STRESS CONCENTRATION AND STRESS INTENSITY FACTORS FOR ELLIPTIC HOLES AND CRACKS IN AN INFINITE PLANE

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INTRODUCTION

The stress analysis of an infinite plane weakened by many holes and cracks is important for engineering applications. During last decade, many contributions had been made[1,2].

Based on references[3-6], the present paper deals with the stress concentration in plane with several arbitrarily distributed elliptic holes (circular hole is the special case of the elliptic one). By the method of complex variables, the stress functions in which the interactions of neighbouring holes have been taken into consideration, can be constructed. With the comformal mapping technique to satisfy the boundary conditions of individual holes, the governing equations of the theory of elasticity are reduced to a set of simultaneous equations. Evidently, the problems with single and multiple cracks can be easily derived by varying the eccentricity of the ellipses to a limit, and an approximate solution of cracking problem may thus be obtained.

The present method is suitable for any combination of holes and distributions. A digital programm is developed for computation. Some numerical examples on stress concentration factor have shown favorable agreement with the photo-elasticity results.

THEORY

1. Stress Function

Let us consider an infinite plane weakened by arbitrarily distributed n ellipses and subjected to uniform tensile stress at infinity (Fig. 1).

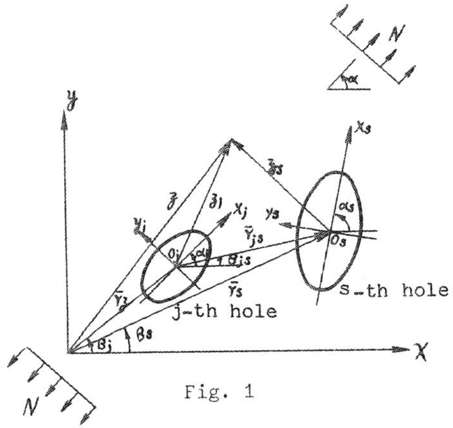

Fig. 1

The stress function is assumed (on the mapped plane) in a form:

$$\left.\begin{aligned} \phi(\zeta) &= \sum_{k=1}^{\infty} \frac{{}^{j}a_k}{\zeta_j^k} + \sum_{\substack{s=1\\ s\neq j}}^{n} \sum_{k=1}^{\infty} \frac{{}^{s}a_k}{\zeta_s^k} + \Gamma\cdot z \\ \psi(\zeta) &= \sum_{k=1}^{\infty} \frac{{}^{j}b_k}{\zeta_j^k} + \sum_{\substack{s=1\\ s\neq j}}^{n} \sum_{k=1}^{\infty} \frac{{}^{s}b_k}{\zeta_s^k} + \Gamma'\cdot z \end{aligned}\right\} \quad (1)$$

The first summation term in the expression is the Laurent's series representation corresponding to the j-th hole. The double summation part represents the interactions of other holes onto the j-th hole, it is written in its respective local coordinates. The third term relates the applied stress at infinity. In the present case, $\Gamma=N/2, \Gamma'=-N/2\, e^{-i2\alpha}$. In order to satisfy the boundary conditions of any one of n holes, for example, the j-th hole, we must transform the expression in eq. (1) to the local coordinate of the j-th hole. Apply mapping function

$$z = w(\zeta) = R\,(\zeta + m/\zeta)$$

in which $R=(a+b)/2$, $m=(a-b)/(a+b)$, a and b are semi-principal axes of an ellipse. Then the inverse function will be

$$\zeta = 1/2R*(z+\sqrt{z^2-4mR^2}), \quad \zeta^{-1} = 1/2mR*(z-\sqrt{z^2-4mR^2}) \quad (2)$$

Substituting (2) to (1), we obtain

$$\left.\begin{aligned}\phi(\zeta)&=\sum_{k=1}^{\infty}\frac{{}^ja_k}{\zeta_j^k}+\sum_{\substack{s=1\\ s\neq j}}^{n}\sum_{k=1}^{\infty}\frac{{}^sa_k}{(2m_sR_s)^k}\left(z_s-\sqrt{z_s^2-4m_sR_s^2}\right)^k+\Gamma . z\\ \psi(\zeta)&=\sum_{k=1}^{\infty}\frac{{}^jb_k}{\zeta_j^k}+\sum_{\substack{s=1\\ s\neq j}}^{n}\sum_{k=j}^{\infty}\frac{{}^sb_k}{(2m_sR_s)^k}\left(z_s-\sqrt{z_s^2-4m_sR_s^2}\right)^k+\Gamma' z\end{aligned}\right\}\quad(3)$$

$$\left.\begin{aligned}z_s&=z_je^{i(\alpha_j-\alpha_s)}-r_{is}e^{i(\beta_{is}-\alpha_s)}\\ z&=z_je^{i\alpha_j}+r_je^{i\beta_j}\end{aligned}\right\}\quad(4)$$

