POWER-LAW ANISOTROPIC HARDENING EFFECT
ON MODE-III CRACK GROWTH

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In this paper the constitutive law for anisotropic hardening suggested by Kadaschevich and Novozhilov[1] is used to obtain the near-tip solutions in mode-III steady crack growth with power-law hardening. A parameter β is introduced to characterize the anisotropy of hardening. A plot is presented to show the dependence of angular distribution of stresses on the parameter β and the hardening exponent n.

BASIC EQUATIONS

Ramberg-Osgood law for power hardening material in simple shear has the form

$$\varepsilon_{12} = \varepsilon^e_{12} + \varepsilon^p_{12}\ , \qquad \varepsilon^e_{12} = \frac{1}{2}\sigma_{12}/G, \qquad \varepsilon^p_{12} = \frac{1}{2}c\sigma^n_{12} \tag{1}$$

the superscripts "e" and "p" denoting the elastic and plastic parts, and G, c, n being material constants, and $n > 1$. Further, we assume that, in the case of simple shear, the translation and expansion of the yielding surface in stress space keep a constant proportion $(1-\beta):\beta$. In the following we shall assume $1 \geq \beta \geq \frac{1}{2}$, so that the stress-free state lies within the subsequent yielding surface. Then the constitutive equations of Kadaschevich and Novozhilov will take the form

$$\dot\varepsilon^p_{ij} = \frac{cn}{2\beta^n}\,\overset{\circ}{\sigma}{}'_{ij}\,\overset{\circ}{T}{}^{n-2}\dot{\overset{\circ}{T}} \tag{2}$$

$$\varepsilon^p_{ij} = \frac{c}{2(1-\beta)^n}\,T_\alpha^{n-1}\alpha_{ij} \tag{3}$$

Here by α_{ij} we denote the stresses corresponding to the center of the yielding surface, by the superdot "$\cdot$" the time-derivative "d/dt", by the supercirclet "o" the "active" component ($\overset{\circ}{\sigma}_{ij} = \sigma_{ij} - \alpha_{ij}$), by the prime "'" the deviator component ($\overset{\circ}{\sigma}{}'_{ij} = \overset{\circ}{\sigma}_{ij} - \frac{1}{3}\overset{\circ}{\sigma}_{kk}\delta_{ij}$) and

$$T_\alpha = \sqrt{\tfrac{1}{2}\alpha_{ij}\alpha_{ij}}\ , \qquad \overset{\circ}{T} = \sqrt{\tfrac{1}{2}\overset{\circ}{\sigma}{}'_{ij}\overset{\circ}{\sigma}{}'_{ij}} \tag{4}$$

Let x, y be the coordinates centered at crack-tip and moving with it along x-direction. For mode-III problems, nonvanishing components are only σ_{xz}, σ_{yz}, α_{xz}, α_{yz} and ε_{xz}, ε_{yz}, but denoted by τ_x, τ_y, α_x, α_y and $\frac{1}{2}\gamma_x$, $\frac{1}{2}\gamma_y$ respectively in this paper. Then the constitutive equation (2) can be reduced to

$$\dot\gamma^p_x = \lambda\overset{\circ}{\tau}_x\ , \qquad \dot\gamma^p_y = \lambda\overset{\circ}{\tau}_y \tag{5}$$

with

$$\overset{\circ}{\tau}_x = \tau_x - \alpha_x, \qquad \overset{\circ}{\tau}_y = \tau_y - \alpha_y, \qquad \overset{\circ}{\tau} = \sqrt{\overset{\circ}{\tau}{}^2_x + \overset{\circ}{\tau}{}^2_y} \tag{6}$$

$$\lambda = \frac{cn}{\beta^n}\,\overset{\circ}{\tau}{}^{n-2}\dot{\overset{\circ}{\tau}}\ , \qquad \text{(for plastic loading)} \tag{7}$$

and (3) leads to

$$\gamma^p_x = \frac{c}{(1-\beta)^n}\,\alpha^{n-1}\alpha_x, \qquad \gamma^p_y = \frac{c}{(1-\beta)^n}\,\alpha^{n-1}\alpha_y \tag{8}$$

with

$$\alpha = \sqrt{\alpha^2_x + \alpha^2_y} \tag{9}$$

The stress components τ_x, τ_y can be expressed in terms of the stress function ϕ namely

$$\tau_x = \frac{\partial\phi}{\partial y}\ , \qquad \tau_y = -\frac{\partial\phi}{\partial x}\ , \qquad \tau = \sqrt{\tau^2_x + \tau^2_y} \tag{10}$$

