THE GENERALIZED J-INTEGRAL OF COMBINED MODES

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I. INTRODUCTION

It is well known that, J-integral provided by Rice is a powerful tool to calculate stress intensity factors. But the usage of this method is restricted by the following conditions:

A. uniform thickness B. absence of body force

C. uniform temperature D. Crack of first mode

Recently, some authors broke through a few of the above restrictions [1], [2], [3].

The purpose of this paper is to develop the concept and method of J-integral completely for determining stress intensity factors of turbo-disk in aeronautical engineering.

II. GENERALIZED J-INTEGRAL OF MIXED MODE AND ENERGY DIFFERENCE RATE

Fig. 1 shows a plate with a notch (a,b). The root of the notch is semi-circular and is denoted by Γ_t. The radius of Γ_t is ρ.

The total energy π of the system is

$$\pi = \int_A (U-Q-B_i u_i) t dA - \int_{C_S} S_i u_i t ds \quad (2.1)$$

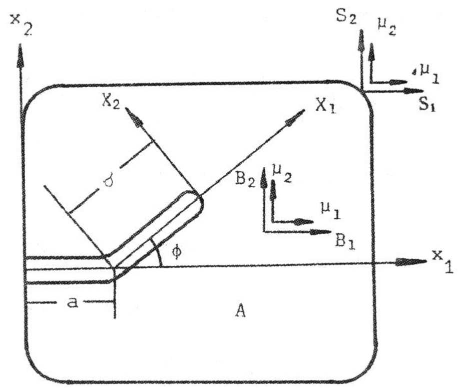

Fig. 1

where, U is intrinsic energy density, Q is heat density, c_s is boundary, t is the thickness of the plate. If the dimension of the notch is changed by Δb, then,

$$\pi + \Delta\pi = \int_{A-\Delta A} [(U+\Delta U) - (Q+\Delta Q) - B_i(u_i+\Delta u_i)]\, tdA - \int_{c_s} S_i(u_i+\Delta u_i)\, tds \tag{2.2}$$

According to the Rice's argument [1], we have

$$\int_{c_s} S_i \Delta u_i tds = \int_{c+\Delta c} (S_i+\Delta S_i)\Delta u_i tds \tag{2.3}$$

and from the principle of virtual work, the following equality is valid

$$\int_{c+\Delta c} (S_i+\Delta S_i)\Delta u_i tds + \int_{A-\Delta A} B_i \Delta u_i tdA = \int_{A-\Delta A} (\sigma_{ij}+\Delta\sigma_{ij})\Delta\varepsilon_{ij} tdA \tag{2.4}$$

From the above equalities, the energy differenc rate G will be equal to

$$G = -\frac{1}{t_b}\cdot\frac{d\pi}{db} = -\frac{1}{t_b}\lim_{\Delta b\to 0}\frac{\Delta\pi}{\Delta b} = \frac{1}{t_b}\left\{\int_{-\Gamma_t} (U-Q-B_i u_i)tdx_2 + \right.$$

$$\left. + \int_A (\sigma_{ij}\frac{d\varepsilon_{ij}}{db} - \frac{dU}{db} + \frac{dQ}{db})\, tdA\right\} \tag{2.5}$$

On account of the definitions of free energy density W and entropy density η, equation (5) can be rewritten as follows

$$G = \frac{1}{t_b}\left\{\int_{-\Gamma_t} WtdX_2 + \int_{-\Gamma_t} (T\eta - Q - B_i u_i)tdX_2 + \int_A (\sigma_{ij}\frac{d\varepsilon_{ij}}{db} - \frac{dW}{db} - \eta\frac{dT}{db})tdA\right\} \tag{2.6}$$

From Fig. 1, we can see that

$$x_1 = X_1\cos\phi - X_2\sin\phi + b\cos\phi + a$$

$$x_2 = X_1\sin\phi + X_2\cos\phi + b\sin\phi \tag{2.7}$$

$$\frac{dP}{db} = \frac{\partial P}{\partial b} - \frac{\partial P}{\partial x_1} = \frac{\partial P}{\partial b} - (\frac{\partial P}{\partial x_1}\cos\phi + \frac{\partial P}{\partial x_2}\sin\phi) \tag{2.8}$$

and from differentiation rules, we have

$$\frac{\partial W}{\partial b} = \frac{\partial W}{\partial \varepsilon_{ij}}\frac{\partial \varepsilon_{ij}}{\partial b} + \frac{\partial W}{\partial T}\frac{\partial T}{\partial b} \tag{2.9}$$

