NEAR-TIP FIELDS FOR PLANE-STRAIN MODE-I STEADY CRACK GROWTH IN LINEAR HARDENING MATERIAL WITH BAUSCHINGER EFFECT

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For linear isotropic hardening material, Amazigo and Hutchinson[1] have obtained the singularity fields at the tip of a steadily growing crack. They neglected plastic reloading along the flank behind the crack tip, which may be an important feature of the plane strain problem. Besides, for most of engineering materials the hardening is anisotropic with Bauschinger effect. In this paper the constitutive law for anisotropic hardening suggested by Kadaschevich and Novozhilov[2] is used to obtain the near-tip fields for plane-strain mode-I steady crack growth, with the reloading zone being considered.

BASIC EQUATIONS

The constitutive equations for linear anisotropic hardening material are taken in the form (see [2])

$$\dot{\varepsilon}^p_{ij} = \frac{1}{2h\overset{o}{\sigma}_e} \dot{\overset{o}{\sigma}}_e \overset{o}{\sigma}{}'_{ij} \tag{1}$$

$$\alpha_{ij} = 2g\varepsilon^p_{ij} \tag{2}$$

where α_{ij} — stresses corresponding to the center of the yielding surface, superscript "p" — plastic strain components, superdot "." — the time-derivative d/dt, supercirclet "o" — active stress components ($\overset{o}{\sigma}_{ij}=\sigma_{ij}-\alpha_{ij}$), prime "," — the deviator components ($\overset{o}{\sigma}{}'_{ij} = \sigma^o_{ij} - \frac{1}{3}\overset{o}{\sigma}_{kk}\delta_{ij}$), and $\overset{o}{\sigma}_e$ — the equivalent active stress,

$$\overset{o}{\sigma}_e = (\frac{3}{2}\overset{o}{\sigma}{}'_{ij}\overset{o}{\sigma}{}'_{ij})^{\frac{1}{2}}$$

$$\dot{\sigma}^o_e = \frac{3}{2}\sigma^{o\prime}_{ij}\dot{\sigma}^o_{ij}/\sigma^o_e = \frac{3}{2}\beta\overset{o}{\sigma}{}'_{ij}\dot{\sigma}_{ij}/\sigma^o_e \qquad (i,j=1,2,3) \qquad (3)$$

Here h and g are material constants, namely

$$\frac{1}{h} = \frac{3}{\beta}\left(\frac{1}{E_t} - \frac{1}{E}\right), \qquad \frac{1}{g} = \frac{3}{1-\beta}\left(\frac{1}{E_t} - \frac{1}{E}\right) \qquad (4)$$

where E — Young's modulus, E_t — tangent modulus, and β — parameter related to anisotropy of hardening with the extreme values $\beta = 1$ for isotropic hardening and $\beta = 0$ for ideal Bauschinger effect.

Let x_1, x_2 be the moving cartesian coordinates with origin at the crack-tip. Denote by $\underset{\sim}{\sigma}$ the stress tensor and $\underset{\sim}{\mathcal{S}} = \dot{\underset{\sim}{\sigma}}$ the rate of stress tensor, then the components of $\underset{\sim}{\mathcal{S}}$ can be expressed in terms of the rate of stress function $\dot{\phi}$ as follows

$$\mathcal{S}_{11} = \dot{\sigma}_{11} = \frac{\partial^2\dot{\phi}}{\partial x_2^2}, \quad \mathcal{S}_{22} = \dot{\sigma}_{22} = \frac{\partial^2\dot{\phi}}{\partial x_1^2}, \quad \mathcal{S}_{12} = \dot{\sigma}_{12} = -\frac{\partial^2\dot{\phi}}{\partial x_1\partial x_2} \qquad (5)$$

or, in polar coordinates (r, θ) centered at the tip,

$$\mathcal{S}_{rr} = \frac{1}{r}\frac{\partial\dot{\phi}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\dot{\phi}}{\partial\theta^2}, \quad \mathcal{S}_{\theta\theta} = \frac{\partial^2\dot{\phi}}{\partial r^2}, \quad \mathcal{S}_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\dot{\phi}}{\partial\theta}\right) \qquad (5)'$$

Note that $\mathcal{S}_{rr} \neq \dot{\sigma}_{rr}$, etc. The components of strain tensor $\underset{\sim}{\varepsilon}$ and rate of strain tensor $\underset{\sim}{\mathcal{E}} = \dot{\underset{\sim}{\varepsilon}}$ can be expressed in terms of components of displacement vector $\underset{\sim}{u}$ and velocity vector $\underset{\sim}{v}$ respectively by formulas of the same form, for instance,

$$\mathcal{E}_{11} = \frac{\partial v_1}{\partial x_1}, \quad \mathcal{E}_{22} = \frac{\partial v_2}{\partial x_2}, \quad \mathcal{E}_{12} = \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) \qquad (6)$$

or, in polar coordinates, (noting again $\mathcal{E}_{rr} \neq \dot{\varepsilon}_{rr}$ etc.)