Identify the time parameter with the increase in crack length, so that in steady state we have for scalars or cartesian components of tensors ( )

$$(\dot{\ }) = \frac{d}{dt}(\quad) = -\frac{\partial}{\partial x}(\quad) \tag{11}$$

Then the time-rate of the compatibility equation

$$\frac{1}{G}\Delta\dot\phi + \frac{\partial}{\partial y}\dot\gamma^p_x - \frac{\partial}{\partial x}\dot\gamma^p_y = 0 \tag{12}$$

can be reduced, by use of (5), (11) and then transforming to polar coordinates centered at crack tip, to

$$\frac{1}{G}\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} - \cos\theta\frac{\partial}{\partial r}\right)\Delta\phi + \lambda\Delta\phi + \frac{\partial\lambda}{\partial r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial\lambda}{\partial\theta}\frac{\partial\phi}{\partial\theta}$$

$$+\frac{1}{r}\left\{\frac{\partial}{\partial r}(\lambda\alpha_\theta r) - \frac{\partial}{\partial\theta}(\lambda\alpha_r)\right\} = 0 \tag{13}$$

After elliminating $\dot{\gamma}^p_x$ and $\dot{\gamma}^p_y$ from (5) and (8), and transforming to polar coordinates, we obtain the constitutive relations*

$$\lambda\mathring{\tau}_r = \frac{c\alpha^{n-1}}{(1-\beta)^n}\left[(n-1)\alpha^{-1}\dot{\alpha}\alpha_r + \dot{\alpha}_r - \frac{\sin\theta}{r}\alpha_\theta\right] \tag{14}$$

$$\lambda\mathring{\tau}_\theta = \frac{c\alpha^{n-1}}{(1-\beta)^n}\left[(n-1)\alpha^{-1}\dot{\alpha}\alpha_\theta + \dot{\alpha}_\theta + \frac{\sin\theta}{r}\alpha_r\right] \tag{15}$$

in which, from (6), (10) and (9)

$$\mathring{\tau}_r = \frac{1}{r}\frac{\partial\phi}{\partial\theta} - \alpha_r\,, \qquad \mathring{\tau}_\theta = -\frac{\partial\phi}{\partial r} - \alpha_\theta$$

$$\mathring{\tau} = \sqrt{\mathring{\tau}_r^2+\mathring{\tau}_\theta^2} \qquad \alpha = \sqrt{\alpha_r^2+\alpha_\theta^2} \tag{16}$$

(13) —(15) constitute the system of nonlinear partial differential equations for unknown functions $\phi(r,\theta)$, $\alpha_r(r,\theta)$ and $\alpha_\theta(r,\theta)$.

CONTIGUITY CONDITIONS

Let Γ be the boundary between two neighboring zones. As in [2,3], we shall use the curvilinear coordinates (n,s) associated with Γ and moving with the crack-tip (Fig.1). Since stresses and strains should be continuous for hardening material, we have the following contiguity conditions across Γ

$$[\phi]_\Gamma = \left[\frac{\partial\phi}{\partial n}\right]_\Gamma = 0 \tag{17}$$

$$[\alpha_n]_\Gamma = [\alpha_s]_\Gamma = 0 \tag{18}$$

* In this paper $\dot{\alpha}_r$, $\dot{\alpha}_\theta$ are used to denote the rates of the components α_r and α_θ , instead of the components of the rate of vector (α_r,α_θ).

where $[\ ]_\Gamma$ denotes discontinuity jump across Γ . As to $\partial^2\phi/\partial n^2$, its jump can be proved as in [2,3] to be related to jump of λ by the relation

$$\frac{1}{G}\left[\frac{\partial^2\phi}{\partial n^2}\right]_\Gamma + \frac{1}{\cos\theta}\mathring{\tau}_s[\lambda]_\Gamma = 0 \tag{19}$$

where $\theta=\theta(s)$ is the angle of inclination of n-line (Fig. 1).

