THE NONLINEAR LINE-SPRING MODEL FOR ELASTIC-PLASTIC ANALYSIS OF SURFACE CRACKS

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Early in 1972, Rice and Levy[1] proposed a line-spring model for the analysis of elastic surface-crack problems and it was found to be quite handy for the estimation of the related LEFM parameters. More recently, Newman and Raju[2] had made a comprehensive three dimensional finite-element analysis on surface cracks and shown that the stress intensity factors determined by the line-spring model are in closer agreement wth their results than other approximate methods. The reliability of line-spring model is thus greatly enhanced. Parks[3,4], and independently, some of our group[5,6], had extended that useful model in the analysis of elastic-plastic surface cracks by introducing a nonlinear line-spring model. It is found that the numerical results is promising. The present paper is a further development and improvement of our previous works[5,6].

NONLINEAR LINE-SPRING CONSTITUTIVE RELATIONS

Consider a surface crack of length $2c_0$ and depth $a_0(x)$ in a large plate of thickness t, subjected to remote uniform loading. We begin by visualizing it as a thin plate containing a through crack with two faces connected by distributed line-springs and loaded with uniformly distributed membrane forces N^∞ and bending moments M^∞ at the far ends, as shown in Fig.1(a). The constitutive relation of the line-spring is simulated by a plane strain edge crack strip, as shown in Fig.1(b).

The nonlinear constitutive relation of the line-spring can be deduced from the D-M model of the edge crack strip. The method given in [7] and [8] is used, and the remote uniform stress is taken as the reference load system. From the stress intensity factor of that system, we derived the weight function of the edge crack strip with crack length a as

$$h(a,z) = h_1(\zeta,z_1)/\sqrt{\pi a} \qquad (1)$$

with $h_1(\zeta,z_1)=(5A(\zeta)\cdot(1-\frac{z_1}{\zeta})^{3/2} + 3B(\zeta)\cdot(1-\frac{z_1}{\zeta})^{\frac{1}{2}} + C(\zeta)\cdot(1-\frac{z_1}{\zeta})^{-\frac{1}{2}})/2\sqrt{2}F(\zeta)$

where $A(\zeta) = I'(\zeta)\zeta^{-1} - 2.5I(\zeta)\zeta^{-2} - -\frac{4}{3}F'(\zeta)\zeta + \frac{4}{3}F(\zeta)$

$B(\zeta) = \frac{1}{3}(7.5I(\zeta)\zeta^{-2} - 16F(\zeta) + + 8F'(\zeta)\zeta)$

$C(\zeta) = 4F(\zeta), I(\zeta) = \pi\sqrt{2}\int_0^{\zeta}(F(\zeta))^2\zeta d\zeta$

$\zeta = a/t, z_1 = z/t, F(\zeta) = g_M(\zeta)/\sqrt{\pi\zeta}$

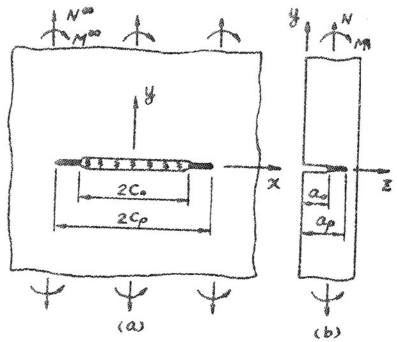

Fig.1

function $g_M(\zeta)$ is taken from reference[9] and $F'(\zeta)$ and $I'(\zeta)$ are the 1st derivative of $F(\zeta)$ and $I(\zeta)$ respectively. Thus, for any arbitrary crack length a, the stress intensity factor induced by uniform tensile stress σ_{YS} acting on the crack surface in the range $a_0 < z < a$ is

$$K_I^{(1)} = \begin{cases} 0 & a \leq a_0 \\ \int_{a_0}^{a} h(a,z)(-\sigma_{YS})dz & a > a_0 \end{cases} \qquad (2)$$