Substituting (4) into (3) and using the mapping function for the j-th hole $z_j = R_j(\zeta_j+\frac{m_j}{\zeta_j})$ we obtain the expression of the stress function in-form of local coordinate of the j-th hole on (ζ_j) plane

$$\begin{aligned}\phi(\zeta_j)=&\sum_{k=1}^{\infty}\frac{{}^ja_k}{\zeta_j^k}+\sum_{\substack{s=1\\ s\neq j}}^{n}\sum_{k=1}^{\infty}\frac{{}^sa_k}{(2m_sR_s)^k}\Big\{\Big[R_j(\zeta_j+\frac{m_j}{\zeta_j})e^{i(\alpha_j-\alpha_s)}-r_{js}e^{i(\beta_{js}-\alpha_s)}\Big]\\ &-\sqrt{[R_j(\zeta_j+\frac{m_j}{\zeta_j})e^{i(\alpha_j-\alpha_s)}-r_{js}e^{i(\beta_{js}-\alpha_s)}]^2-4m_sR_s^2}\Big\}^k+\Gamma\cdot\Big[R_j(\zeta_j+\frac{m_j}{\zeta_j})e^{i\alpha_j}+r_je^{i\beta_j}\Big]\\ \psi(\zeta_j)=&\sum_{k=1}^{\infty}\frac{{}^jb_k}{\zeta_j^k}+\sum_{\substack{s=1\\ s\neq j}}^{n}\sum_{k=1}^{\infty}\frac{{}^sb_k}{(2m_sR_s)^k}\Big\{\Big[R_j(\zeta_j+\frac{m_j}{\zeta_j})e^{i(\alpha_j-\alpha_s)}-r_{js}e^{i(\beta_{js}-\alpha_s)}\Big]-\\ &\sqrt{[R_j(\zeta_j+\frac{m_j}{\zeta_j})e^{i(\alpha_j-\alpha_s)}-r_{js}e^{i(\beta_{js}-\alpha_s)}]^2-4m_sR_s^2}\Big\}^k+\Gamma'\cdot\Big[R_j(\zeta_j+\frac{m_j}{\zeta_j})e^{i\alpha_j}+r_je^{i\beta_j}\Big]\end{aligned}\quad(5)$$

2. Boundary Condition

The boundary condition of the j-th hole is

$$\phi(\sigma_j)+w(\sigma_j)\frac{\overline{\phi'(\sigma_j)}}{\overline{w'(\sigma_j)}}+\overline{\psi(\sigma_j)}=F(\sigma_j)+C_j\quad(6)$$

The conjugated form of eq. (6) is

$$\overline{\phi(\sigma_j)}+\overline{w(\sigma_j)}\frac{\phi'(\sigma_j)}{w'(\sigma_j)}+\psi(\sigma_j)=\overline{F(\sigma_j)}+\overline{C_j}\quad(7)$$

On the boundary of the j-th hole, $\sigma_j = e^{i\theta_j}$. The free boundary condition gives $F(\sigma_j)=\overline{F(\sigma_j)} = 0$. In case of n holes, there exist (n-1) constants,

but they can be eliminated.

With $F(\sigma_j) = 0$, the boundary condition equation (6), by Harnack's theorem is equivalent to the expression as:

$$\oint\frac{f(\sigma_j)}{\sigma_j-\zeta}\,d\sigma_j = 0$$

Substituting $\frac{1}{\sigma_j-\zeta} = -\sum_{p=0}^{\infty}\frac{\sigma_j^p}{\zeta^{p+1}}$ into above equation, we obtain

$$-\sum_{p=0}^{\infty}\frac{1}{\zeta_j^{p+1}}\oint\sigma_j^p\cdot f(\sigma_j)\,d\sigma_j = 0$$

Since ζ is the region of the σ_j, $\zeta_j^{p+1}\neq 0$, hence

$$\oint_{\sigma_j}\sigma_j^p\, f(\sigma_j)d\sigma_j = 0,\ (p = 0,1,2,\dots,\infty)$$

First we substitute $(\sigma_j)^p\, d\sigma_j = i\, e^{i(p+1)\theta}d\theta_j$ and then perform the integration within the interval of $0\sim 2\pi (p=0, 1, 2, \dots\infty)$. Separating the real and imeginary parts of the equation, we obtain two sets of linear equations. Similarly, from equation (7) we get another sets of linear equations. Then we have a system of total number of 4K linear equations used to satisfy exactly the boundary condition of the j-th hole. The rest of holes are treated exactly in the same way as for the j-th hole, so a total of 4nK number of linear equations are obtained which matches with the number of unknown coefficients.