The various zones in the x-y plane are shown in Fig.2, with I — elastic zone, II — primary plastic zone, III — unloading wake zone and IV — secondary reloading zone. The near-tip zones are II, III, IV only. Similar to the case of isotropic hardening ($\beta=1$) considered in [2,3], λ can be proved to be continuous across the unloading boundary Γ_B (i.e. $\lambda=0$ at Γ_B). At the reloading boundary Γ_D we have the condition

$$\mathring{\tau}\big|_{\Gamma_D}(y) = \mathring{\tau}\big|_{\Gamma_B}(y) \tag{20}$$

ASYMPTOTIC SOLUTIONS

The unknown functions ϕ, α_r are α_θ are assumed in an asymptotic form

$$\phi = r\left(\ln\frac{A}{r}\right)^s\sum_{m=0} f_m(\theta)\left(\ln\frac{A}{r}\right)^{-m} \tag{21}$$

$$\begin{Bmatrix}\alpha_r\\ \alpha_\theta\end{Bmatrix} = (1-\beta)\left(\ln\frac{A}{r}\right)^s\sum_{m=0}\begin{Bmatrix}\alpha_{rm}(\theta)\\ \alpha_{\theta m}(\theta)\end{Bmatrix}\left(\ln\frac{A}{r}\right)^{-m} \tag{22}$$

where A is a constant which can not be determined from the near-tip asymptotic analysis. Then it follows from (10), (16) and (7)

$$\{\tau_r,\tau_\theta\} = \left(\ln\frac{A}{r}\right)^s\sum_{m=0}\{\tau_{rm}(\theta),\ \ \tau_{\theta m}(\theta)\ \}\left(\ln\frac{A}{r}\right)^{-m} \tag{23}$$

$$\{\mathring{\tau}_r,\mathring{\tau}_\theta\} = \left(\ln\frac{A}{r}\right)^s\sum_{m=0}\{\mathring{\tau}_{rm}(\theta),\ \ \mathring{\tau}_{\theta m}(\theta)\}\left(\ln\frac{A}{r}\right)^{-m} \tag{24}$$

$$\{\tau^2,\alpha^2,\mathring{\tau}^2\} = \left(\ln\frac{A}{r}\right)^{2s}\sum_{m=0}\{K_m(\theta),\ (1-\beta)^2H_m(\theta),\ \mathring{K}_m(\theta)\}\left(\ln\frac{A}{r}\right)^{-m} \tag{25}$$

$$\lambda = \frac{1}{r}\left(\ln\frac{A}{r}\right)^{s(n-1)}\sum_{m=0}\lambda_m(\theta)\left(\ln\frac{A}{r}\right)^{-m} \tag{26}$$

$$\tau_{rm}(\theta) = f'_m(\theta), \quad \tau_{\theta 0}(\theta) = -f_0(\theta) \quad \tau_{\theta 1} = -f_1(\theta) + sf_0(\theta), \cdots \tag{27}$$

$$\mathring{\tau}_{rm}(\theta) = \tau_{rm} - (1-\beta)\alpha_{rm}, \quad \mathring{\tau}_{\theta m}(\theta) = \tau_{\theta m} - (1-\beta)\alpha_{\theta m} \tag{28}$$

$$K_0(\theta) = f_0^2 + f_0'^2 \qquad K_1(\theta) = 2f'_0 f'_1 + 2f_0(f_1 - sf_0), \cdots \tag{29}$$