$$\mathcal{E}_{rr} = \frac{\partial v_r}{\partial r}, \quad \mathcal{E}_{\theta\theta} = \frac{1}{r}\left(\frac{\partial v_\theta}{\partial\theta} + v_r\right), \quad \mathcal{E}_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r}v_\theta\right) \qquad (6)'$$

Referring to the results for isotropic hardening ($\beta = 1$) in [1], we

shall look for solutions of the form, corresponding to dominant singularity

$$\dot{\phi} = A_o r^{s+1} f_o(\theta) \qquad (7)$$

$$\{\mathcal{S}_{ij}, \dot{\sigma}^o_e\} = A_o r^{s-1}\{t_{ij}(\theta), t^o(\theta)\} \qquad (8)$$

$$\{\sigma_{ij}, \sigma^o_e, \overset{o}{\sigma}{}'_{ij}\} = A_o r^s\{\Sigma_{ij}(\theta), \Sigma^o(\theta), S_{ij}(\theta)\} \qquad (9)$$

$$\{v_1, v_2\} = A_o r^s\{g_o(\theta), h_o(\theta)\} \qquad (10)$$

$$\{u_1, u_2\} = A_o r^{s+1}\{G_o(\theta), H_o(\theta)\} \qquad (11)$$

$$\mathcal{E}_{\lambda\omega} = A_o r^{s-1}\psi_{\lambda\omega}(\theta), \quad \varepsilon_{\lambda\omega} = A_o r^s E_{\lambda\omega}(\theta) \qquad (12)$$

where A_o is an amplitude factor, s and functions of θ are to be determined, and from (3),

$$S_{ii}(\theta) = 0, \quad \Sigma^o(\theta) = \{\frac{3}{2}S_{ij}(\theta)S_{ij}(\theta)\}^{\frac{1}{2}}$$

$$t^o(\theta) = \frac{3}{2}\beta S_{ij}(\theta)\,t_{ij}(\theta)/\Sigma^o(\theta) \qquad (13)$$

For plane strain, $\varepsilon_{33} = \varepsilon^e_{33} + \varepsilon^p_{33} = 0$ (superscript "e" denoting elastic components), we have $\psi_{33}(\theta) = E_{33}(\theta) = 0$, and plastic imcompressibility requires

$$E_{\lambda\lambda}(\theta) = \frac{1-2\nu}{E}\Sigma_{ii}(\theta), \quad \psi_{\lambda\lambda}(\theta) = \frac{1-2\nu}{E}t_{ii}(\theta) \quad (\lambda=1,2) \qquad (14)$$

From (5)', using (7), (8), and denoting $d/d\theta$ by "," hereafter,

$$t_{rr}(\theta) = (s+1)f_o(\theta) + f''_o(\theta)$$

$$t_{\theta\theta}(\theta) = s(s+1)f_o(\theta), \quad t_{r\theta} = -sf'_o(\theta) \qquad (15)$$

from which cartesian components $t_{ij}(\theta)$ can be easily obtained. Similarly, (6), with (10), (12), leads to

$$\psi_{11}(\theta) = s\cos\theta g_o(\theta) - \sin\theta g'_o(\theta)$$

$$\psi_{22}(\theta)=s\ \sin\theta h_o(\theta)+\cos\theta h_o'(\theta) \tag{16}$$
$$\psi_{12}(\theta) = \tfrac{1}{2}\{(g_o'(\theta)+s\ h_o(\theta))\cos\theta+(s\ g_o(\theta)-h_o'\ (\theta))\ \sin\theta\}$$

Identify the time parameter t with the increase in crack length, so that in steady state we have for scalars or tensors ( )

$$(\ \dot{}\ ) = \frac{d}{dt}(\ ) = -\frac{\partial}{\partial x}(\ \ ) \tag{17}$$

Applied to stress tensor $\underset{\sim}{\sigma}$ and strain tensor $\underset{\sim}{\varepsilon}$, (17) gives, respectively,

$$\sin\theta\ \Sigma_{ij}'(\theta) = s\ \cos\theta\ \Sigma_{ij}(\theta) + t_{ij}(\theta) \tag{18}$$

$$\sin\theta\ E_{\lambda\omega}'(\theta) = s\ \cos\theta E_{\lambda\omega}(\theta) + \psi_{\lambda\omega}(\theta)\quad (\lambda,\omega=1,2) \tag{19}$$