Furthermore, for any elastic material, the following constitutive equations are valid

$$\frac{\partial W}{\partial \varepsilon_{ij}} = \sigma_{ij}, \quad \frac{\partial W}{\partial T} = -\eta \tag{2.10}$$

On account of the above equalities, G can be rewritten as follows

$$G = G_1\cos\phi + G_2\sin\phi \tag{2.11}$$

where

$$G_1 = \frac{1}{t_b}\left\{\int_{-\Gamma_t} Wtdx_2 + \int_{-\Gamma_t} (T\eta - Q + B_i u_i)tdx_2 - \right.$$

$$\left. - \int_A (\sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial x_1} - \frac{\partial W}{\partial x_1} + \frac{\partial W}{\partial T}\frac{\partial T}{\partial x_1})tdA\right\}$$

$$G_2 = \frac{1}{t_b}\left\{-\int_{-\Gamma_t} Wtdx_1 - \int_{-\Gamma_t} (T\eta - Q + B_i u_i)tdx_1 - \right.$$

$$\left. - \int_A (\sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial x_2} - \frac{\partial W}{\partial x_2} + \frac{\partial W}{\partial T}\frac{\partial T}{\partial x_2})tdA\right\} \tag{2.12}$$

From the principle of virtual work and Green's formula, we have

$$\int_A \sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial x_1} tdA = \int_A B_i\frac{\partial u_i}{\partial x_1} tdA + \int_c S_i\frac{\partial u_i}{\partial x_1} tds$$

$$\int_A \sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial x_2} tdA = \int_A B_i\frac{\partial u_i}{\partial x_2} tdA + \int_c S_i\frac{\partial u_i}{\partial x_2} tds \tag{2.13}$$

$$\int_A \frac{\partial W}{\partial x_1} tdA = \int_c Wtdx_2 - \int_A W\frac{\partial t}{\partial x_1} dA$$

$$\int_A \frac{\partial W}{\partial x_2} tdA = -\int_c Wtdx_1 - \int_A W\frac{\partial t}{\partial x_2} dA \tag{2.14}$$

By means of equations (13) and (14), we can transform G_1 and G_2 into following forms

$$G_1 = \frac{1}{t_b}\left\{\int_{c-\Gamma_t} Wtdx_2 - \int_A \left(W\frac{\partial t}{\partial x_1} + \frac{\partial W}{\partial T}\frac{\partial T}{\partial x_1} t\right)dA - \right.$$
$$\left. - \int_c S_i \frac{\partial u_i}{\partial x_1} tds - \int_A B_i \frac{\partial u_i}{\partial x_1} tdA \right\} + G^*_1$$

$$G_2 = \frac{1}{t_b}\left\{- \int_{c-\Gamma_t} Wtdx_1 - \int_A \left(W\frac{\partial t}{\partial x_2} + \frac{\partial W}{\partial T}\frac{\partial T}{\partial x_2} t\right)dA - \right.$$
$$\left. - \int_c S_i \frac{\partial u_i}{\partial x_2} tds - \int_A B_i \frac{\partial u_i}{\partial x_2} tdA \right\} + G^*_2 \qquad (2.15)$$

where,

$$t_b G^*_1 = \int_{-\Gamma_t} (T\eta - Q + B_i u_i) tdx_2$$
$$t_b G^*_2 = \int_{-\Gamma_t} (T\eta - Q + B_i u_i) tdx_1 \qquad (2.16)$$