The stress intensity factor due to the membrane force N and the bending moment M acting at the ends of the edge crack plate is given by

$$K_I^{(2)} = \sqrt{t}(\sigma_M g_M(\zeta) + \sigma_B g_B(\zeta)) \qquad (3)$$

in which $g_B(\zeta)$ is taken from reference[9] and $\sigma_M=N/t$, $\sigma_B=6M/t^2$. Then, the equation for the determination of a_p can be obtained by the combined action of the two load system to make $K = K_I^{(1)} + K_I^{(2)} = 0$, that is

$$\sigma_M g_M(\zeta_p) + \sigma_B g_B(\zeta_p) - \sigma_{YS} h_2(\zeta_p,\zeta_0) = 0 \qquad (4)$$

where

$$h_2(\zeta,\zeta_0) = \int_{\zeta_0}^{\zeta} h_1(\zeta,z_1)dz/\sqrt{\pi\zeta} = [A(1-\frac{\zeta_0}{\zeta})^{5/2} + B(1-\frac{\zeta_0}{\zeta})^{3/2} + + C(1-\frac{\zeta_0}{\zeta})^{1/2}]/\sqrt{2}g_M(\zeta)/\zeta$$

From the Paris displacement formula[9], we get the line-spring constitutive equations as follows

$$\delta = [2t(\alpha_{MM}\sigma_M + \alpha_{MB}\sigma_B) - 2t\sigma_{YS}\int_{\zeta_0}^{\zeta_p} g_M(\zeta)h_2(\zeta,\zeta_0)d\zeta]/E'$$

$$\theta = [12(\alpha_{BM}\sigma + \alpha_{BB}\sigma_B) - 12\sigma_{YS}\int_{\zeta_0}^{\zeta_p} g_B(\zeta)h_2(\zeta,\zeta_0)d\zeta]/E' \qquad (5)$$

where δ and θ are relative displacement and rotation of two ends of the edge crack strip respectively, α_p is the depth of the D-M model crack, $E'=E/(1-\nu^2)$, E=Young's modulus, ν=poisson ratio, and $\alpha_{\lambda\mu}=\int_0^{\zeta_0} g_\lambda(\zeta)g_\mu(\zeta)d\zeta$, $\lambda, \mu=M,B$, equations (5) can be written in matrix form

$$\{\overline{q}\} = 2\sigma_{YS}([C]\{\overline{\sigma}\} - \int_{\zeta_0}^{\zeta_p} h_2(\zeta,\zeta_0)\begin{Bmatrix} g_M(\zeta) \\ g_B(\zeta) \end{Bmatrix} d\zeta)/E \qquad (6)$$

where $\{\overline{q}\}^T = [\overline{\delta},\overline{\theta}]$, $\overline{\delta} = \delta/t$, $\overline{\theta} = \theta/6$, $\{\overline{\sigma}\}^T = [\overline{\sigma}_M,\overline{\sigma}_B]$; $\overline{\sigma}_M = \sigma_M/\sigma_{YS}$, $\overline{\sigma}_B = \sigma_B/\sigma_{YS}$, and $[C] = \begin{bmatrix} \alpha_{MM} & \alpha_{MB} \\ \alpha_{BM} & \alpha_{BB} \end{bmatrix}$

After the ligament is completely yielded, the constitutive relation in incremental form is

$$d\{\overline{\sigma}\} = [S^{ep}]\, d\{\overline{q}\} \qquad (7)$$

where $[S^{ep}] = [S^e] - [S^e]\{\frac{\partial\phi}{\partial\overline{\sigma}}\}[\frac{\partial\phi}{\partial\overline{\sigma}}]\cdot[S^e]/[\frac{\partial\phi}{\partial\overline{\sigma}}][S^e]\{\frac{\partial\phi}{\partial\overline{\sigma}}\}$, $\{\frac{\partial\phi}{\partial\overline{\sigma}}\}^T = [\frac{\partial\phi}{\partial\overline{\sigma}_M}\ \frac{\partial\phi}{\partial\overline{\sigma}_B}]$, $\phi(\overline{\sigma}_M,\overline{\sigma}_B,\zeta_0)$ is the generalized yielding function of the edge crack strip, and $\phi(\overline{\sigma}_M,\overline{\sigma}_B,\zeta_0) = 0$ is the generalized yield surface. Based upon Rice's slip-line analysis for the edge crack strip, to simplify the numerical calculations, the yield surfac is linearized to the following form $\phi(\overline{\sigma}_M,\overline{\sigma}_B,\zeta_0) = \overline{\sigma}_\varepsilon - \alpha(\zeta_0) = 0$, where the equivalent stress $\overline{\sigma}_e = \overline{\sigma}_M + \frac{1}{3}k(\zeta_0)\overline{\sigma}_B$. Thus, $[S^{ep}]$ is constant, and the constitutive relation (7) can be integrated into the total deformation expression:

$$\{\overline{\sigma}\} = \{\overline{\sigma}\}_0 + [S^{ep}](\{\overline{q}\} - \{\overline{q}\}_0) \qquad (8)$$

in which $\{\bar{\sigma}\}_0$ and $\{\bar{q}\}_0$ are the generalized stress and displacement respectively when the ligament starts to yield.