$$H_0(\theta) = \alpha_{r0}^2 + \alpha_{\theta 0}^2 \qquad H_1(\theta) = 2\alpha_{r0}\alpha_{r1} + 2\alpha_{\theta 0}\alpha_{\theta 1}, \ldots \tag{30}$$

$$\mathring{K}_0(\theta) = \mathring{\tau}_{r0}^2 + \mathring{\tau}_{\theta 0}^2 \qquad \mathring{K}_1(\theta) = 2\mathring{\tau}_{r0}\mathring{\tau}_{r1} + 2\mathring{\tau}_{\theta 0}\mathring{\tau}_{\theta 1}, \ldots \tag{31}$$

$$\lambda_0(\theta) = \frac{cn}{2\beta^n}\mathring{K}_0^{(n-3)/2}\mathring{K}'_0 \sin\theta$$

$$\lambda_1(\theta) = \frac{cn}{2\beta^n}\mathring{K}_0^{(n-5)/2}\left\{\frac{n-3}{2}\mathring{K}'_0\mathring{K}_1\sin\theta + \mathring{K}_0(\mathring{K}'_1\sin\theta + 2s\mathring{K}_0\cos\theta)\right\} \tag{32 a,b}$$

. . . .

here and hereafter the prime "'" is used to denote $d/d\theta$. The exponent s of the logarithmic singularity is determined from the requirement that the elastic and plastic strains should be coupled in the equation of compatibility (44) for the unloading zone III, and the result is

$$s = \frac{2}{n-1} \tag{33}$$

The leading terms in asymptotic expansions of the basic equations (13)—(15) are

$$\{\lambda_0(f'_0 - (1-\beta)\alpha_{r0})\}' = 0 \tag{34}$$

$$c\frac{n-1}{2}H_0^{(n-3)/2}H'_0\alpha_{r0}\sin\theta + cH_0^{(n-1)/2}(\alpha'_{r0} - \alpha_{\theta 0})\sin\theta = \lambda_0\mathring{\tau}_{r0} \tag{35}$$

$$c\frac{n-1}{2}H_0^{(n-3)/2}H'_0\alpha_{\theta 0}\sin\theta + cH_0^{(n-1)/2}(\alpha'_{\theta 0} + \alpha_{r0})\sin\theta = \lambda_0\mathring{\tau}_{\theta 0} \tag{36}$$

Noting that by (32a) $\lambda_0(0)=\lambda_0(\pi)=0$, we obtain the solutions of (34)—(36) for plastic zone II in the form

$$\lambda_0(\theta) = 0 \tag{37}$$

$$\left.\begin{aligned} \alpha_{r0}(\theta) &= C_1\cos\theta + C_2\sin\theta \\ \alpha_{\theta 0}(\theta) &= -C_1\sin\theta + C_2\cos\theta \end{aligned}\right\} \tag{38}$$

and in the same form but with constants C_1, C_2 replaced by C_1^*, C_2^* for secondary plastic zone IV. And $f_0(\theta)$ must satisfy

$$\frac{1}{2}\mathring{K}'_0 = (f'_0 - (1-\beta)\alpha_{r0})(f''_0 + f_0) = 0 \tag{39}$$

For both plastic zones II and IV, the second asymptotic approximation of (13)—(15) will then be

$$\{\lambda_1(f'_0 - (1-\beta)\alpha_{r0})\}' = 0 \tag{40}$$

$$c\frac{n-1}{2}H_0^{(n-3)/2}H'_1\alpha_{r0}\sin\theta + cH_0^{(n-1)/2}\{(\alpha'_{r1} - \alpha_{\theta 1})\sin\theta + (s+2)\alpha_{r0}\cos\theta\} = \lambda_1\mathring{\tau}_{r0} \tag{41}$$

$$c\frac{n-1}{2}H_0^{(n-3)/2}H'_1\alpha_{\theta 0}\sin\theta + cH_0^{(n-1)/2}\{(\alpha'_{\theta 1} + \alpha_{r1})\sin\theta + (s+2)\alpha_{\theta 0}\cos\theta\} = \lambda_1\mathring{\tau}_{\theta 0} \tag{42}$$

