THE STRESS STRAIN FIELDS AT CRACK TIP AND STRESS INTENSITY FACTORS IN REISSNER'S PLATE

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ABSTRACT

By using Reissner's theory, a general solution for stress strain fields at crack tip in a bending cracked plate is obtained. Using the near-tip expansions, the stress intensity factors in finite size plates for symmetic and anti-symmtric cases are calculated.

INTRODUCTION

The study of bending cracked plate is one of the fundamental problems in engineering. In earlier literature the classical theory was used. In recent years more investigators began to study the problem with Reissner's theory. In [2] and [3] the singularity of Reissner's plate was studied; in [4] the expansions of stress strain fields at crack tip for symmetric case were obtained. In [5] the expression of the first several terms including mode I, II and III were proposed. In [6] an asymptotic solution of zero order was given. For a finite size plate in bending, in [7], [8] and [9] the stress intensity factors for mode I were calculated using Reissner's theory. In [10], [11], the stress intensity factors in infinite plate with uniform twisting moment was calculated using integral transformation. But so far, the solution for mixed mode in a finite plate is not available.

THE GOVERNING EQUATIONS OF A CRACKED PLATE IN BENDING

A plate containing a semi-infinite crack in bending is shown in Fig.1. Based on Reissner's theory, the governing equations can be expressed in terms of three generalized displacements ψ_x, ψ_y and W as follows:

$$D(\frac{\partial^2\psi_x}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2\psi_x}{\partial y^2} + \frac{1+\nu}{3}\frac{\partial^2\psi_y}{\partial x\partial y}) + C(\frac{\partial w}{\partial x} - \psi_x) = 0 \qquad (2.1)$$

$$D(\frac{1+\nu}{2}\frac{\partial^2\psi_x}{\partial x\partial y} + \frac{1-\nu}{2}\frac{\partial^2\psi_y}{\partial x^2} + \frac{\partial^2\psi_y}{\partial y^2}) + C(\frac{\partial w}{\partial y} - \psi_y) = 0 \qquad (2.2)$$

$$C(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - \frac{\partial\psi_x}{\partial x} - \frac{\partial\psi_y}{\partial y}) + p = 0 \qquad (2.3)$$

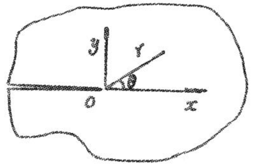

Fig. 1

where $D = \frac{Eh^3}{12(1-\nu^2)}$ bending stiffness

$C = \frac{5}{6}Gh$ shearing stiffness

ψ_x, ψ_y are rotations of the line segment perpendicular to the middle surface before bending. Among them, ψ_x is the rotation in xz plane, ψ_y is the rotation in yz plane, W is the deflection and p is the lateral load per unit area.

According to [12], let

$$\psi_x = \frac{\partial F}{\partial x} + \frac{\partial f}{\partial y}$$

$$\psi_y = \frac{\partial F}{\partial y} - \frac{\partial f}{\partial x} \qquad (2.4)$$

Substituting (2.4) into (2.1) and (2.2), we have:

$$\frac{\partial}{\partial x}[D\nabla^2F + C(W-F)] + \frac{\partial}{\partial y}[\frac{D}{2}(1-\nu)\nabla^2 f - Cf] = 0$$

$$\frac{\partial}{\partial x}[D\nabla^2F + C(W-F)] - \frac{\partial}{\partial x}[\frac{D}{2}(1-\nu)\nabla^2 f - Cf] = 0 \qquad (2.5)$$

This is Cauchy-Riemann equations, from which it follows that:

$$\frac{D}{2}(1-\nu)\nabla^2 f - Cf + i[D\nabla^2F + C(W-F)] = C\Phi(x+iy) \qquad (2.6)$$

where $\Phi(x+iy)$ is an analytic function. In [12] $\Phi=0$ is assumed. It is correct for cases without singularity. As the crack tip is a singular point, generally speaking, $\Phi(x+iy)\neq 0$.