When $\rho \to 0$, the notch becomes a crack and

$$\lim_{\rho\to 0} G^*_1 = 0, \quad \lim_{\rho\to 0} G^*_2 = 0 \qquad (2.17)$$

because there is no singular point of temperature field. Then,

$$G_1 = \lim_{\rho\to 0} \frac{1}{t_b}\left\{\int_{c-\Gamma_t} Wtdx_2 - \int_A \left(W\frac{\partial t}{\partial x_1} + \frac{\partial W}{\partial T}\frac{\partial T}{\partial x_1} t\right)dA - \right.$$
$$\left. - \int_c S_i \frac{\partial u_i}{\partial x_1} tds - \int_A B_i \frac{\partial u_i}{\partial x_1} tdA \right\}$$

$$G_2 = \lim_{\rho\to 0} \frac{1}{t_b}\left\{- \int_{c-\Gamma_t} Wtdx_1 - \int_A \left(W\frac{\partial t}{\partial x_2} + \frac{\partial W}{\partial T}\frac{\partial T}{\partial x_2} t\right)dA - \right.$$
$$\left. - \int_c S_i \frac{\partial u_i}{\partial x_2} tds - \int_A B_i \frac{\partial u_i}{\partial x_2} tdA \right\} \qquad (2.18)$$

III. GENERALIZED J-INTEGRAL OF MIXED MODE AND ITS PATH INDEPENDENCE PROPERTY

Let Γ be a curve ABCDE as shown in Fig. 2. Both the starting paint A and the terminating point E are distant the same infinitesimal length ε from the crack tip. Let Ω denote the area surrounded by Γ. Then the generalized J-integral of mixed mode can be defined as follows

$$J = J_1\cos\phi + J_2\sin\phi \qquad (3.1)$$

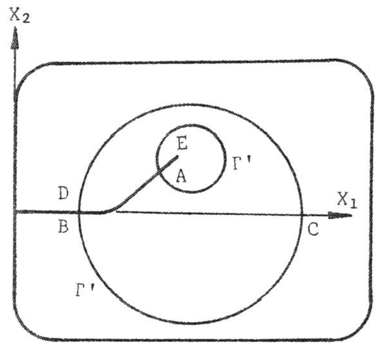

Fig. 2

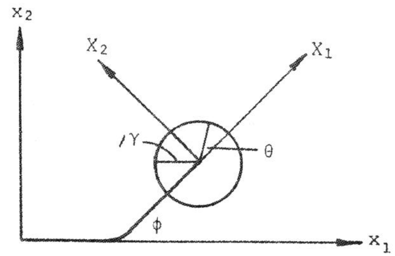

Fig. 3

in the above equation

$$J_1 = \lim_{\varepsilon\to 0} \frac{1}{t_b}\left\{\int_\Gamma Wtdx_2 - \int_\Omega W\frac{\partial t}{\partial x_1}dA - \int_\Omega \frac{\partial W}{\partial T}\frac{\partial T}{\partial x_1} tdA - \right.$$
$$\left. - \int_\Gamma S_i \frac{\partial u_i}{\partial x_1} tds - \int_\Omega B_i \frac{\partial u_i}{\partial x_1} tdA \right\} \qquad (3.2)$$

$$J_2 = \lim_{\varepsilon\to 0} \frac{1}{t_b}\left\{ - \int_\Gamma Wtdx_1 - \int_\Omega W\frac{\partial t}{\partial x_1}dA - \int_\Omega \frac{\partial W}{\partial T}\frac{\partial T}{\partial x_2} tdA - \right.$$
$$\left. - \int_\Gamma S_i \frac{\partial u_i}{\partial x_2} tds - \int_\Omega B_i \frac{\partial u_i}{\partial x_2} tdA \right\} \qquad (3.3)$$

Next, we shall prove that J_1 and J_2 are indepedent of the path of integration. Therefore, we introduce a curve Γ' which is similiar to Γ, and an area Ω' surrounded by Γ'. Let

$$\Gamma^* = \Gamma - \Gamma' \quad , \quad \Omega^* = \Omega - \Omega' \qquad (3.4)$$

then,

$$J - J' = (J_1 - J'_1)\cos\phi + (J_2 - J'_2)\sin\phi \qquad (3.5)$$

$$J_1 - J_1' = \frac{1}{t_b}\Big\{\int_{\Gamma^*} W t dx_2 - \int_{\Omega^*} W\frac{\partial t}{\partial x_1} dA - \int_{\Omega^*} \frac{\partial W}{\partial T}\frac{\partial T}{\partial x_1} t dA - \\ - \int_{\Gamma^*} S_i \frac{\partial u_i}{\partial x_1} t ds - \int_{\Omega^*} B_i \frac{\partial u_i}{\partial x_1} t dA\Big\} \tag{3.6}$$