As to the third approximation, only that of (13) will be needed, which has the form

$$\frac{1}{G}\{(f'''_0 + f'_0)\sin\theta + (f''_0 + f_0)\cos\theta\} + \{\lambda_1(f'_1 - (1-\beta)\alpha_{r1})\}' + \{\lambda_2(f'_0 - (1-\beta)\alpha_{r0})\}' - (s+1)\lambda_1\{f_0 + (1-\beta)\alpha_{\theta 0}\} = 0 \tag{43}$$

Here we omit all the details and only write down the leading terms of final solutions in various zones, which satisfy the equations (39)—(43) in plastic zones II ($0 \leq \theta \leq \theta_p$) and IV ($\pi - \theta_s \leq \theta \leq \pi$), also the equation of compatibility in unloading wake zone III ($\theta_p \leq \theta \leq \pi - \theta_s$):

$$\frac{1}{G}\Delta\phi + \frac{d}{dy}\gamma_{\mathbf{x}}^{p} = 0 \tag{44}$$

besides, the solutions meet all contiguity conditions (17)—(20)

Stress Fields. For plastic zone II ($0 \leq \theta \leq \theta_p$),

$$\tau_{x0}/F = -\sin\theta, \qquad \tau_{y0}/F = \frac{1-\beta}{\beta} + \cos\theta \tag{45}$$

For unloading wake zone III ($\theta_p \leq \theta \leq \pi - \theta_s$),

$$\tau_{x0}/F = -\sin\theta_p + \frac{1}{R}\ln\frac{\sin\theta_p}{\sin\theta}, \quad R = \frac{\pi - \theta_p - \theta_s}{(1-\beta)/\beta + \cos\theta_p} \tag{46}$$

$$\tau_{y0}/F = \frac{1}{R}(\pi - \theta_s - \theta),$$

And for secondary plastic zone IV ($\pi-\theta_s \leqq \theta \leqq \pi$),

$$\tau_{x_0}/F = \sqrt{2\beta-1}/\beta \qquad \tau_{y_0}/F = 0 \tag{47}$$

Plastic Strain Fields. In plastic zones II we have

$$\gamma_x^p = (\ln\frac{A}{r})^{s+1}\gamma_{x_0}^p(\theta) , \qquad \gamma_y^p = (\ln\frac{A}{r})^{s+2}\ \gamma_{y_0}^p \tag{48}$$

$$G\gamma_{x_0}^p/F = -\{\sin\theta - Q\lambda_1^*(\theta)F(\theta)\}/(s+1)$$

$$G\gamma_{y_0}^p/F = Q \quad , \qquad \lambda_1^*(\theta) = \frac{\beta^n}{cF^{n-1}}\lambda_1(\theta) \tag{49}$$

and in plastic zone IV, we obtain

$$G\gamma_{x_0}^p/F = -\{\sin\theta_p - Q\lambda_1^*(\theta_p)\mathfrak{F}(\theta_p)\}/(s+1) \tag{50}$$

$$G\gamma_{x_0}^p/F = Q.$$

For unloading wake zone III,

$$\gamma_x^p = a_0(\ln\frac{A}{y})^{s+1}, \qquad \gamma_y^p = b_0(\ln\frac{A}{y})^{s+2} \tag{51}$$

with Ga_0/F, Gb_0/F the same values as (50).

