensile stress in an infinite or a semi-infinite plate, Proc. 1971 Int. Conf. Mech. Behav. Mater., 2, pp. 346-356 (1972).

[8] Nisitani, H., Tension of a strip with symmetric edge cracks or elliptic notches, Trans. JSME, Vol. 41, No. 349, pp. 2518-2526 (1975).

[9] Nisitani, H. and Isida, M., Stress intensity factor for the tension of an infinite plate containing an elliptic hole with two symmetric edge cracks, Vol. 39, No. 317, pp. 7-14 (1973).

[10] Murakami, Y. and Nisitani, H., The stress intensity factors for the cracked hollow spin disk, Trans. JSME, Vol. 41, No. 348, pp. 2255-2264 (1975).

[11] Nisitani, H. and Murakami, H., Stress intensity factors of an elliptical crack or a semi-elliptical crack subjected to tension, Int. J. Frac., Vol. 10, No. 3, pp. 353-368 (1974).

[12] Nisitani, H., Stress intensity factor for the tension of a semi-infinite plate having an oblique or a bent edge crack, Trans. JSME, Vol. 41, No. 344, pp. 1103-1109 (1975).

[13] Murakami, Y., Application of the body force method to the calculation of stress intensity factors for a crack in the arbitrarily shaped plate, Engng. Frac. Mech., Vol. 10, pp. 497-513 (1978).

[14] Murakami, Y., Prediction of crack propagation path (Analysis and experiment), Trans. JSME, Vol. 46, No. 407, pp. 729-738 (1980).

[15] Nisimura, A., Aoki, S. and Sakata, M., Stress intensity factor for a semi-elliptical crack in an internally pressurized cylinder, Proc. Third Int. Conf. Pressure Vessel Technology, Part II, Tokyo, pp. 517-526 (1977-4).

[16] Murakami, Y. and Nemat-Nasser, S., Interacting dissismilar semi-elliptical surface flaws under tension and bending, Engng. Frac. Mech., Vol. 16, No. 3, pp. 373-386 (1982).

[17] Murakami, Y. and Nemat-Nasser, S., Growth and stability of interacting surface flaws of arbitrary shape, Engng. Frac. Mech., Vol. 17, No. 3, pp. 193-210 (1983).

[18] Isida, M. and Noguchi, H., Tension and bending of plates with a semi-elliptical surface crack, Trans. JSME, Vol. 48, No. 429, pp. 607-619(1982).

[19] Isida, M. and Noguchi, H., Prelim Proc. JSME, No. 820-11, p. 56 (1982).

[20] Murakami, Y. and Isida, M., Prelim Proc. JSME, No. 838-1, pp. 7-9 (1983).

[21] Isida, M., Chen Dai-heng and Nisitani, H., Plane problems of an arbitrary array of cracks emanating from the edge of an elliptical hole, Prelim Proc. JSME, No. 820-11, pp. 53-55 (1982).

[22] Nisitani, H. and Chen Dai-heng, Stress intensity factors for a partial semi-elliptic crack in a shaft under tension, Prelim Proc. JSME, No. 838-2, pp. 10-12 (1983).

[23] Nisitani, H. and Chen Dai-heng, Stress intensity factors for a semi-elliptic surface crack emanating from the inside of an infinitely thick cylinder subjected to internal pressure, (1983), to be published.

[24] Erdogan, F. and Sih, G.C., Trans. ASME, Ser. D, Vol. 85, No. 4, pp. 519-527 (1963).