$$J_2 - J_2' = \frac{1}{t_b}\Big\{ - \int_{\Gamma^*} W t dx_1 - \int_{\Omega^*} W\frac{\partial t}{\partial x_2} dA - \int_{\Omega^*} \frac{\partial W}{\partial T}\frac{\partial T}{\partial x_2} t dA - \\ - \int_{\Gamma^*} S_i \frac{\partial u_i}{\partial x_2} t ds - \int_{\Omega^*} B_i \frac{\partial u_i}{\partial x_2} t dA\Big\} \tag{3.7}$$

According to Green's formula and (2.4) we have

$$\int_{\Gamma^*} W t dx_2 = \int_{\Omega^*} \sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial x_1} t dA + \int_{\Omega^*} \frac{\partial W}{\partial T}\cdot\frac{\partial T}{\partial x_1} t dA + \int_{\Omega^*} W\frac{\partial t}{\partial x_1} dA \tag{3.8}$$

$$-\int_{\Gamma^*} W t dx_1 = \int_{\Omega^*} \sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial x_2} t dA + \int_{\Omega^*} \frac{\partial W}{\partial T}\cdot\frac{\partial T}{\partial x_2} t dA + \int_{\Omega^*} W\frac{\partial t}{\partial x_2} dA \tag{3.9}$$

Furthermore by the principle of virtual work, it can be shown that

$$\int_{\Omega^*} \sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial x_1} t dA = \int_{\Gamma^*} S_i\frac{\partial u_i}{\partial x_1} t ds + \int_{\Omega^*} B_i\frac{\partial u_i}{\partial x_1} t dA \tag{3.10}$$

$$\int_{\Omega^*} \sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial x_2} t dA = \int_{\Gamma^*} S_i\frac{\partial u_i}{\partial x_2} t ds + \int_{\Omega^*} B_i\frac{\partial u_i}{\partial x_2} t dA \tag{3.11}$$

From the above equations, we can prove that

$$J_1 = J_1' \qquad J_2 = J_2' \tag{3.12}$$

IV. GENERALIZED J-INTEGRAL OF MIXED MODE AND STRESS INTENSITY FACTORS

Now let us establish the relations between J-integrals and stress intensity factors, for the cracked plate shown in Fig.3. We take the boundary of a cracked circle centered at crack tip with infinitisimal radius as Γ.

We proved that, for a plate with variable thickness subjected to both surface tractions and body forces in the non-uniform temperature

field, the stress and displacement field in the vicinity of crack tip are same with the fields of a plate with uniform thickness, without body forces in the uniform temperature field.

In the plane stress problems, the stress, displacement and strain energy fields are given as follows

$$\sigma^*_{ij} = \sqrt{\frac{1}{2r}}\, a_{p,ij}\, K_p; \quad u^*_i = \frac{1}{2\mu}\sqrt{\frac{r}{2}} b_{p,i}\, K_p; \quad W = \frac{1}{r}\, C_{pq} K_p K_q \tag{4.1}$$

From the above equations we can obtain

$$\int_\Gamma W t dx_2 = t_b \int_{-\pi}^{\pi} W r\cos(\phi+\theta) d\theta = t_b K_p K_q\{\cos\phi A_{1,pq} - \sin\phi A_{2,pq}\} \tag{4.2}$$

and on acount of that $W(r,\pi) = W(r,-\pi)$, we have

$$\int_\Gamma W t dx_1 = -t_b \int_{-\pi}^{\pi} W r\sin(\phi+\theta) d\theta = -t_b K_p K_q\{\sin\phi A_{1,pq} + \cos\phi A_{2,pq}\} \tag{4.3}$$

where,

$$A_{1,11} = \int_{-\pi}^{\pi} C_{11}\cos\theta d\theta, \qquad A_{1,22} = \int_{-\pi}^{\pi} C_{22}\cos\theta d\theta$$

$$A_{1,12} = A_{1,21} = \int_{-\pi}^{\pi} C_{12}\cos\theta d\theta, \qquad A_{2,12} = A_{2,21} = \int_{-\pi}^{\pi} C_{12}\sin\theta d\theta \tag{4.4}$$

$$A_{2,11} = \int_{-\pi}^{\pi} C_{11}\sin\theta d\theta, \qquad A_{2,22} = \int_{-\pi}^{\pi} C_{22}\sin\theta d\theta$$