Some of the equations in (18) and (19) are integrable after substituting (15) and (16) into them, and lead to

$$\begin{aligned} \Sigma_{12}(\theta) &= (s+1)\sin\theta\ f_o(\theta)+\cos\theta\ f_o'(\theta) \\ \Sigma_{22}(\theta) &= -(s+1)\cos\theta\ f_o(\theta)+\sin\theta\ f_o'(\theta) \\ E_{11}(\theta) &= -\ g_o(\theta) \end{aligned} \tag{20}$$

the remaining equations in (18) and (19) are

$$\sin\theta\ \Sigma_{11}'(\theta) = s\ \cos\theta\ \Sigma_{11}(\theta) + t_{11}(\theta) \tag{21}$$

$$\sin\theta\ \Sigma_{33}'(\theta) = s\ \cos\theta\ \Sigma_{33}(\theta) + t_{33}(\theta) \tag{22}$$

$$\sin\theta\ E_{12}'(\theta) = s\ \cos\theta\ E_{12}(\theta) + \psi_{12}(\theta) \tag{23}$$

$$\sin\theta\ E_{22}'(\theta) = s\ \cos\theta\ E_{22}(\theta) + \psi_{22}(\theta) \tag{24}$$

With $E_{22}(\theta)$, $t_{33}(\theta)$ obtained from (14), eq. (24) will be identically satisfied. From (8), (9), (12) and Hooke's law, the constitutive eq. (1) is reduced to

$$\psi_{\lambda\omega}(\theta) = \frac{1+\nu}{E}\ t_{\lambda\omega}(\theta) - \frac{\nu}{E}\ t_{ii}(\theta)\delta_{\lambda\omega}+ \frac{\mu}{2h}\ t^o(\theta)S_{\lambda\omega}(\theta)/\Sigma^o(\theta) \tag{25}$$

where μ= 1 for plastic loading, and μ= 0 for elastic responses, and

$$S_{\lambda\omega}(\theta) = (1+2g\ \frac{1+\nu}{E})\Sigma_{\lambda\omega}(\theta)-(1/3+\ \frac{2g}{E}\ \nu)\Sigma_{ii}(\theta)\delta_{\lambda\omega}-2gE_{\lambda\omega}(\theta) \tag{26}$$

(21)–(23) and (25) are the six governing equations for plastic zone (μ=1) as well as for unloading zone (μ=0). The functions $G_o(\theta)$, $H_o(\theta)$ for displacements can be determined through the following relations

$$\begin{aligned} \sin\theta\ G_o'\ (\theta) &= (s+1)\cos\theta\ G_o(\theta) + g_o(\theta) \\ \sin\theta\ H_o'(\theta) &= (s+1)\ \cos\theta\ H_o(\theta) + h_o(\theta) \end{aligned}$$

CONTIGUITY CONDITIONS

The crace-tip geometry is shown in Fig. 1. For hardening materials, stresses and strains should be continuous across boundary Γ between neighboring zones. Denoting the jump in a quantity across Γ by $[\,]_\Gamma$, we have the contiguity conditions

$$[f_o(\theta)]_\Gamma= [f_o'(\theta)]_\Gamma= [g_o(\theta)]_\Gamma= [h_o(\theta)]_\Gamma= 0 \tag{27}$$

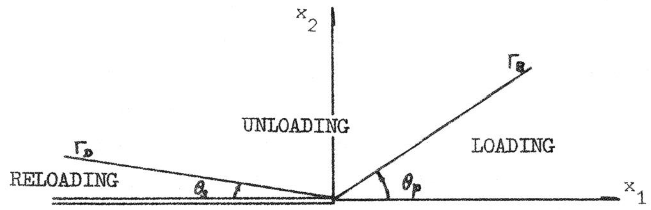

Fig. 1 Crack-tip geometry

At unloading boundary Γ_B, an additional contiguity condition, should be added to (27):

$$\dot{\sigma}_e^o(\theta_p+0) = \dot{\sigma}_e^o(\theta_p-0) = 0 \tag{28}$$

The rigorous proof of (28) is omitted here, but it is intuitively acceptable. Accordingly, all components of stress rate and strain rate should also be continuous across Γ_B, and then we have

$$[f_o''(\theta)]_{\Gamma_B} = [g_o'(\theta)]_{\Gamma_B} = [h_o'(\theta)]_{\Gamma_B} = 0 \tag{29}$$

The location of reloading boundary Γ_D is determined from

$$\overset{o}{\sigma}_e(x_2)|_{\Gamma_D} = \overset{o}{\sigma}_e(x_2)|_{\Gamma_B} \quad \text{for same } x_2 \tag{30}$$