Separating real and imaginary part in (2.6), we have:

$$\nabla^2 f - 4k^2 f = 4k^2\mathrm{Re}\Phi \qquad (2.7)$$

$$W = F - \frac{D}{c}\nabla^2F + \mathrm{Im}\Phi \qquad (2.8)$$

where $$4k^2 = \frac{2C}{D(1-\nu)} \qquad (2.9)$$

Substituting (2.4) and (2.8) into (2.3), we have:

$$D\nabla^2\nabla^2F = P \qquad (2.10)$$

The governing equations (2.7)—(2.10) are equivalent to (2.1)—(2.3). For cracked plate, the bending fracture problems are reduced to solving two equations in terms of F, f with the boundary conditions.

The solution of equations (2.7) and (2.8) can be expressed by the sum of a particular solution and the general solution of the corresponding homogeneous equations.

The particular solution can be chosen as follows:

$$f_1 = -\mathrm{Re}\Phi \quad , \quad F_1 = 0 \quad , \quad W_1 = \mathrm{Im}\Phi \qquad (2.11)$$

The homogeneous equations corresponding to (2.7) and (2.8) are:

$$\nabla^2 f - 4k^2 f = 0 \qquad (2.7')$$

$$W = F - \frac{D}{C}\nabla^2F \qquad (2.8')$$

where $\Phi(x+iy)$ is an analytic function. It can be expanded in series.

$$\Phi(x+iy) = \sum_\mu(\beta_\mu + i\alpha_\mu)z^\mu = \sum_\mu(\beta_\mu + i\alpha_\mu)r^\mu(\cos\mu\theta + i\sin\mu\theta) \qquad (2.12)$$

In polar coordinates, the generalized displacements and generalized stresses can be expressed as follows:

$$\psi_r = \frac{\partial F}{\partial r} + \frac{1}{r}\frac{\partial f}{\partial\theta} \quad , \quad \psi_\theta = \frac{1}{r}\frac{\partial F}{\partial\theta} - \frac{\partial f}{\partial r} \qquad (2.13)$$

$$M_r = -D[\nu\nabla^2F + (1-\nu)\frac{\partial^2F}{\partial r^2} + (1-\nu)\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial f}{\partial\theta})] \qquad (2.14)$$

$$M_\theta = -D[\nabla^2F - (1-\nu)\frac{\partial^2F}{\partial r^2} - (1-\nu)\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial f}{\partial\theta})] \qquad (2.15)$$

$$M_{r\theta} = -D(1-\nu)[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial F}{\partial\theta}) + (2k^2f_0 - \frac{\partial^2 f}{\partial r^2})] \qquad (2.16)$$

$$Q_r = -C(\frac{D}{C}\frac{\partial}{\partial r}\nabla^2F + \frac{1}{r}\frac{\partial f_0}{\partial r^2}) \qquad (2.17)$$

$$Q_\theta = C(\frac{\partial f_0}{\partial r} - \frac{D}{C}\frac{1}{r}\frac{\partial}{\partial\theta}\nabla^2F) \qquad (2.18)$$

where f_0 is the general solution of (2.7') and

$$f = f_0 - \mathrm{Re}\Phi \tag{2.19}$$

According to the singularity analysis [2][3], stresses $\sigma_x, \sigma_y, \tau_{xy}, \tau_{yz}, \tau_{xz}$, as well as $M_x, M_y, M_{xy}, Q_x, Q_y$ should be of $O(r^{-\frac{1}{2}})$. This demands that F, f should be of $O(r^{3/2})$ and W should be of $O(r^{\frac{1}{2}})$. These are the singularity conditions at the crack tip.

THE EIGENFUNCTION EXPANSION OF STRESS FIELDS AT CRACK TIP

When p=0, let $F(r,\theta) = \sum_\lambda r^{\lambda+1} F(\theta)$ (3.1)

Substituting (3.1) into (2.10), we have:

$$F(r,\theta) = \sum_\lambda r^{\lambda+1}[K_\lambda \cos(\lambda-1)\theta + L_\lambda \sin(\lambda-1)\theta + M_\lambda \cos(\lambda+1)\theta + N_\lambda \sin(\lambda+1)\theta] \tag{3.2}$$

From (2.8) and (2.11), we obtain

$$W = F - \frac{D}{C}\nabla^2 F + \sum_\mu r^\mu(\alpha_\mu \cos\mu\theta + \beta_\mu \sin\mu\theta) \tag{3.3}$$

