SOME THEOREMS IN MATHEMATICAL THEORY
OF NONLINEAR FRACTURE DYNAMICS

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I. INTRODUCTION

Mathematical theory of fracture, especially its nonlinear part has been an attractive and important area in engineering and applied sciences. In practical applications, various kinds of nonlinear media, such as nonlinear elastic, elastic-plastic and visco-elastic-plastic ones, may be used in static, quasi-static and dynamical cases, and sometimes one will face to handle coupled systems for these nonlinear media, as the thermo-mechanical ones, for example. One of the main purposes for this mathematical research of fracture is to develop available fracture criteria for engineering uses.

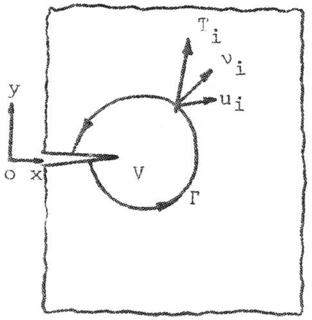

Fig. 1

Some important works initiated the research on mathematical theory of nonlinear fracture problems[1,2]. In 1968, J.R. Rice proposed the famous path-independent integral J:

$$J = \int_\Gamma Wdy - T_i \frac{\partial u_i}{\partial x} ds \qquad (1)$$

here W is the strain energy density:

$$W = \int \sigma_{ij} d\, e_{ij} \qquad (2)$$

σ_{ij}, e_{ij} are the stress, strain tensors respectively. T_i is the traction vector along the integration path around the crack tip, and u_i is the displacement. Since then an important nonlinear fracture criterion based on J has been formed and various extensions to dynamic case have been

made[4].

In this paper we give further discussions on path-independent integrals and fracture criteria in nonlinear mathematical theory of fracture. New path-independent integrals are worked out for fracture dynamics of coupled thermo-mechanical systems of nonlinear media. Mechanical meaning of these integrals are shown to be related with the dynamical crack extension force which is firstly given here by integral expressions. Thus it is possible to form nonlinear dynamic fracture criteria based on the present research.

II. CRACK PROPAGATION IN NONLINEAR ELASTIC MEDIA

Let u_i, e_{ij}, σ_{ij}, ρ, T, v_i, h_i be the displacements, strain, stress tensor, density, temperature, velocity and heat flux. Write the constitutive law as

$$\sigma_{ij} = f'_{ij}(e_{kl}, T) = f_{ij}(e_{kl}) - \beta_{ij}\theta, \quad \theta = T - T_0 \qquad (3)$$

here β_{ij} is the thermal moduli. Let k_{ij} be heat conduction coefficients, c_v be the heat capacity per unit mass at constant strain. Introduce H_i, proportional to the entropy displacement, such that

$$h_i = \partial H_i / \partial t \qquad (4)$$

and

$$H_i = 0, \quad \text{when } \theta = e_{ij} = 0 \qquad (5)$$

In the subsequent we denote the elastic strain as e^e_{ij} and denote the elastic displacement as u_i.

For the crack propagation in nonlinear elastic media, we may propose the following:

Theorem 1. The integral

$$Y_1 = \int_{t_0}^{t_1} \left(\int_\Gamma (W+Q-X_i u_i - K)dy - \left(T_i \frac{\partial u_i}{\partial x} + \frac{\theta \nu_i}{T_0} \frac{\partial H_i}{\partial x}\right) ds \right) dt$$

$$+ \int_V \rho v_i \frac{\partial u_i}{\partial x} dv \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \int_V \frac{1}{T_0} \lambda_{ij} \dot{H}_j \frac{\partial H_i}{\partial x} dv dt \qquad (6)$$

is path-independent for any path Γ around the crack tip (Fig. 1) and any $t_1 > t_0 \geqq 0$. Here

$$W = \int f_{ij} de_{ij} \tag{7}$$

is the strain energy density under uniform temperature,

$$Q = \int \frac{\rho c_v \theta}{T_0} d\theta \tag{8}$$

is the heat that may be transformed into useful work.

$$K = 1/2 \, \rho v_i v_i \tag{9}$$

is the kinetic energy,

$$\lambda_{ij} = (k_{ij})^{-1} \tag{10}$$

is the inverse of matrix (k_{ij}). The domain V is bounded by Γ and crack surfaces. Here we assume that X_i is independent of x.

If we consider moving paths $\Gamma(t)$, then we could obtain the following:

Theorem 2. The integral

$$Y_2 = \int_{t_0}^{t_1} \left\{ \int_{\Gamma(t)} [W+Q+(\rho a_i - X_i)u_i] dy - \left(T_i \frac{\partial u_i}{\partial x} - \frac{\theta}{T_0} \nu_i \frac{\partial H_i}{\partial x}\right) ds \right\} dt - \int_{t_0}^{t_1} \int_{v(t)} \rho u_i \frac{\partial a_i}{\partial x} dv dt + \int_{t_0}^{t_1} \int_{v(t)} \frac{1}{T_0} \lambda_{ij} \dot{H}_j \frac{\partial H_i}{\partial x} dv dt \tag{11}$$

or simply

$$Y_3 = \int_{\Gamma(t)} [W+Q+(\rho a_i - X_i)u_i] dy - \left(T_i \frac{\partial u_i}{\partial x} - \frac{\theta}{T_0} \nu_i \frac{\partial H_i}{\partial x}\right) ds - \int_{v(t)} \rho u_i \frac{\partial a_i}{\partial x} dv + \int_{V(t)} \frac{1}{T_0} \lambda_{ij} \dot{H}_j \frac{\partial H_i}{\partial x} dv \tag{12}$$

is path-independent for any $\Gamma(t)$ around crack tip and $t_1 > t > t_0 \geqq 0$.