NUMERICAL SOLUTION

By symmetry the boundary conditions at $\theta=0$ are

$$f_o'(0) = 0, \quad g_o'(0) = 0, \quad h_o(0) = 0 \tag{31}$$

The traction-free conditions at $\theta=\pi$ require

$$f_o(\pi) = f_o'(\pi) = 0 \tag{32}$$

Having closed-form solutions in the unloading zone (with $\mu=0$), the basic equations are integrated numerically over the loading and reloading plastic zones. The values of $f''(0)$ and the exponent of singularity s are assumed to start the numerical integration from $\theta=0$, and the values of these two parameters are refined by iteration until the boundary conditions (32) at $\theta=\pi$ are satisfied with a prescribed accuracy. The near-tip stress and strain fields are computed for varying parameters $\alpha=E_t/E$, β and ν. Here are shown only some results for $\nu=1/3$. The singularity exponents s are shown in Table 1 and Fig. 2 for isotropic hardening ($\beta=1$) and one case of anisotropic hardening ($\beta=0.5$). The angular distribution of equivalent active stress $\Sigma^o(\theta)$, components of stress $\Sigma_{rr}(\theta)$, $\Sigma_{\theta\theta}(\theta)$, $\Sigma_{r\theta}(\theta)$ and rate of plastic strain $\psi^p_{rr}(\theta)$, $\psi^p_{\theta\theta}(\theta)$, $\psi^p_{r\theta}(\theta)$ are shown in Fig. 3,4 and 5 respectively. The normalizing condition is taken as

$$\max\{\Sigma^o(\theta),\ \theta\leq\theta_p\} = 1$$

Comparison is made with the corresponding results ($\beta=1$) obtained in [1]. The numerical results show the significant role of the anisotropy parameter β. Fig. 3 shows great discrepancy in angular distribution of equivalent stress from Amazigo and Hutchinson's results [1] for very low hardening case ($\alpha=0.01$), and it is expected that in this case the plastic strain are underestimated in [1] without consideration of the reloading zone.

Table 1 (ν= 1/3)

α	β = 1			β = 0.5		
	s	θ_p	θ_s	s	θ_p	θ_s
0.5	−0.442 (−0.442)[1)]	1.717 (1.717)	0	−0.393	2.327	0.0056
0.3	−0.373 (−0.373)	1.876 (1.875)	0	−0.318	2.440	0.1165
0.1	−0.197 (−0.197)	2.153 (2.174)	0.1115	−0.212	2.548	0.3230
0.05	−0.142 (−0.136)	2.279 (2.393)	0.3331	−0.174	2.568	0.4017
0.01	−0.0797 (−0.0887)	2.360 (2.736)	0.6026	−0.134	2.590	0.4783

1) Values in parentheses are taken from [1] which neglected reloading zone

REFERENCES

[1] Amazigo, J.C. and Hutchinson, J.W., J. Mech. Phys. Solids, Vol. 25 (1977), 81–97.

[2] Kadaschevich, U. I. and Novozhilov, V.V., Appl. Math. and Mech., Vol. 22 (1958) (in Russian).

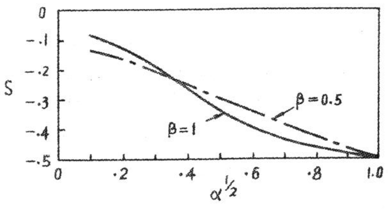

Fig.2 Order of singularity of stresses and strains.

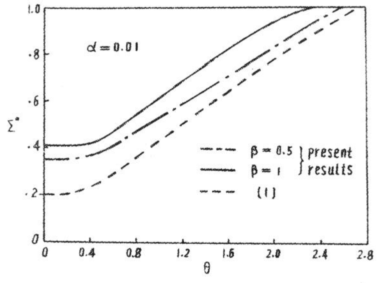

Fig.3 Distributions of equivalent active stress

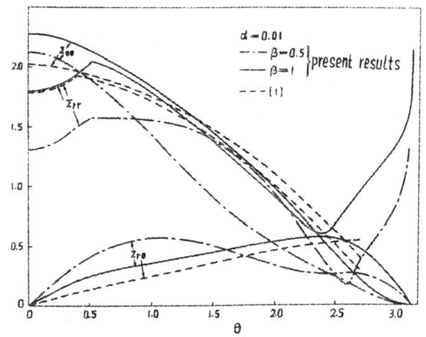

Fig.4 Stress distribution.

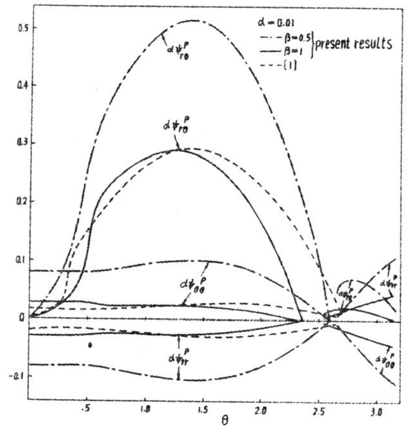

Fig.5 Distributions of rates of plastic strain.