According to singularity conditions, in order to satisfy that W is of $O(r^{\frac{1}{2}})$, let $\mu = \lambda - 1$ (3.4)

$$W = \sum_\lambda r^{\lambda+1}[K_\lambda \cos(\lambda-1)\theta + L_\lambda \sin(\lambda-1)\theta + M_\lambda \cos(\lambda+1)\theta + N_\lambda \sin(\lambda+1)\theta] + \sum_\lambda r^{\lambda-1}[(\alpha_{\lambda-1} - \frac{D}{C}4\lambda K_\lambda)\cos(\lambda-1)\theta + (\beta_{\lambda-1} - \frac{D}{C}4\lambda L_\lambda)\sin(\lambda-1)\theta] \tag{3.5}$$

Function f_0 is the general solution of Helmholtz's equation (2.7'). It can be expressed in modified Bessel function.

$$f_0 = (C+D\theta)I_0(2kr) + \Sigma\{[A_\lambda I_\lambda(2kr) + A_{-\lambda}I_{-\lambda}(2kr)]\sin\lambda\theta + [B_\lambda I_\lambda(2kr) + B_{-\lambda}I_{-\lambda}(2kr)]\cos\lambda\theta\} \;(\lambda \text{ is integer}) + \Sigma\{[A'_\lambda I_\lambda(2kr) + A'_{-\lambda}K_\lambda(2kr)]\sin\lambda\theta + [B'_\lambda I_\lambda(2kr) + B'_{-\lambda}K_\lambda(2kr)]\cos\lambda\theta\} \;(\lambda \text{ is integer}) \tag{3.6}$$

where, C, D, A_λ, $A_{-\lambda}$, B_λ, $B_{-\lambda}$, ··· are arbitrary unknowns. Owing to the boundary conditions, we should take D=0. From the condition of finite rotations, we should drop out the modified Bessel functions of second kind $K_\lambda(2kr)$. Keeping in mind that λ will take positive zero as well as negative values, the function f_0 can be expressed in modified Bessel functions of first kind only.

For symmetric case

$$f_\lambda = \sin\lambda\theta I_\lambda(2kr) = \sin\lambda\theta \sum_{m=0,1,\cdots} \frac{k^{2m} r^{\lambda+2m}}{m!\phi(\lambda,m)} \tag{3.7}$$

and for anti-symmetric case

$$\tilde{f}_\lambda = \cos\lambda\theta I_\lambda(2kr) = \cos\lambda\theta \sum_{m=0,1,\cdots} \frac{k^{2m} r^{\lambda+2m}}{m!\phi(\lambda,m)} \tag{3.8}$$

where $\phi(\lambda,m) = (\lambda+1)(\lambda+2)\cdots(\lambda+m)$ (for $m \geq 1$), $\phi(\lambda,m)=1$ (for m=0)

The linear combination of (3.7) (3.8) is also a solution of (2.7'), the general solution of (2.7') can be expressed as the following linear combination:

$$f_0 = \sum_\lambda \sum_{n=0,1,2\cdots} (A_{\lambda-1+2n} f_{\lambda-1+2n} + B_{\lambda-1+2n} \tilde{f}_{\lambda-1+2n}) \tag{3.9}$$

Substituting (2.11) and (3.9) into (2.19) the expression of f is obtained as follows:

$$f = \sum_\lambda \sum_{n=0,1,\cdots} [A_{\lambda-1+2n}\sin(\lambda-1+2n)\theta \sum_{m=0,1\cdots} \frac{k^{2m} r^{\lambda-1+2(n+m)}}{m!\phi(\lambda-1+2n,m)} + B_{\lambda-1+2n}\cos(\lambda-1+2n)\theta \sum_{m=0,1\cdots} \frac{k^{2m} r^{\lambda-1+2(n+m)}}{m!\phi(\lambda-1+2n,m)}] + \sum_\lambda r^{\lambda-1}[\alpha_{\lambda-1}\sin(\lambda-1)\theta - \beta_{\lambda-1}\cos(\lambda-1)\theta] \tag{3.10}$$

The boundary conditions are:

when $\theta = \pm\pi$, $M_\theta = M_{r\theta} = Q_\theta = 0$ (3.11)

Substituting (3.2) and (3.10) into (3.11) a set of linear algebraic equations is obtained for the unknown coefficients of the expansions. Let

$$\lambda = \pm\frac{n}{2} \qquad n = 0, 1, 2, \cdots \tag{3.12}$$

From the condition of finite strain energy, λ should be positive.