D-M MODEL SOLUTION FOR A THROUGH CRACK IN THIN PLATE

Besides the aforementioned line-spring constitutive Eqs. (6) and (7) (or (8)) and the additional equation (4), we are still in need of the behavior equations of the plate with through crack. We adopt the D-M model to simplify the elastic-plastic analysis, as shown in Fig. 1(a). In that figure, a through crack plate is subjected to the membrane force N^∞ and bending moment M^∞ at the far ends and the length of plastic zone is $(C_p - C_0)$. Accordingly, we have a D-M model of crack length $2C_p$ and is connected by the distributed line-springs throughout its crack length. In the range $0 < |x| < c_0$, the crack surface is loaded by the spring reaction forces $N(x)$ and $M(x)$, while in the yielding zone $(c_0 < |x| < c_p)$ the surface forces N' and M' become constant and uniformly distributed. For such a case, the behavior equation given in Rice-Levy's paper[1] is directly applicable. Thus, we have

$$\{\bar{q}\} = 4\sigma_{YS}c_0[B]\Big(\sqrt{\bar{c}_p^2-\bar{x}^2}\,\{\bar{\sigma}^\infty\} - \int_0^1 (F(\bar{x},\bar{t})\{\bar{\sigma}\}\,d\bar{t} - \int_1^{\bar{c}_p} F(\bar{x},\bar{t})\{\bar{\sigma}'\}d\bar{t}\Big)/Et \tag{9}$$

where $[B] = \begin{bmatrix}1 & 0\\ 0 & \gamma\end{bmatrix}$, $\gamma = (1+\nu)/(9+3\nu)$, $[\bar{\sigma}^\infty] = [\bar{\sigma}_M^\infty, \bar{\sigma}_B^\infty]$, $\bar{\sigma}_M^\infty = \frac{\sigma^\infty}{\sigma_{YS}} = \frac{N^\infty}{\sigma_{YS}t}$, $\bar{\sigma}_B^\infty = \frac{\sigma_B^\infty}{\sigma_{YS}} = \frac{6M^\infty}{\sigma_{YS}t^2}$, $[\bar{\sigma}'] = [\bar{\sigma}_M', \bar{\sigma}_B']$, $\bar{\sigma}_M' = \frac{N'}{\sigma_{YS}t}$, $\bar{\sigma}_B' = \frac{6M'}{\sigma_{YS}t^2}$, $\bar{c}_p = \frac{c_p}{c_0}$, $\bar{x} = \frac{X}{c_0}$

and $F(\bar{x},\bar{t}) = \ln\left(\sqrt{\bar{c}_p^2-\bar{x}^2} + \sqrt{\bar{c}_p^2-\bar{t}^2}\right)/\left(\sqrt{\bar{c}_p^2-\bar{x}^2} - \sqrt{\bar{c}_p^2-\bar{t}^2}\right)$

In equation (9), relating $\{\bar{q}\}$ and $\{\bar{\sigma}\}$ in the cracked thin plate, three unknown quantities $\bar{\sigma}_M'$, $\bar{\sigma}_B'$ and $\bar{c}_p$ are involved so that three additional equations are needed. Similar to the ingenious method of Erdogan[10], we furnish the complementary equations by introduing three axilliary conditions, namely: that both of the membrane and bending stress intensity factors for the crack tip of the D-M model are vanishing, i.e. $K_M=0$ and $K_B=0$; and that both of the membrane and bending stress σ_M' and σ_B' on the crack surface should satisfy the generalized yield condition of the uncracked strip, that is

$$\phi_1(\sigma_M', \sigma_B') = (\sigma_M'/\sigma_{YS})^2 + \tfrac{2}{3}(\sigma_B'/\sigma_{YS}) - 1 = 0 \tag{10}$$