For the curve given in Fig. 3, it can be shown that

$$\int_\Gamma S_i\frac{\partial u_i}{\partial x_1} t ds = L_{ij} L_{ik} L_{m_1} t_b B_{jkmpq} K_p K_q \tag{4.5}$$

$$\int_\Gamma S_i\frac{\partial u_i}{\partial x_1} t ds = L_{ij} L_{ik} L_{m_2}\, t_b B_{jkmpq} K_p\, K_q \tag{4.6}$$

where

$$L_{11} = L_{22} = \cos\phi, \quad L_{21} = -L_{12} = \sin\phi$$

$$B_{jkmpq} = \frac{1}{8\mu}\int_{-\pi}^{\pi} a_{p,j} n^1{}_n (b_{pqk}\xi_m + \frac{db_{qk}}{d\theta}\eta_m) d\theta \tag{4.7}$$

$$l_1 = \cos\theta, \quad l_2 = \sin\theta; \quad \xi_m = \frac{\partial r}{\partial x_m}, \quad \eta_m = 2r\frac{\partial\theta}{\partial x_m}$$

We can prove that, when $r \to 0$, all the area integrals in J-integral will approach to zero, i.e.

$$\lim_{r\to 0}\int_{\Omega} W\frac{\partial t}{\partial x_j}\,dA = 0\ , \qquad \lim_{r\to 0}\int_{\Omega}\frac{\partial W}{\partial T}\frac{\partial T}{\partial x_j}\,t\,dA = 0$$

$$\lim_{r\to 0}\int_{\Omega} B_i\frac{\partial u_i}{\partial x_j}\,t\,dA = 0 \qquad (4.8)$$

From the above analysis, we can obtain the following relations

$$J_1 = \frac{\pi}{E}(K_1^2+K_2^2)\cos\phi + \frac{\pi}{E}2K_1K_2\sin\phi$$

$$J_2 = \frac{\pi}{E}(K_1^2+K_2^2)\sin\phi - \frac{\pi}{E}2K_1K_2\cos\phi \qquad (4.9)$$

When $\phi=0$, the above relations become

$$J_1 = \frac{\pi}{E}(K_1^2+K_2^2)\ , \qquad J_2 = -2\frac{\pi}{E}K_1K_2 \qquad (4.10)$$

V. CALCULATION RESULTS

We used this method to calculate the stress intensity factors of a plate of variable thickness by finite element technique. The results are shown in Table 1

Table 1

Γ	a=15mm J_1	β=15° J_2	a=15mm J_1	β=30° J_2
1	2.13962	−0.20187	1.90402	−0.30067
2	2.14086	−0.20231	1.91473	−0.29721
3	2.14879	−0.20342	1.90762	−0.29282
m	2.14309	−0.20253	1.90879	−0.29690
	K_1=118.318	K_2=22.853	K_1=92.891	K_2=46.953

Authors of [4] adopted our method to calculate the stress intensity factors of a rotating disk by triangular elements and isoparametric elements. Now, the results are shown in tables 2 and 3.

Table 2

Γ	a=0,9mm(45°) J_1 kg/mm	J_2 kg/mm	a=1,0mm(40°) J_1 kg/mm	J_2 kg/mm
1	0.011216	−0.007377	0.013867	−0.007298
2	0.011252	−0.007366	0.013841	−0.007419
3	0.011201	−0.007314	0.013800	−0.007516
e	0.25%	0.5%	0.26%	1.50%

Table 3

elements	a=2.0mm(45°) J_1 kg/mm	J_2 kg/mm	a=2.5mm(40°) J_1 kg/mm	J_2 kg/mm
triangular	0.013157	−0.00941	0.018075	−0.006867
isoparametic	0.012906	−0.01020	0.017828	−0.007061

From the above two Tables we can see that the results are satisfactrily well.

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