The relations between coefficients in eigenfunction expansion are as follows

1. If λ is fractional, the coefficients of the particular solution are

$$\alpha_{\lambda-1} = \frac{D}{C}4\lambda K_\lambda \tag{3.13}$$

$\beta_{\lambda-4}$ are arbitrary unknowns. We divide $\beta_{\lambda-1}$ into two parts and let the first part

$$\beta_{\lambda-1} = \frac{D}{C}4\lambda L_\lambda \tag{3.14}$$

The second part is denoted by $\tilde{\beta}_{\lambda-1}$, of which only $\tilde{\beta}_{\frac{1}{2}+2n}$ are different from zero.

For $\alpha_{\lambda-1}$ and the first part of $\beta_{\lambda-1}$, we have:

$$A_{\lambda-1} = -\frac{D}{C}4\lambda K_\lambda \tag{3.15}$$

$$A_{\lambda-1+2n} = -\frac{4D\lambda(-k^2)^n}{cn!\,\phi(\lambda+n-2,n)}K_\lambda \tag{3.16}$$

$$B_{\lambda-1} = \frac{D}{C}4\lambda L_\lambda \tag{3.17}$$

$$B_{\lambda-1+2n} = -\frac{D}{C}\left\{\frac{4(\lambda-1)(-k^2)^n}{n(\lambda-1+n)(n-2)!\,\phi(\lambda+n,n-1)}L_\lambda + \right. \\ \left. + \frac{2(\lambda+1)(1-\nu)(-k^2)^n}{(n-1)!\,\phi(\lambda-1+n,n)}B_{\lambda+1}\right\} \tag{3.18}$$

$$M_\lambda = -\frac{4+(\lambda-1)(1-\nu)}{(\lambda+1)(1-\nu)}K_\lambda \tag{3.19}$$

For $\tilde{\beta}_{\frac{1}{2}}$, the general solution of (2.7') is:

$$f_0 = \sum_{n=0,1,\ldots}\tilde{\beta}_{\frac{1}{2}+2n}\cos(\tfrac{1}{2}+2n)\theta\sum_{m=0,1\ldots}\frac{k^{2m}r^{\frac{1}{2}+2(n+m)}}{m!\,\phi(\frac{1}{2}+2n,m)} \tag{3.20}$$

$$\tilde{\beta}_{\frac{1}{2}+2n} = \frac{1}{2}\frac{(-k^2)^n}{n!\,\phi(n-\frac{1}{2},n+1)}\tilde{\beta}_{\frac{1}{2}} \tag{3.21}$$

2. If λ is an integer, the coefficients of the particular solution $\alpha_{\lambda-1}$ are arbitrary unknowns. We divide $\alpha_{\lambda-1}$ into two parts. Let the first part

$$\alpha_{\lambda-1} = \frac{D}{C}4\lambda K_\lambda \tag{3.22}$$

The second part is denoted by $\tilde{\alpha}_{\lambda-1}$, of which only $\tilde{\alpha}_{1+2n}$ are different from zero.

$$\beta_{\lambda-1} = \frac{D}{C}4\lambda L_\lambda \tag{3.23}$$

For the first parts of $\alpha_{\lambda-1}$ and $\beta_{\lambda-1}$, we have:

$$A_{\lambda-1} = -\frac{D}{C}4\lambda K_\lambda \tag{3.24}$$

$$A_{\lambda-1+2n} = \frac{D}{C}\left\{\frac{4(\lambda-1)(-k^2)^n}{n(\lambda-1+n)(n-2)!\,\phi(\lambda+n,n-1)}K_\lambda - \frac{2(\lambda+1)(1-\nu)(-k^2)^n}{(n-1)!\,\phi(\lambda-1+n,n)}A_{\lambda+1}\right\} \tag{3.25}$$

$$B_{\lambda-1} = \frac{D}{C}4\lambda L_\lambda \tag{3.26}$$

$$B_{\lambda-1+2n} = \frac{4D\lambda(-k^2)^n}{Cn!\,\phi(\lambda+n-2,n)}L_\lambda \tag{3.27}$$

$$N_\lambda = -\frac{4+(\lambda-1)(1-\nu)}{(\lambda+1)(1-\nu)}L_\lambda \tag{3.28}$$