From $K_M=0$ and $K_B=0$, we have

$$\bar{\sigma}_M' = \Big[\frac{\pi}{2}\bar{\sigma}_M^\infty - \int_0^1 \bar{\sigma}_M(\bar{x})/(\bar{c}_p^2-\bar{x}^2)^{\frac{1}{2}}\,d\bar{x}\Big]/s$$

$$\bar{\sigma}_B' = \Big[\frac{\pi}{2}\bar{\sigma}_B^\infty - \int_0^1 \bar{\sigma}_B(\bar{x})/(\bar{c}_p^2-\bar{x}^2)^{\frac{1}{2}}\,d\bar{x}/s \tag{11}$$

where $s=\frac{\pi}{2} - \arcsin(c_0/c_p)$. Substituting equations (11) into equation (10), we obtain the equation to determine $\bar{c}_p$:

$$\Big[\frac{\pi}{2}\bar{\sigma}_M^\infty - \int_0^1 \bar{\sigma}_M(\bar{x})/(\bar{c}_p^2-\bar{x}^2)^{\frac{1}{2}}d\bar{x}\Big]^2/s^2 + 2\Big[\frac{\pi}{2}\bar{\sigma}_B^\infty - \int_0^1 \bar{\sigma}_B(\bar{x})/(\bar{c}_p^2-\bar{x}^2)^{\frac{1}{2}}d\bar{x}\Big]3s-1=0 \tag{12}$$

Substituting equations (11) into equation (9), gives

$$\{\bar{q}\} = 4\sigma_{YS}c_0[B](I(\bar{x})\{\bar{\sigma}^\infty\} - \int_0^1 H(\bar{x},\bar{t})\,\{\bar{\sigma}(\bar{t})\}d\bar{t}/Et \tag{13}$$

where $I(\bar{x}) = (\bar{c}_p^2 - \bar{x}^2)^{\frac{1}{2}} - \pi\int_0^{\bar{c}_p} F(\bar{x},\bar{t})d\bar{t}/2s$, $H(\bar{x},\bar{t}) = F(\bar{x},\bar{t}) - \int_1^{\bar{c}_p} F(\bar{x},\bar{t})d\bar{t}/s(\bar{c}_p^2 - \bar{t}^2)^{\frac{1}{2}}$.

GOVERNING EQUATIONS OF SURFACE CRACK PROBLEM AND FORMULA FOR CTOD

Before ligament yielding occurs in the edge crack strip, substitute equation (6) into equation (13), we obtain

$$\{\bar{q}\} + 2\xi_0[B]\int_0^1 H(\bar{x},\bar{t})[C]^{-1}\{\bar{q}\}d\bar{t}/(1-\nu^2) = 4\frac{\sigma_{YS}\xi_0}{E}[B]\Big(\bar{c}_p I(\bar{x})\{\bar{\sigma}^\infty\} - \int_0^1 H(\bar{x},\bar{t})[C]^{-1}\{\bar{g}\}d\bar{t}\Big) \tag{14}$$

where $\xi_0 = c_0/t$, $\{\bar{g}\} = \int_{\xi_0}^{\xi_p} h_2(\zeta,\zeta_0)\begin{Bmatrix}g_M\\ g_M\end{Bmatrix}d\zeta$. This is a set of integral equations to determine $\{\bar{q}\}$, which contains two unknown quantities ζ_p and $\bar{c}_p$. Thus, equation (14), (4) and (12) are governing equations for the case of surface crack problems before the ligament yielding. If ζ_p and $\bar{c}_p$ are given, equation (14) is a Fredholm integral equation of second kind with a singular kernel.

After the ligament yielding, the constitutive relations in incremental form have to be used. So we deduce the incremental form of equation (13) and (12) by differentiation. These equations and equation (7) form a complete set of governing equations in incremental form for the case of surface crack problem after the ligament yielding. When the linearized yield surface or equivalent stress $\overline{\sigma}_e$ is acceptable, the constitutive relation, Eq. (7) can be replaced by Eq. (8), and the resulting set of governing equations for the case of ligament yielding become Eqa. (8), (13) and (12).

Making use of the D-M model for the edge crack strip, we can readily calculate the crack tip opening displacement δ_t. Before the ligament yielding, from Paris displacement formula[9]

$$\delta_t = 2t\sigma_{YS}\int_{\zeta_0}^{\zeta_p}(\overline{\sigma}_M g_M(\zeta) + \overline{\sigma}_B g_B(\zeta) - h_2(\zeta,\zeta_0)h_1(\zeta,\zeta_0)/\sqrt{\pi\zeta}\,d\zeta/E' \qquad (15)$$

After the ligament yielding, the increment of crack tip opening displacement $d\delta_t$ and the generalized displacement increment $d\delta$ and $d\theta$ of the line-spring have the following relationship: $d\delta_t = d\delta + (t/2-a)d\theta$. If we adopt the total deformation expression, δ_t can be computed from following formula

$$\delta_t = \delta_{t0} + (\delta-\delta_0) + (t/2-a)(\theta-\theta_0)$$

where, δ_{t0}, δ_0 and θ_0 are values at the begining of ligament yielding.