3. Calculation of Stress Intensity Factors for Crack

With the result of stress concentration factors of the varions ellipses, following fomula is used to compute the stress intensity factor of the crack[7]

$$K_I = \frac{1}{2}\sqrt{\pi}\lim_{\rho\to 0}\sqrt{\rho}\cdot\sigma_{max}$$

here, $\rho = b^2/a$. For single elliptic hole, a reasonable accuracy may be obtained by extrapolating K_I' curve calculated from the equation $K_I'=\frac{1}{2}\sqrt{\pi\rho}\cdot\sigma_{max}$ for $\rho>0$. The extrapolation curve has fairly good linearity for $(b+2a)/\sqrt{a}$ as shown in the following table a=1, N=1

b/a	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3
$(\sigma_\theta)_{max}$	3.000	3.222	3.500	3.857	4.333	5.000	6.000	7.666
K_I'	2.658	2.570	2.481	2.393	2.304	2.215	2.127	2.038

Hence, the value of K_I can be calculated by linear extrapolation formula

$$K_I = \frac{C_2(K_I')_1 - C_1(K_I')_2 + 2\sqrt{a}\,[(K_I')_2 - (K_I')_1]}{C_2 - C_1} \qquad (8)$$

here, $C_2 > C_1$; C_1, C_2 and $(K_I')_1$, $(K_I')_2$ are values of $(b+2a)/\sqrt{a}$ and K_I' at the first and second points, respectively. Using above table and formula (8), we obtain $K_I = 1.77$, while the theoretical value of $K_I = \sqrt{\pi a}\, N = 1.77$.

For the case of a crack surrounded by many holes, the K_I value of the crack may be computed as follows: the crack is first replaced by various ellipses of different b-values. We can calculate the corresponding K_I' -values. The K_I-values of the crack can then be determined through formula (8). Since the stress concentration factors can reflect the interactions between holes, the calculation shows that the value of b may not necessary be very small for fairly good result. For cases of multi-crack, if K_I of certain crack is intended, the same procedure is introduced with the equivalent transformation of other cracks into a set of proper ellipses.

NUMERICAL EXAMPLES

Example one: as shown in figure (2), only changing the value of b of the first hole, we obtain K_I', from which the K_I of the crack can be extrapolated. The calculating results are tabulated in the following table.

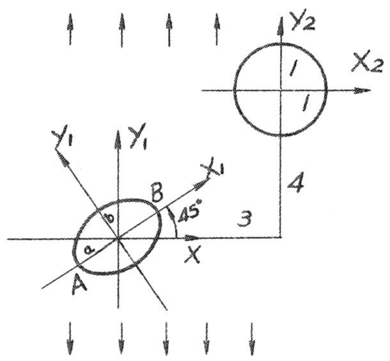

Fig.2 $N = 1$

1-st hole	a=1 b=1		a=0.9 b=1		a=0.8 b=1	
Point place	A	B	A	B	A	B
σ_θ	2.178	2.208	2.340	2.373	2.543	2.579
K_I'	1.930	1.953	1.866	1.893	1.803	1.828

with extrapolation formula, we obtain

$(K_I)_A = 1.29$, $(K_I)_B = 1.31$

Example two:

Five equal circular holes are disposed uniformly on an infinite plate, as shown in figure (3). We compare two results as follows: the first result is calculated by the method of this paper and the other by photo-elastic method in reference[8]. Here, the distance between the center of side holes from that of the central hole is 4.73 R (R=1). The magnitude of σ_θ at three points on the periphery of the central hole are tabulated. Apparently, the agreement is fairly good.

Point place	A	B	C
Cal. value	2.866	1.052	-0.909
Expt. value	2.82	1.013	-0.94

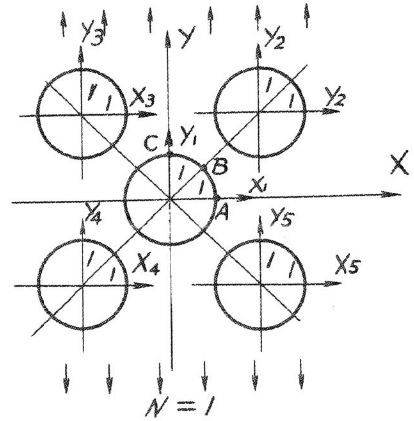

Fig. 3

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