III. CRACK PROPAGATION IN ELASTIC-PLASTIC SOLIDS

We introduce the integral

$$Y_4 = \int_{t_0}^{t_1} \left(\int_{\Gamma} (W_e+Q-K-X_i u_i) dy - \left(T_i \frac{\partial u_i}{\partial x} - \frac{\theta}{T_0} \nu_i \frac{\partial H_i}{\partial x} ds\right) \right) dt + \int_{t_0}^{t_1} \int_{V_p} (\sigma_{ij} + \beta_{ij}\theta) \partial e^p_{ij}/\partial x \, dv dt + \int_{t_0}^{t_1} \int_V \frac{1}{T_0} \lambda_{ij} \dot{H}_j \frac{\partial H_i}{\partial x} dv dt + \int_V \rho v_i \frac{\partial u_i}{\partial x} dv \Big|_{t_0}^{t_1} \tag{13}$$

Here W_e is the elastic strain energy density,

$$W_e = \int f_{ij} de^e_{ij} \tag{14}$$

V_p is the plastic region within path Γ, e^p_{ij} is the plastic strain.

Now we have the following:

Theorem 3. The integral Y_4 is path-independent for any path Γ around the crack tip and $t_1 > t_0 \geqq 0$ in the case of elastic-plastic crack propagation.

$$Y_5 = \int_{t_0}^{t_1} \left\{ \int_{\Gamma(t)} [W_e+Q+(\rho a_i - X_i)u_i] dy - \left(T_i \frac{\partial u_i}{\partial x} - \frac{\theta}{T_0} \nu_i \frac{\partial H_i}{\partial x}\right) ds \right\} dt + \int_{t_0}^{t_1} \int_{V_p(t)} (\sigma_{ij} + \beta_{ij}\theta) \frac{\partial}{\partial x} e^p_{ij} dv dt + \int_{t_0}^{t_1} \int_{v(t)} \frac{1}{T_0} \lambda_{ij} \dot{H}_j \frac{\partial H_i}{\partial x} dv dt - \int_{t_0}^{t_1} \int_{v(t)} \rho u_i \frac{\partial a_i}{\partial x} dv dt \tag{15}$$

or simply

$$Y_6 = \int_{\Gamma(t)} (W_e+Q+(\rho a_i - X_i)u_i) dy - \left(T_i \frac{\partial u_i}{\partial x} - \frac{\theta}{T_0} \nu_i \frac{\partial H_i}{\partial x}\right) ds + \int_{V_p(t)} (\sigma_{ij} + \beta_{ij}\theta) \frac{\partial}{\partial x} e^p_{ij} dv + \int_{V(t)} \frac{1}{T_0} \lambda_{ij} \dot{H}_j \frac{\partial H_i}{\partial x} dv - \int_{v(t)} \rho u_i \frac{\partial a_i}{\partial x} dv \tag{16}$$

is path-independent for any path $\Gamma(t)$ around the crack tip and any $t_1 > t_0 \geqq 0$.

Theorem 5. The integral

$$Y_7 = \int_{t_0}^{t_1} \Big(\int_{\Gamma(t)} (W_e + Q_e + (\rho a_i - X_i) u_i^e) dy - \Big(T_i \frac{\partial u_i^e}{\partial x} - \frac{\theta}{T_0} \nu_i \frac{\partial H_i}{\partial x}\Big) ds\Big) dt - \int_{t_0}^{t_1} \int_v \rho u_i^e \frac{\partial a_i}{\partial x} dvdt + \int_{t_0}^{t_1} \int_v \frac{1}{T_0} \lambda_{ij} \dot{H}_j \frac{\partial H_i}{\partial x} dvdt \quad (17)$$

or simply

$$Y_8 = \int_{\Gamma(t)} (W_e + Q_e + (\rho a_i - X_i) u_i^e) dy - \Big(T_i \frac{\partial u_i^e}{\partial x} - \frac{\theta}{T_0} \nu_i \frac{\partial H_i}{\partial x}\Big) ds - \int_{v(t)} \rho u_i^e \frac{\partial a_i}{\partial x} dv + \int_v \frac{1}{T_0} \lambda_{ij} \dot{H}_j \frac{\partial H_i}{\partial x} dv \quad (18)$$

is path-independent for any path $\Gamma(t)$ around the crack tip and any $t_1 > t_0 \geqq 0$. Here

$$Q_e = \int \frac{\rho c_v \theta d\theta^e}{T_0}$$

$$\theta_e = -\frac{1}{\rho c_v}(H_{i,i} + T_0 \beta_{ij} e_{ij}^e) \quad (19)$$

IV. Y-INTEGRAL AND DYNAMICAL CRACK EXTENSION FORCE

Here we have the following

Theorem 6. For dynamical notched crack extension in coupled thermo-mechanical system of nonlinear media, there is the following relation between path-independent integral Y_8 and the dynamical crack extension force $\tilde{G}$:

$$\tilde{Y}_8 = \tilde{G} = \int_{\Gamma_t} (W_e + Q_e + (\rho a_i - X_i) u_i^e) dy \quad (20)$$

In limiting case for the crack notch width tending to zero, a relation for the sharp crack results:

$$G = Y_8 \quad (21)$$

Results here may also be extended to 3-dimensional case. For simplicity, all proofs are omitted here.

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