NUMERICAL EXAMPLES

Ex. (1). Infinite plate with a surface crack loaded by uniaxial uniform tension N^∞ at infinity. Fig.2 and Fig.3 show the stress distributions $\overline{\sigma}_M(\overline{x})$ and $\overline{\sigma}_B(\overline{x})$ in the line-spring by solid lines, the results of

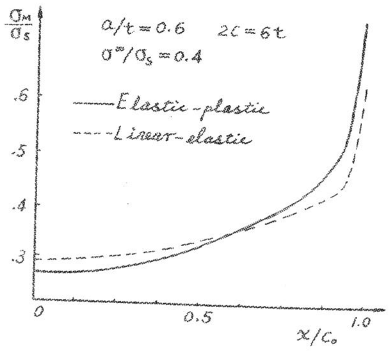

Fig. 2

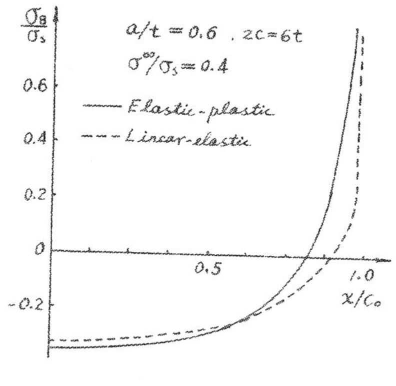

Fig. 3

linear elastic solutions are plotted in dotted lines. One can find that the effect of plasticity reduces the stress $\overline{\sigma}_M$ and $\overline{\sigma}_B$ in the vicinity of x=0. Fig. 4 shows the normalized values of J at x=0 versus load σ_M^∞ for semielliptical surface cracks of constant length 2c=6t. Results obtained by Parks[3] and the linear elasticity solutions are also included in that figure for comparison.

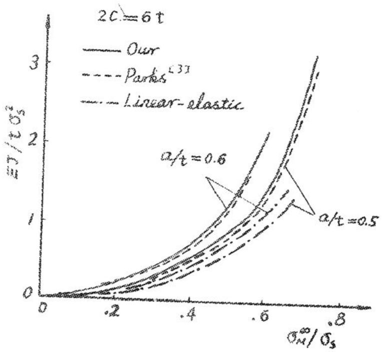

Fig. 4

Ex. (2). Thin cylindrical shell with a long axial surface crack subjected to uniform internal pressure. Consider a cylindrical shell of mean radius R and wall thickness t, with a longitudinal surface crack on the inner face, subjected to a uniform internal pressure p. If the internal surface crack is relatively long, then the problem reduces to a plane strain case with a single line-spring element representing the surface crack. Numerical results showing the effect of crack depth a/t on J~p curves for a shell with R/t = 10 is given in Fig.5. For comparison, Parks' results[3] are also included. It can be seen that Parks' curves have an abrupt transition before and after the ligament yielding, whereas our curves carry on smoothly. However, after the ligament yielding, the J-integral increases rapidly with the increase in pressure and is accompanied by the redistribution of N and M which drift up the yield surface toward the mid-ligament tension vertex[3].

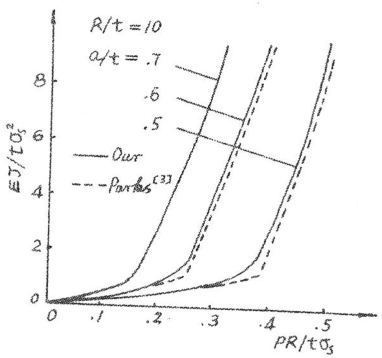

Fig. 5

CONCLUSION

A nonlinear line-spring model for the elastic-plastic analysis of surface cracks is proposed in this paper. It is based on the D-M model and is a direct extension of the ingenious linear line-spring model. It was found that the method is quite suitable for the determination of crack opening displacements and may be used to calculate the tearing modulus T_δ for stability analysis.

Parks has advanced a similar nonlinear line-spring model for the elastic-plastic analysis[3,4] and got some interesting results. However, he ignored the large scale yielding before the ligament yielding and left a gap of transition. Furthermore, he adopted the equations of fully elastic behavior for the main body of the plate or shell and confined the plastic deformation to the line-spring. The differences of both numerical results are clearly shown in Fig. 4 and 5.

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