The near-tip fields are found to be within two extreme states, the state "Tr" and the state "Nb"[2,3], and denote the quantities with them by subscripts "tr" and "b", respectively. These two states differ by the greatest amount as n=1, and coincide as $n\to\infty$. The angles θ_p and θ_s lie between the limiting values:

$$\theta_{p,tr}(\beta) \leqq \theta_p \leqq \theta_{pb}(\beta,n)$$

$$\theta_{s,tr}(\beta) \geqq \theta_s \geqq \theta_{sb}(\beta,n) \tag{52}$$

The constants F and Q are related by

$$Q = cF^{n-1}G/\beta^n \tag{53}$$

and

$$Q_{tr}(\beta,n) \geqq Q \geqq Q_b(\beta,n) \tag{54}$$

$\mathcal{F}(\theta)$ is defined as

$$\mathfrak{F}(\theta) = \frac{1}{F}\{f_1'(\theta) - (1-\beta)\alpha_{r_1}(\theta)\} = \frac{1}{F}\,\mathring{\tau}_{r1}(\theta), \tag{55}$$

and in zone II ($0 \leqq \theta \leqq \theta_p$) it satisfies a nonlinear first-order differential equation which follows from (41—43). The state "Tr" is one extreme beyond which the unloading condition is violated in wake zone III, while the state "Nb" is the other extreme characterized by the condition that θ_{pb} is the maximum possible angle of zone II, i.e. $F'(\theta_{pb}) = \infty$. For illustration, only the angular distribution of stresses τ_{x0}/F, τ_{y0}/F is shown in Fig. 3, from which it is evident that the difference between the two extreme states is rather small.

In the above solutions, the condition of vanishing λ at the unloading boundary Γ_B can not be satisfied. It is shown in [3] that for isotropic hardening (β=1) this condition ($\lambda|_{\Gamma_B}=0$) is satisfied by the state "Nb" superposed with an inner boundary layer near Γ_B. This leads to the expectation that the state "Nb" will be the true state.

REFERENCES

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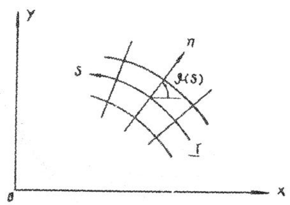

Fig.1 n,s coordinates associated with Γ

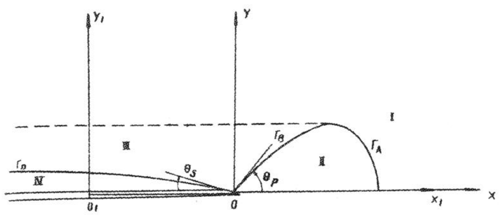

Fig.2 Zones in the x-y plane

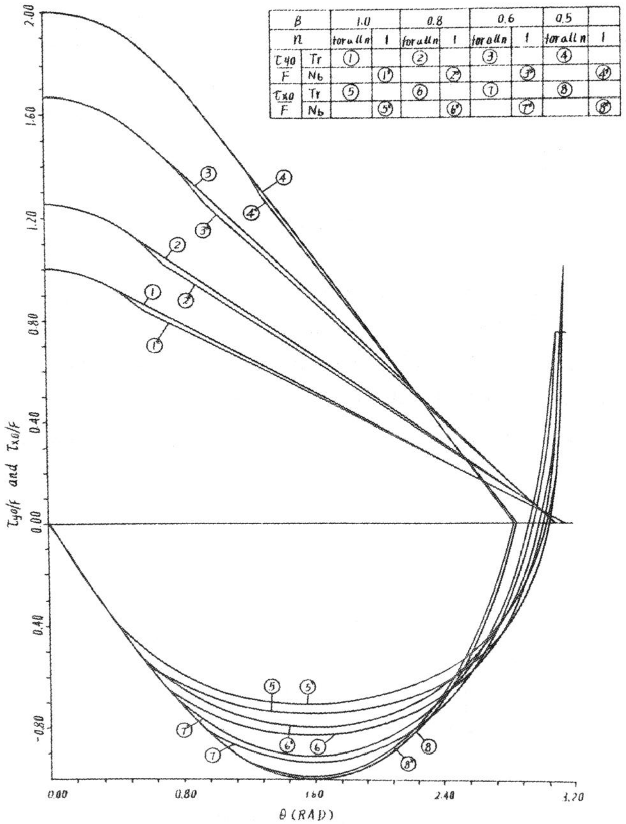

Fig.3 Angular distribution of stresses.