For $\tilde{\alpha}_1$, the general solution of (2.7') are:

$$f_0 = -\sum_{m=0,1\ldots}\tilde{\alpha}_{1+2n}\sin(1+2n)\theta\sum_{m=0,1\ldots}\frac{k^{2m}r^{1+2(n+m)}}{m!\,\phi(1+2n,m)} \tag{3.29}$$

$$\tilde{\alpha}_{1+2n} = \frac{(-k^2)^n}{(2n+1)!}\tilde{\alpha}_1 \tag{3.30}$$

In the above equations the basic unknowns are K_λ, $\tilde{\alpha}_1$ and $A_{\lambda+1}$ for symmetric case and the basic unknowns are L_λ, $\tilde{\beta}_{\frac{1}{2}}$ and $B_{\lambda+1}$, for anti-symmetric case. The others can be found by using corresponding recurrence relations.

Substituting (3.13)—(3.30) into (3.2), (3.10), the expressions of F, f can be found. Substituting F, f into (2.13)—(2.18), we can obtain the expressions of ψ_r, ψ_θ, W as well as M_r, $M_{r\theta}$, M_θ, Q_r, Q_θ.

NUMERICAL EXAMPLES

Example 1. Finite size plate with uniform bending moment

The results in [9] are improved in this paper. The difference is that in this paper we take more terms in the displacement expansions than in [9]. The results show that the difference is 1—2% only. The graphs and results are shown in Fig. 2 and Fig. 3 respectively.

Example 2. Finite Size plate with uniform twisting moment

In [10] and [11] the infinite plate with uniform twisting moment was studied. Up to now, the solution of this problem for finite plate is not available.

The graphs and numerical results of calculation are shown in Fig.4 and Fig. 5. In Fig. 5, the solution for a/L=0 is obtained by extrapolating method, which represents the solution for infinite plate and compares favourably with that in [11].

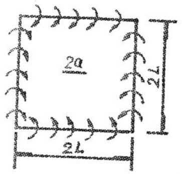

Fig.2

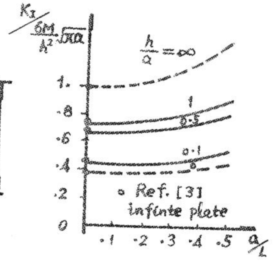

Fig.3

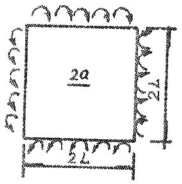

Fig. 4

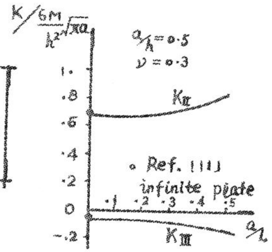

Fig. 5

CONCLUSIONS

1. This paper gives the general expansions of elastic stress-strain fields at the crack tip in modes I, II and III for plates of Reissner type, the expansions can serve as a basis for numerical methods for calculations of SIF in plates, such as boundary collcation, variational method, asymptotic method and higher order finite element method.

2. It is pointed out in this paper that the governing equations given in [12] for plates expressed in three generalized displacements are valid only for uncracked plates. For cracked plates it is necessary to use Eqs. (2.7)–(2.10) given in this paper.

3. For symmetric case (mode I), if we include in our calculations the terms of $O(r^{5/2})$ in the expansions of ψ_x, ψ_y and W, the good results could be obtained.

4. As the ratio a/L increases, both stress intensity factors K_{II} and K_{III} increase.

5. For symmetric case (mode I), the size of the special element should be taken 0.1–0.4 $[a,h]_{min}$, where $[a,h]_{min}$ is the minimum value among crack semilength a and thickness h. For anti-symmetric cases (modes II and III), it should be taken 0.04–0.06 $[a,h]_{min}$.

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